The chromatic polynomial and its roots

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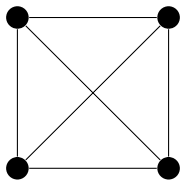
The chromatic polynomial

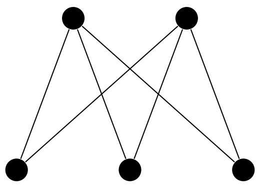
A colouring of a graph $G = (V, E)$ is a function $c : V \to S_c$ such that $\forall uv \in E : c(u) \neq c(v)$.

$$P(G, t) = \text{Number of colourings of } G \text{ using at most } t \text{ colours.}$$

Examples:



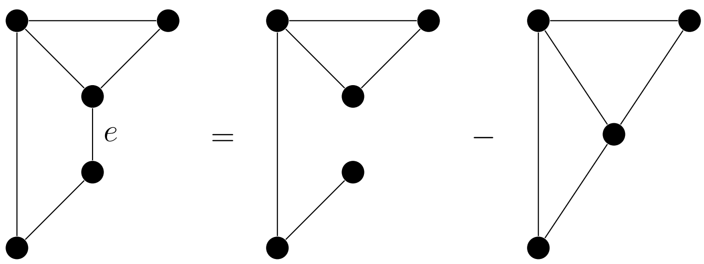
$$\begin{aligned} P(K_4, t) &= t(t-1)(t-2)(t-3) \\ &= t^4 - 6t^3 + 11t^2 - 6t \end{aligned}$$



$$\begin{aligned} P(K_{2,3}, t) &= t(t-1)^3 + t(t-1)(t-2)^3 \\ &= t(t-1)(t^3 - 5t^2 + 10t - 7) \\ &= t^5 - 6t^4 + 15t^3 - 17t^2 + 7t \end{aligned}$$

Deletion-Contraction:

$$P(G, t) \quad = \quad P(G \setminus e, t) \quad - \quad P(G/e, t)$$



By induction on $|E|$, $P(G, t)$ is a polynomial!

Exercise – Explain: Why is it enough to consider simple graphs?

The structure of a chromatic polynomial

Using the deletion-contraction relation, we can prove many facts about the chromatic polynomial:

$$P(G,t) = \sum_{i=0}^{n} a_i t^i$$

- $n = |V|$ and $a_n = 1$, $a_{n-1} = -|E|$
- The a_i alternate in sign and are whole numbers.
- $a_0 = 0$ and $a_i = 0 \Rightarrow a_{i-1} = 0$
- $\min\{i \mid a_i \neq 0\}$ is the number of connected components of G.
- ...

Log-Concavity (Huh, 2012):

$$a_i^2 > |a_{i+1}a_{i-1}|$$

$P(G,t)$ holds much information on G, leading to the following open problems:

- For a polynomial $p, \exists\ G : p = P(G,t)$?
- *(Chromatic unicity)* For which graphs $G : P(G,t) = P(H,t) \Rightarrow G \simeq H$?

Exercise – Construct two graphs $G \not\simeq H$ with $P(G,t) = P(H,t)$.

Zeroes of Chromatic Polynomials

A complex number z is a *chromatic root* if $\exists$ a graph $G : P(G,z) = 0$.

(Jackson, 1993) 1 is the only chromatic root in $(0, \frac{32}{27}]$.

(Thomassen, 1996) Chromatic roots are dense in the interval $(\frac{32}{27}, +\infty)$.

(Sokal, 2000) Chromatic roots are dense in the complex plane.

(Cameron & Morgan, 2016) Conjecture: For an algebraic integer α, $\exists\ n \in \mathbb{N} : \alpha + n$ is a chromatic root.

Exercise – Use a property of the chromatic polynomial to prove there are no chromatic roots in $(-\infty, 0)$.