

Evolution of parton densities in Quantum Chromodynamics

Master thesis

Proefschrift ter verkrijging van de graad van Master in de Fysica

2021

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Abstract

In this thesis an introduction to quantum chromodynamics (QCD) and explicit calculations of renormalization group evolution in QCD will be given. QCD is the theory that explains the Strong Interaction, one of the fundamental forces in nature, responsible for keeping quarks and gluons bound together in hadrons, and for binding protons and neutrons in nuclei. In the introduction to QCD this thesis will treat the QCD colour charge, the Lagrangian, the gauge symmetry related to QCD, the full quantisation and the renormalisation of the theory. Then the application of QCD to high energy collisions will be explained. Since in high energy collisions between hadrons the Strong Interaction is almost always involved, QCD is necessary as the theoretical background for such collisions, like those in the LHC. The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations for quark and gluon densities (also called parton distribution functions) in a hadron will be treated in terms of perturbative kernels given by quark and gluon splitting functions, computable as power series expansions in the QCD coupling. We will explain the physical meaning of the splitting functions as probabilities for QCD radiation processes from colour-charged particles. We will illustrate the dependence of the splitting functions on the longitudinal momentum transferred during the radiative process within the DGLAP framework. Then, by extending this framework, we will discuss how the dependence of the splitting functions on the transverse momenta involved in the radiative process can be computed. To this end, we will employ the high-energy factorization method in QCD. We will illustrate the application of this method by explicitly calculating the gluon-to-quark splitting function, at first order in the QCD coupling, including the dependence on both longitudinal momentum and transverse momentum. Finally, we will present numerical calculations of parton densities by using a computer code which solves the evolution equations by a Monte Carlo method.

Samenvatting

In deze thesis wordt een introductie tot quantum chromodynamica (QCD) en expliciete berekeningen van renormalisatiegroep evolutie in QCD gegeven. QCD is de theorie achter de Sterke Kernkracht, één van de fundamentele krachten in de natuur, verantwoordelijk voor het binden quarks en gluonen in hadronen, en voor het binden van protonen en neutronen in atoomkernen. In de introductie tot QCD zal deze thesis de QCD kleurlading, de Lagrangiaan, de gauge symmetrie gerelateerd aan QCD, en de volledige kwantisatie en renormalisatie van de theorie geven. Daarna wordt de toepassing van QCD op hoge energie botsingen uitgelegd. Omdat in hoge energie botsingen tussen hadronen de Sterke Kernkracht altijd betrokken is, is QCD nodig als theoretische achtergrond voor zulke botsingen. (Zoals de botsingen in de LHC) De Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolutievergelijkingen voor quark- en gluondichtheden (ook parton distributie functies genoemd) in een hadron zullen worden beschreven door perturbatieve kernels gegeven door quark- en gluonsplittingfuncties. Deze splittingfuncties zijn berekenbaar als machtreeksen in de QCD koppeling. We zullen de fysieke betekenis van de splittingfuncties uitleggen als waarschijnlijkheden voor QCD stralingsprocessen van deeltjes met kleurlading. We zullen tonen dat de splittingfuncties afhankelijk zijn van de longitudinale impuls die wordt overgebracht tijdens het stralingsproces in het DGLAP kader. Dan zullen we dit kader uitbreiden, en wordt uitgelegd hoe we kunnen bereken hoe de splittingfuncties afhankelijk zijn van de transversale impulsen die aanwezig zijn in het stralingsproces. Om dit te doen zullen we de methode van hoge-energie-factorisatie toepassen in QCD. We zullen de toepassing van deze methode uitleggen door expliciet de gluon-naar-quark splittingfunctie uit te rekenen, tot op eerste orde in de QCD koppeling, en met inbegrip van de afhankelijkheid van zowel de longitudinale als transversale impuls. Uiteindelijk zullen we numerieke berekeningen van partondichtheden laten zien, door het gebruik van een computerprogramma dat de evolutievergelijkingen oplost met behulp van een Monte Carlo methode.

Acknowledgement

I would like to thank several people first, before going to the main body of the thesis. They all contributed in some way, for me to finish this project.

I would like to express my greatest gratitude towards my thesis advisor, Prof. Dr. F Hautmann, for his advice and patience, and for making this project possible.

I would like to thank the members of the theoretical research group that I was involved in during this project. Especially Ola Lelek and Lissa Keersmaekers. They helped me a lot for the numerical part of this thesis.

I would like to thank my family for their love and support during the time of this project. Especially my mother and sister.

I also want to thank everyone in general who read some part of the thesis for checking spelling and grammatics.

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Conventions and units

Units:

Throughout this thesis, we will use units in which $c = \hbar = 1$.

This will simplify most equations, and most units can be considered equivalent to each other up to some power:

$$[\text{length}] = [\text{time}] = [\text{mass}]^{-1} = [\text{energy}]^{-1}$$

We will express both mass and energy in eV (electronvolt).

Special Relativity, tensors, and indices:

The metric we use will always be the Minkowski metric with the following convention:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The inverse metric $g^{\mu\nu}$ will have the same components as $g_{\mu\nu}$.

Greek indices will always denote full spacetime indices and run from 0 to 3, where 0 is the time component and 1, 2, 3 are the spatial components (which we will also call space components). Sometimes we will only consider the spatial components of a tensor, and in this case we will use lowercase Roman indices i, j, k , etc (these run over 1, 2, 3).

In case of spacetime indices, it matters if they appear as superscript or subscript. These are respectively contravariant and covariant indices. Four-vectors will usually be denoted by italic letters. Three-vectors will be denoted by non-italic boldface letters. As example we give the position vector to demonstrate these conventions:

$$x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x}); \quad x_\mu = g_{\mu\nu}x^\mu = (x^0, -\mathbf{x}) = (t, -\mathbf{x})$$

$$\mathbf{x}^i = (x^1, x^2, x^3)$$

The exception for boldface symbols will be the identity matrix $\mathbf{1} = \mathbf{1}_{n \times n}$

In some cases we will not write the Greek or Roman index if we mean the whole vector and if it is unlikely to cause confusion. For example with $f(x)$ we would mean a function of the spacetime position vector x .

For the four-gradient we have

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla \right) = \left(\frac{\partial}{\partial t}, \nabla \right)$$

where ∇ is the three-gradient, which is the usual 3D-space gradient. A time derivative of a function will be denoted by a dot on top:

$$\partial_0 \varphi(x) = \frac{\partial}{\partial t} \varphi(x) = \dot{\varphi}(x)$$

The notation for the d'Alembertian will be

$$\square = \partial^\mu \partial_\mu$$

All non-spacetime indices will also be denoted by Roman indices, and for these indices it doesn't matter if they appear as superscript or subscript.

Repeated indices will always be summed over, unless stated otherwise. For scalar products we will sometimes also omit the indices.

$$p^\mu x_\mu = \sum_\mu p^\mu x_\mu = p \cdot x = px$$

Functions and fields:

A field is a tensor function of spacetime, with one "value" for each point in spacetime. Instead of a simple tensor, it can also be a combination of multiple tensors.

For functions, and especially functions of spacetime, we will often omit the arguments in our notation: $f = f(x)$

Fourier transformation:

$$f(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} f(p) ; \quad f(p) = \int d^4 x e^{ipx} f(x)$$

Dirac-delta function:

$$\int d^n x \delta^{(n)}(x) = 1$$

$$\int d^n x e^{ipx} = (2\pi)^n \delta^{(n)}(p)$$

Other:

The commutator: $[A, B] = AB - BA$

The anticommutator: $\{A, B\} = AB + BA$

The elementary electric charge e will be considered positive, and thus the opposite of the electron charge: $e := |e|$

In Feynman diagrams time will usually run from left to right, unless stated otherwise.

1 Generalities on QCD

This chapter contains the general understanding and construction of QCD, and SU(N) gauge field theories in general, and compared to QED. We will build up, quantise, and renormalise the Lagrangian, and give the Feynman rules. This chapter is heavily based on the QFT books [4] and [5].

1.1 QCD as part of the Standard Model

This section contains a general overview of the Standard Model (SM) and the fundamental interactions, and why quantum chromodynamics (QCD) is needed.

1.1.1 Content of the Standard Model

The Standard Model of particle physics is the theory describing the known fundamental particles and known fundamental forces (except gravity, if you would consider this a force). The Standard Model is based on quantum field theory (QFT). QFT is a theoretical framework that unifies special relativity, quantum mechanics and classical field theory. It is able to describe quantum mechanics in Minkowski spacetime, and where the number of particles does not need to remain constant (due to creation and annihilation). The Standard Model is a specific choice of such quantum field theories, which describes the known particle physics in our universe.

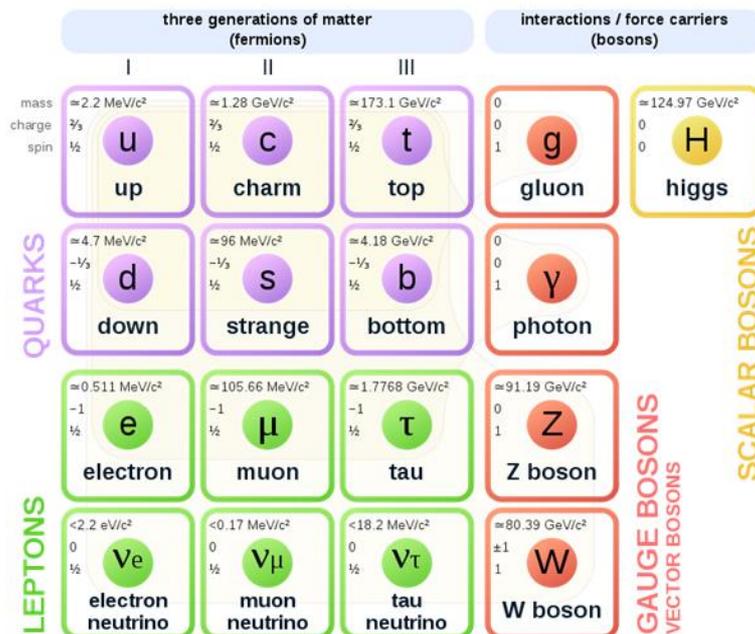


Figure 1.1: Elementary particles of the Standard Model, with property values as of 2008. Figure from [6]

The particles in the Standard Model can be divided into bosons and fermions. Bosons have integer spin and obey Bose-Einstein statistics. Fermions have half-integer spin and obey Fermi-Dirac statistics. It appears that matter is made of the elementary fermions, while the forces are mediated by the elementary bosons.

The elementary fermions all have spin $1/2$ and are divided into leptons and quarks. The different lepton and quark species are called “flavours”. There are six lepton flavours (electron, muon, tau, electron-neutrino, muon-neutrino and tau-neutrino) and six quark flavours (up, down, charm, strange, top, bottom). Both leptons and quarks can be divided into what we call “families” or “generations”. There are three lepton families, each consisting of a charged lepton (electron, muon, tau) and its associated neutrino (electron-neutrino, muon-neutrino, tau-neutrino). For the quarks, there are also three families. The first family consists of the up and down quarks, the second family consists of the charm and strange quarks, and the third family consists of the top and bottom quarks. All fermions have mass, possibly with the exception of one of the neutrino mass eigenstates. Neutrinos are extremely light, and the only evidence for neutrino masses is due to neutrino oscillations. We can only observe neutrino flavour eigenstates, but these are not equal to the neutrino mass eigenstates. Since this only tells us something about squared mass differences, we only know that at least two of the three mass eigenstates have non-zero mass.

The elementary bosons can be divided into gauge bosons and the Brout-Englert-Higgs boson. The gauge bosons all have spin 1 and are the carriers of the three main fundamental forces. These forces are electromagnetism (EM), weak interaction (also called weak force or weak nuclear force), and strong interaction (also called strong force or strong nuclear force). The photon is the gauge boson of electromagnetism and interacts with everything that has an electric charge. W and Z bosons are the gauge bosons of the weak interaction, the interaction related to weak isospin (weak charge) and responsible for the radioactive decay of atoms. There are two W bosons, W^+ with a positive charge and W^- with a negative charge. Gluons are the gauge bosons of the strong interaction, which is an interaction associated with what we call “colour charges”. There are eight different gluons. The strong interaction between quarks and gluons, which is mediated by gluons, also gives rise to the strong interaction between hadrons. The strong interaction is what keeps quarks together in a proton or neutron, and it also keeps protons and neutrons together in the nucleus of an atom. These three fundamental interactions are caused by gauge symmetries. Of the gauge bosons, only the W and Z bosons have mass. The photon and gluons are massless.

The Brout-Englert-Higgs (BEH) boson has spin 0 and is responsible for the mass of the elementary particles in the Standard Model. Even though the interactions associated with the BEH boson are often not considered one of the fundamental forces, it can technically be considered one.

Every particle also has an antiparticle, which is the same as the particle but with opposite charges (at least for electric and colour charges, but for weak isospin this is not always true). In the case of the photon, Z boson, gluons and BEH boson, they are their own antiparticle.

Gravity is not a part of the Standard Model. Gravity is often considered a fundamental force, but whether it's a force or not depends on the theory. The currently accepted theory for gravity is general relativity, in which gravity isn't a force at all.

1.1.2 Fundamental interactions

In quantum field theory (QFT), every particle can be described by a quantized field. Spin 0 bosons are described by scalar fields, spin 1 bosons are described by vector fields, and spin 1/2 fermions are described by spinor fields. The theory is described by a Lagrangian containing free field terms and interaction terms. Every basic interaction in the theory corresponds to a specific interaction term in the Lagrangian, which is a product between several fields. In the context of Feynman diagrams, such an interaction term can be thought of as a vertex in which the involved particles meet. In the case of a force arising from gauge symmetry, a gauge boson interacts with everything that carries a charge associated with the gauge symmetry.

Electromagnetism (EM) is described by quantum electrodynamics (QED), which is a type of quantum field theory called an abelian gauge field theory. It describes interactions associated with an electric charge, due to a local U(1) gauge symmetry. The photon is the gauge boson that mediates this force, which means it interacts with everything that has an electric charge.

Weak interaction can be described by a type of QFT called a non-abelian gauge field theory. It describes interactions associated with weak isospin, due to a local SU(2) gauge symmetry. There are three weak isospin charges, of which the T_3 is the most important one. The W and Z gauge bosons interact with everything that has weak isospin, which are the left-handed fermions (spin and momentum have opposite direction) and right-handed antifermions (spin and momentum have the same direction).

Strong interaction is described by quantum chromodynamics (QCD), which is also a non-abelian gauge field theory. It describes interactions associated with what we call "colour charges", due to a local SU(3) gauge symmetry. The gluons are the gauge bosons that mediate this force, which means they interact with everything that has colour charge. The only particles that carry colour charge are quarks and gluons themselves. For the quarks there are three different colour charges, while for the gluons there are eight.

Both QED and weak interaction can be unified to electroweak interaction (EW). This interaction uses a local $U(1) \times SU(2)$ symmetry and is related to weak isospin and weak hypercharge. For the strong interaction, we have not observed a similar unification (although some theories beyond the Standard Model unify strong interactions with the other two interactions at very high energies).

The Brout-Englert-Higgs (BEH) boson is responsible for the mass of the W and Z bosons via spontaneous electroweak symmetry breaking. The BEH mechanism completes the theory of electroweak interaction. In addition to this, the masses of the elementary fermions can also be explained by an interaction with the BEH field, via Yukawa couplings. The BEH boson is massive and interacts with everything that has mass, including itself. Since the vacuum expectation value is nonzero for the scalar field that breaks electroweak symmetry, it will provide mass to the coupled fields even in the absence of any BEH field excitations.

Particle		Spin	Electric charge	Weak isospin T_3	Colour charge	Interacts via
Leptons	e^-, μ^-, τ^-	1/2	-1	-1/2 or 0	0	EM, Weak, BEH
	ν_e, ν_μ, ν_τ	1/2	0	+1/2	0	Weak, BEH?
Antileptons	e^+, μ^+, τ^+	1/2	+1	+1/2 or 0	0	EM, Weak, BEH
	$\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$	1/2	0	-1/2	0	Weak, BEH?
Quarks	u, c, t	1/2	+2/3	+1/2 or 0	r, g, or b	all
	d, s, b	1/2	-1/3	-1/2 or 0		
Antiquarks	$\bar{u}, \bar{c}, \bar{t}$	1/2	-2/3	-1/2 or 0	$\bar{r}, \bar{g}, \text{ or } \bar{b}$	all
	$\bar{d}, \bar{s}, \bar{b}$	1/2	+1/3	+1/2 or 0		
Photon	γ	1	0	0	0	EM*, EW**
W bosons	W^+	1	+1	+1	0	EM, Weak, EW**, BEH
	W^-	1	-1	-1	0	
Z boson	Z	1	0	0	0	Weak, EW**, BEH
Gluons	g	1	0	0	$(r\bar{b} + b\bar{r})/\sqrt{2}$	Strong
					$(r\bar{g} + g\bar{r})/\sqrt{2}$	
					$(g\bar{b} + b\bar{g})/\sqrt{2}$	
					$-i(r\bar{b} - b\bar{r})/\sqrt{2}$	
					$-i(r\bar{g} - g\bar{r})/\sqrt{2}$	
					$-i(b\bar{g} - g\bar{b})/\sqrt{2}$	
					$(r\bar{r} - b\bar{b})/\sqrt{2}$	
$(r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}$						
BEH boson	H^0	0	0	-1/2	0	BEH

Table 1.1: Standard Model particles with their spin, charges and interactions they take part in.

* Solely as a force carrier, not by self-interaction or other gauge bosons from the same force.

** Has an interaction from electroweak theory that it wouldn't have without this unification.

Also note that the weak isospin 0 versions of the fermions and antifermions don't interact via weak interactions.

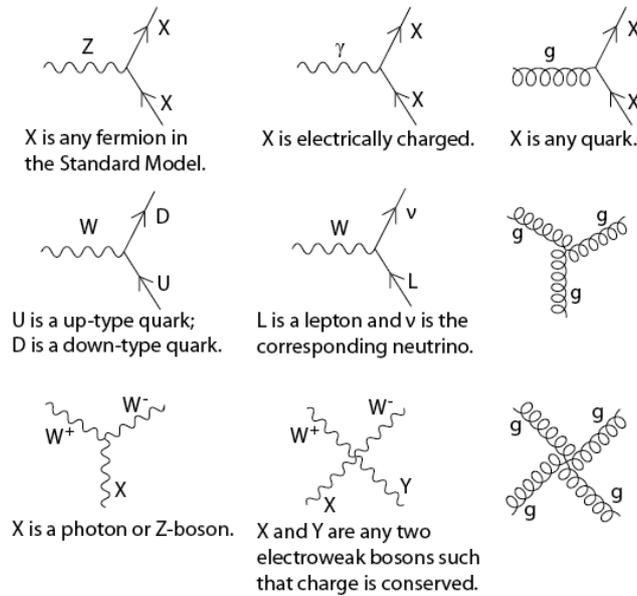


Figure 1.2: Interaction vertices due to gauge symmetries. (Interactions involving BEH boson omitted)
Figure from [7]

The photon and gluons are massless, which means the electromagnetic and strong interactions have infinite range. The W and Z bosons are massive, which means the weak interaction is short range. The coupling strengths of the interactions depend on the energy at which the interaction is measured. For the electromagnetic interaction, the coupling strength increases with energy. But for the strong and weak interactions, it decreases with energy.

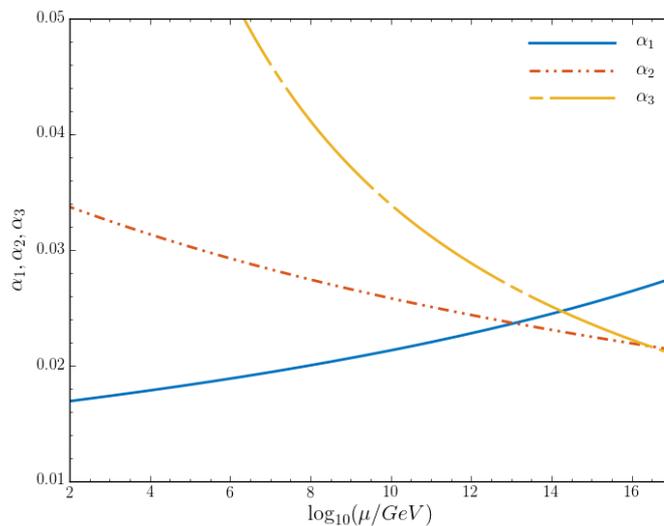


Figure 1.3: Coupling strengths in function of energy scale μ : α_1 for QED, α_2 for weak interaction, and α_3 for QCD. Figure from [8]

Whether the interaction results in an attractive or repulsive potential between charges in the nonrelativistic limit depends on the type of exchanged boson. For a Yukawa

coupling with a scalar boson, this nonrelativistic potential will always be attractive. For a (gauge) interaction via a vector boson, this nonrelativistic potential will be repulsive between like charges but will be attractive between unlike charges.

Exchanged particle	ff or $\bar{f}\bar{f}$	$f\bar{f}$
Scalar (Yukawa)	attractive	attractive
Vector (gauge)	repulsive	attractive

1.1.3 Why colour and QCD?

In the 1950s and onwards, a lot of new particles were discovered called hadrons. Among the hadrons there were fermions, called baryons, and there were bosons, called mesons. (While the proton and neutron were discovered decennia earlier, they are also hadrons.) More and more hadrons were being discovered, and Gell-Mann and Zweig suggested in 1964 that hadrons are not fundamental particles, but are composed of constituent particles called quarks. (At that time there were three known quarks: up, down and strange. The other three were too massive to show up in experiments when the idea of quarks was introduced. The latest and heaviest one, the top quark, was theorized in 1975 and discovered in 1995.) Mesons are bound states of a quark and an antiquark, and baryons are bound states of three quarks (and antibaryons consist of three antiquarks). This (pre-QCD) quark model for hadrons was able to describe the observed hadron spectrum. Some examples of baryons are the proton (uud) and neutron (udd). Some examples of mesons are the π^+ ($u\bar{d}$), π^- ($d\bar{u}$) and π^0 ($u\bar{u}$ and $d\bar{d}$).

But there are also some problems with this model. Free quarks have never been observed. Bound states of two quarks have also never been observed. Why only baryons and mesons? Are quarks real particles, or just convenient mathematical constructs and not real particles at all? Other problems are the symmetry of the hadron wavefunction and violation of the Pauli exclusion principle.

A hadron's wavefunction could be decomposed into three parts: A space wavefunction, a spin wavefunction, and a flavour wavefunction.

$$|\psi\rangle = |\psi_{\text{space}}\rangle \otimes |\psi_{\text{spin}}\rangle \otimes |\psi_{\text{flavour}}\rangle \quad (1.1.1)$$

For baryons, since they're fermions, the overall wavefunction needs to be antisymmetric under permutations of the quarks.

The Ω^- baryon (sss), which contains three identical strange quarks with their spin all in the same direction (so the total spin is $3/2$), causes a problem. Because space, spin, and flavour wavefunctions are all symmetric under interchange of two of the quarks, we have a symmetric wavefunction for the Ω^- instead of an antisymmetric one.

A similar situation happens with the Δ^{++} baryon (uuu), which contains three identical up quarks with total spin 3/2. This causes the same problem as with the Ω^- .

Since these problematic baryons have three quarks with identical quantum numbers, they also violate the Pauli exclusion principle. To solve these problems, an extra quantum number was suggested. Since the problematic baryons have three seemingly identical quarks, each of them should differ in this new quantum number to obey the Pauli exclusion principle and to make the overall wavefunction antisymmetric. So this new quantum number should have at least three different values, or in other words, it must have a triplet representation. This new quantum number is the “colour state”. The wavefunction of a hadron becomes:

$$|\psi\rangle = |\psi_{\text{space}}\rangle \otimes |\psi_{\text{spin}}\rangle \otimes |\psi_{\text{flavour}}\rangle \otimes |\psi_{\text{colour}}\rangle \quad (1.1.2)$$

With the colour wavefunction antisymmetric under interchange of two quarks.

Because mesons are made of a quark and antiquark, this triplet representation must be complex. We can write the colour wavefunctions as follows:

$$\text{Baryon: } \psi_{\text{colour}} \sim \varepsilon^{ijk} q_i q_j q_k$$

$$\text{Antibaryon: } \psi_{\text{colour}} \sim \varepsilon^{ijk} \bar{q}_i \bar{q}_j \bar{q}_k$$

$$\text{Meson: } \psi_{\text{colour}} \sim \delta^{ij} q_i \bar{q}_j$$

This hints to a SU(N) symmetry group with the number of colours $N \geq 3$. The exact number of colours comes from measuring cross sections. For example, comparing the cross section for $e^- e^+ \rightarrow q \bar{q}$ with the cross section for $e^- e^+ \rightarrow \mu^- \mu^+$. Experiments yield $N=3$.

This theory of colour was developed by Han, Nambu and Greenberg in 1964-1965. Note that there are only colour neutral hadrons. In other words, only singlet states.

High energy electron-proton scattering experiments showed that hadrons were indeed made up of smaller point-like constituents, by Bjorken and Feynman in 1968-1969. These constituents are called “partons”. Further deep inelastic scattering showed more and more structure of the proton. Eventually, these experiments show that at very high energies, the proton does not only consist of three valence quarks but also gluons (the gluon was theorised earlier by Gell-Mann) and “sea quarks” (quark-antiquark pairs).

Quantum chromodynamics was developed by Fritzsche, Gell-Mann and Leutwyler in 1973 as a quantum field theory for the strong interaction, by employing Yang-Mills theory for SU(3). This theory proves successful in explaining the observations and making predictions.

At high energy, a phenomenon called “asymptotic freedom” is observed. This means that the results of these high energy experiments can be explained by applying perturbation theory, while this is impossible at low energies. The strength of the strong interaction decreases with increasing energy. This property allows the partons to be considered nearly “free” at very high energy.

The inability to isolate a single colour charge is called “confinement”. This is understood by QCD as follows: If you try to separate a single colour charge from a hadron, the force will not decrease with distance and remain large. Because it remains large, it will create quarks and antiquarks to turn this separating charge and the original hadron into colour neutral hadrons again. While it is usually understood like this, there is not an analytic proof yet.

To summarise, full QCD can explain:

- The hadron spectrum.
- How baryon wavefunctions can be antisymmetric even if they consist of three identical quark flavours with identical spin direction, and how this doesn't violate the Pauli exclusion principle.
- Partons.
- Asymptotic freedom.
- Hard scattering experiments, like deep inelastic scattering and Drell-Yan.
- Confinement: why a free, separated quark or gluon has never been observed. But confinement has not yet been proven analytically.
- Why hadrons are colour singlet states (colour neutral). This is related to confinement.

1.2 QCD Lagrangian

In this section, we start from the free Dirac field and develop the gauge field theories for both $U(1)$ and $SU(N)$ gauge symmetries. The gauge theory for $SU(3)$ will be the theory for QCD. The theories in this section are not quantum field theories yet, in other words, they are not quantised by “second quantisation” yet. These theories will be referred to as “classical”, despite that they might be compatible with quantum mechanics as wave equations. The full quantisation to a quantum field theory will be done in section 1.3.

1.2.1 The free Dirac field

We start with the free Dirac field, which is a field that describes free massive spin $1/2$ particles that are not their own antiparticles (such as the electron, positron, quarks,

etc), in a special relativistic way. The Lagrangian density for the free Dirac field is given by:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi \quad (1.2.1)$$

In this equation, and throughout the whole thesis, we use units in which $c = \hbar = 1$.

$\psi = \psi(x)$ is a spin 1/2 spinor field, and m is the rest mass. The slash notation in the equation denotes a contraction with the gamma matrices: $\not{A} = \gamma^\mu A_\mu$

The gamma matrices are a set of four $n \times n$ matrices that satisfy the anticommutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{n \times n} \quad (1.2.2)$$

In our case, this will be 4×4 matrices. We will often use $\mathbf{1}$ as a notation for $\mathbf{1}_{n \times n}$, and omit it in multiplications, because it is usually understood that we are dealing with matrices if gamma matrices appear in our equations.

The gamma matrices also obey the following conditions $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$, where the i components are the spatial components. (For more about gamma matrices, see Ref [5].)

The spinors ψ and $\bar{\psi}$ will each have n components (4 in our case). $\bar{\psi}$ is the adjoint field for ψ and is given by:

$$\bar{\psi} = \psi^\dagger \gamma^0 \quad (1.2.3)$$

Throughout this book we will use the Weyl (also called chiral) representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (1.2.4)$$

With σ^i the Pauli matrices. This representation allows us to write the spinor ψ as:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad (1.2.5)$$

Where ψ_L and ψ_R are the left handed and right handed Weyl spinors, each having two components.

Under a Lorentz transformation, the spinor ψ will transform as

$$\psi_a(x) \rightarrow \psi'_a(x') = \psi_a(x) - \frac{i}{4} \varepsilon_{\mu\nu} \sigma^{\mu\nu}_{ab} \psi_b(x) \quad (1.2.6)$$

With the components of the spinor fields and gamma matrices labeled with a and b , ε the Levi-Civita tensor, and with $\sigma^{\mu\nu}_{ab}$ proportional to the commutator of the gamma matrices as follows:

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \quad (1.2.7)$$

Now we go back to the Dirac Lagrangian and derive the Euler-Lagrange equations of motion for ψ .

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (1.2.8)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad (1.2.9)$$

Where the $\partial \partial_\mu \psi$ in the denominator means that it is a partial derivative with respect to $\partial_\mu \psi$.

Applying this to $\mathcal{L}_{\text{Dirac}}$ we get:

$$i \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0 \quad (1.2.10)$$

$$(i \gamma^\mu \partial_\mu - m) \psi = 0 \quad (1.2.11)$$

These are the Dirac equations for $\bar{\psi}$ and ψ respectively. They can be interpreted as the relativistic wave equations for these spin 1/2 fields.

From the Lagrangian density, we can derive the Hamiltonian density, by using

$$\mathcal{H}_{\text{Dirac}} = \pi_a \dot{\psi}_a + \bar{\pi}_a \dot{\bar{\psi}}_a - \mathcal{L}_{\text{Dirac}} \quad (1.2.12)$$

Where the dot denotes a partial derivative with respect to time, and π_a and $\bar{\pi}_a$ are the fields conjugate to ψ_a and $\bar{\psi}_a$

$$\pi_a = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial \dot{\psi}_a} = i \psi_a^\dagger; \quad \bar{\pi}_a = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial \dot{\bar{\psi}}_a} = 0 \quad (1.2.13)$$

If we plug this in the equation for the Hamiltonian density, we get:

$$\mathcal{H}_{\text{Dirac}} = \bar{\psi} \left(-i \gamma^j \frac{\partial}{\partial x^j} + m \right) \psi \quad (1.2.14)$$

If we want the Hamiltonian, all we need to do is integrate the Hamiltonian density over space:

$$H_{\text{Dirac}} = \int d^3x \mathcal{H}_{\text{Dirac}}(x) = \int d^3x \bar{\psi}(x) \left(-i \gamma^j \frac{\partial}{\partial x^j} + m \right) \psi(x) \quad (1.2.15)$$

This Hamiltonian equals the total energy of the free Dirac field, and will be constant.

1.2.2 U(N) and SU(N) groups

Symmetries play a very important role in field theories, and the continuous unitary groups U(N) and SU(N) are the fundamental reason for gauge theories in the Standard Model. Both U(N) and SU(N) are Lie groups, groups that are differentiable manifolds. We will now give an overview of the main properties of these groups, but we will be mostly focusing on SU(N) and U(1) since only these are of importance in this thesis.

U(N) is the unitary group, the group of $N \times N$ unitary matrices, which means:

$$\forall G \in U(N) : G^\dagger G = G G^\dagger = \mathbf{1}_{N \times N} \quad (1.2.16)$$

In other words, the Hermitian conjugate is equal to the inverse $G^{-1} = G^\dagger$

This means that there are only N^2 independent components, and thus U(N) is a group of dimension N^2 .

U(1) is equivalent to the circle group and can be represented as the complex numbers with modulus (absolute value) equal to 1

$$\forall G \in U(1) : G = e^{i\alpha} \quad \alpha \in \mathbb{R} \quad (1.2.17)$$

SU(N) is the special unitary group, the group of $N \times N$ unitary matrices with determinant equal to 1. Just like with U(N), this means that the Hermitian conjugate is equal to the inverse, but the additional condition that $\det(G) = 1$ means that there are only $N^2 - 1$ independent components. Thus SU(N) is a group of dimension $N^2 - 1$. We can represent the group elements as:

$$\forall G \in SU(N) : G = e^{i\alpha^a T^a} \quad (1.2.18)$$

α^a are real numbers, T^a are the generators for the group, and the index a runs from 1 to $N^2 - 1$. The generators are traceless Hermitian matrices, and form a Lie algebra for SU(N), with commutation relations:

$$[T^a, T^b] = i f^{abc} T^c \quad (1.2.19)$$

f^{abc} are the structure constants and are totally antisymmetric, which means that swapping any two indices will result in a minus sign, and this can be expressed by using the notation $[abc]$ for total antisymmetrisation of the indices abc :

$$f^{abc} = f^{[abc]} \quad (1.2.20)$$

There are different possible representations for the generators T^a . The fundamental representation T_F^a , which we will often just call T^a , is a $N \times N$ representation. The antifundamental representation $T_{\bar{F}}^a$ is defined as $T_{\bar{F}}^a = -T_F^{a*}$, and thus also a $N \times N$ representation. The adjoint representation T_A^a is a $(N^2 - 1) \times (N^2 - 1)$ representation for which we have $(T_A^a)^{bc} = -i f^{abc}$.

We define the Casimir invariant C_R for a representation R as

$$T_R^a T_R^a = C_R \mathbf{1}_{d_R} \quad (1.2.21)$$

where $\mathbf{1}_{d_R}$ is the $d_R \times d_R$ identity matrix, and with d_R the dimension of the representation R .

We define the trace invariant T_R for a representation R as

$$\text{Tr}(T_R^a T_R^b) = T_R \delta^{ab} \quad (1.2.22)$$

They are related to each other by $C_R d_R = T_R d_G$, with $d_G = N^2 - 1$ (the dimension of the group SU(N)).

For the fundamental and adjoint representations we have:

$$T_F = \frac{1}{2}; \quad C_F = \frac{N^2 - 1}{2N}; \quad T_A = C_A = N \quad (1.2.23)$$

A special case is SU(1), which is the trivial group. It has the identity as its only element. The elements of U(N) and SU(N) can be interpreted as (multidimensional) phase transformations. U(1) is an abelian group, while SU(N>1) and U(N>1) are non-abelian groups. U(1) will be the group underlying electromagnetism, while SU(3) will be the group underlying QCD. For SU(3), the group generators in the fundamental representation can be written as

$$T^a = T_F^a = \frac{1}{2} \lambda^a \quad (1.2.24)$$

With λ^a the Gell-Mann matrices. (See ref [5])

1.2.3 Global phase invariance and Noether current

We will now take the free Dirac Lagrangian and check if it is invariant under global phase transformations. These transformations will be elements of U(N) or SU(N) groups. The free Dirac Lagrangian is:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\cancel{\partial} - m)\psi \quad (1.2.1)$$

But this time we will consider a spinor field ψ that is a combination of N spinor fields ψ_r , and has the following form:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix} \quad (1.2.25)$$

Where the components ψ_r themselves are spinor fields, each of them having 4 components, and r runs from 1 to N. The adjoint field $\bar{\psi}$ will have the form:

$$\bar{\psi} = (\bar{\psi}_1 \quad \bar{\psi}_2 \quad \dots) \quad (1.2.26)$$

(Keep in mind that all these fields are functions of spacetime coordinates)

We will check if this Lagrangian is invariant under global SU(N) transformations. We will transform the fields as follows:

$$\psi \rightarrow \psi' = G\psi \quad (1.2.27)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger \quad (1.2.28)$$

With $G = e^{i\alpha^a T^a}$ and $G^\dagger = G^{-1} = e^{-i\alpha^a T^a}$ (Remember that T^a are Hermitian matrices, and thus $T^{a\dagger} = T^a$).

T^a is in the fundamental representation, and α^a are real constants. (The word “global” refers to α^a being constant.)

If we plug this in our free Dirac Lagrangian, we get

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = \bar{\psi}G^\dagger(i\cancel{\partial} - m)(G\psi) = i\bar{\psi}G^\dagger G\cancel{\partial}\psi - m\bar{\psi}G^\dagger G\psi \quad (1.2.29)$$

And since $G^\dagger G = 1$, we just get the original Lagrangian again:

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = i\bar{\psi}\cancel{\partial}\psi - m\bar{\psi}\psi = \mathcal{L}_{\text{Dirac}} \quad (1.2.30)$$

We can conclude that the free Dirac Lagrangian is invariant under a global SU(N) symmetry. This is also true for U(N). Generally, a Dirac field of the form we considered (with N component fields), will always be invariant under global U(1) (with its generator the N×N identity matrix), global SU(N), and global U(N).

If a field theory has a symmetry (from a continuous symmetry group), then according to Noether’s theorem there is a conserved current for each generator of the symmetry group. For a symmetry transformation G that results in

$$\psi \rightarrow \psi' = G\psi = \psi + \delta\psi \quad (1.2.31)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger = \bar{\psi} + \delta\bar{\psi} \quad (1.2.32)$$

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = \mathcal{L}_{\text{Dirac}} + \delta\mathcal{L}_{\text{Dirac}} \quad (1.2.33)$$

Noether theorem tells us that the following equation holds for the Dirac field [4 (p32)]:

$$\delta\mathcal{L}_{\text{Dirac}} = \partial_\mu \left(\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\partial_\mu\psi} \delta\psi + \delta\bar{\psi} \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\partial_\mu\bar{\psi}} \right) = \partial_\mu \left(\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\partial_\mu\psi} \delta\psi \right) = 0 \quad (1.2.34)$$

since for our global phase transformations $\delta\mathcal{L}_{\text{Dirac}} = 0$

This means that the expression inside the brackets is a conserved current.

We can consider infinitesimal transformations since we are dealing with continuous groups, which means

$$\psi \rightarrow \psi' = G\psi = e^{i\alpha^a T^a} \psi \approx (1 + i\alpha^a T^a)\psi = \psi + \delta\psi \quad (1.2.35)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger = \bar{\psi}e^{-i\alpha^a T^a} \approx \bar{\psi}(1 - i\alpha^a T^a) = \bar{\psi} + \delta\bar{\psi} \quad (1.2.36)$$

So if we plug it in the equation related to Noether's theorem, we get

$$\partial_\mu(i\bar{\psi}\gamma^\mu\delta\psi) = \alpha^a\partial_\mu(\bar{\psi}\gamma^\mu T^a\psi) = 0 \quad (1.2.37)$$

This is true for any arbitrary α^a , which means

$$\partial_\mu(\bar{\psi}\gamma^\mu T^a\psi) = 0 \quad (1.2.38)$$

with $\bar{\psi}\gamma^\mu T^a\psi$ being the conserved currents. This will also hold for transformations that aren't infinitesimal, by considering them an infinite amount of infinitesimal transformations. If we integrate the timelike ($\mu = 0$) current over space, we get a conserved charge for each T^a [4 (p32)]

$$F^a = \int d^3x \bar{\psi}\gamma^0 T^a\psi = \int d^3x \bar{\psi}^\dagger \gamma^0 \gamma^0 T^a\psi = \int d^3x \bar{\psi}^\dagger T^a\psi = \text{constant} \quad (1.2.39)$$

These are what we call the colour charges.

1.2.4 Local phase invariance for an abelian gauge group (U(1))

What if instead of a global transformation $G = e^{i\alpha^a T^a}$, with α^a constant, we would try to do the same for a local transformation where $\alpha^a = \alpha^a(x)$? We will first try this with the U(1) symmetry group, which is an abelian group, before we try this with the more complicated SU(N) symmetry groups.

Again we start with the free Dirac Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\cancel{\partial} - m)\psi \quad (1.2.1)$$

and this time we will check if it is invariant under local U(1) transformations. We will transform the fields as follows:

$$\psi \rightarrow \psi' = G\psi = e^{i\alpha}\psi \quad (1.2.40)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger = \bar{\psi}e^{-i\alpha} \quad (1.2.41)$$

With $G = e^{i\alpha}$ and $G^\dagger = G^* = G^{-1} = e^{-i\alpha}$, with $\alpha = \alpha(x)$.

Because in this case α isn't a constant anymore but depends on the spacetime coordinates, we will do a phase transformation of the field where the phase

transformation itself can be different in different spacetime locations. (This is what the word “local” refers to.) This in contrast with a global phase transformation where we transform the field by the same phase everywhere in spacetime.

If we plug this in our free Dirac Lagrangian, we get

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = \bar{\psi}G^\dagger(i\partial - m)(G\psi) = i\bar{\psi}G^\dagger\partial(G\psi) - m\bar{\psi}G^\dagger G\psi \quad (1.2.42)$$

Again we have $G^\dagger G = 1$, so we get

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = i\bar{\psi}G^\dagger\partial(G\psi) - m\bar{\psi}\psi \quad (1.2.43)$$

We notice that the mass term $-m\bar{\psi}\psi$ transforms into itself again and is invariant. For the kinetic term, we use $\partial = \gamma^\mu\partial_\mu$ and

$$\partial_\mu G = ie^{i\alpha}\partial_\mu\alpha = iG\partial_\mu\alpha \quad (1.2.44)$$

And for the kinetic term we get:

$$i\bar{\psi}\partial\psi \rightarrow i\bar{\psi}G^\dagger\partial(G\psi) = i\bar{\psi}\partial\psi - \bar{\psi}(\partial\alpha)\psi \neq i\bar{\psi}\partial\psi \quad (1.2.45)$$

This means that the kinetic term $i\bar{\psi}\partial\psi$ is not invariant under local U(1), which means that the Lagrangian is not invariant either:

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = i\bar{\psi}\partial\psi - \bar{\psi}(\partial\alpha)\psi - m\bar{\psi}\psi = \mathcal{L}_{\text{Dirac}} - \bar{\psi}(\partial\alpha)\psi \quad (1.2.46)$$

The free Dirac Lagrangian is not invariant under local U(1) symmetry, but there is a way to adjust this Lagrangian and force it to be invariant. Since the problem arises in the kinetic term due to the partial derivative, we will define a new type of derivative instead:

$$D_\mu := \partial_\mu + V_\mu \quad (1.2.47)$$

We call this the covariant derivative. The vector V_μ will not be a differential operator, but a multiplicative one. While the partial derivative of the field transforms as

$$\partial_\mu\psi \rightarrow \partial_\mu(G\psi) \neq G\partial_\mu\psi \quad (1.2.48)$$

We will demand that the covariant derivative transforms in the same way as ψ

$$D_\mu\psi \rightarrow D_\mu'(G\psi) = GD_\mu\psi \quad (1.2.49)$$

For the vector V_μ we will also have a transformation $V_\mu \rightarrow V_\mu'$

If we work out $D_\mu'(G\psi) = GD_\mu\psi$, we will get

$$V_\mu' = GV_\mu G^\dagger - (\partial_\mu G)G^\dagger \quad (1.2.50)$$

G and V_μ commute since G is an element of an abelian group, so we have

$$V_\mu \rightarrow V'_\mu = V_\mu - (\partial_\mu G)G^\dagger = V_\mu - i\partial_\mu \alpha \quad (1.2.51)$$

We can do something similar for the transformation of $(D_\mu \psi)^\dagger$, and find the transformation of V_μ^\dagger

$$V_\mu^\dagger \rightarrow V_\mu^{\dagger'} = V_\mu^\dagger - (\partial_\mu G^\dagger)G = V_\mu^\dagger + i\partial_\mu \alpha \quad (1.2.52)$$

We find for the difference between the transformed field and the original field:

$$\delta V_\mu^\dagger = -\delta V_\mu = i\partial_\mu \alpha \quad (1.2.53)$$

This means that δV_μ is purely imaginary. We can choose V_μ to be purely imaginary as well, and will define it as

$$V_\mu = -igA_\mu \quad (1.2.54)$$

where A_μ is a real vector, and g a real number that is called the coupling constant. The covariant derivative can now be written as

$$D_\mu = \partial_\mu - igA_\mu \quad (1.2.55)$$

Now we are ready to replace the free Dirac Lagrangian by a similar Lagrangian in which we replace the partial derivatives by covariant derivatives.

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \mathcal{L}_{\text{Dirac}} + g\bar{\psi}\not{A}\psi \quad (1.2.56)$$

The fields and Lagrangian will now transform under local U(1) as

$$\psi \rightarrow \psi' = G\psi \quad (1.2.57)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger \quad (1.2.58)$$

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{i}{g}(\partial_\mu G)G^\dagger = A_\mu + \frac{1}{g}\partial_\mu \alpha \quad (1.2.59)$$

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} \quad (1.2.60)$$

Here we obtained the transformation law for A_μ by plugging it in the transformation law for V_μ .

Our new Lagrangian is now invariant under local U(1) symmetry and is equal to the free Dirac Lagrangian plus an extra term that contains a new field A_μ . We call this field the gauge field. Because we introduced a new field, we will need to add a new term for A_μ in the Lagrangian. We will take A_μ to be a massless field, because the gauge fields for QED and QCD are massless. A massive gauge field would add additional problems in the theory which would require spontaneous symmetry breaking and the introduction of a Brout-Englert-Higgs field. We would have to do this if we were doing Electroweak Interactions in this thesis, but we will stick to QCD and

QED in this thesis. Since our gauge field is massless, the Lagrangian term for the gauge field will be a purely kinetic term. This kinetic term is a free Lagrangian term for a massless abelian vector field and is given by

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1.2.61)$$

Where $F^{\mu\nu}$ is the field strength tensor, an antisymmetric tensor given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (1.2.62)$$

This is also the free Lagrangian for electromagnetism, with A^μ the 4D vector potential:

$$A^\mu = (\phi, \mathbf{A}) \quad (1.2.63)$$

with ϕ the potential, and \mathbf{A} the 3D vector potential. The Euler-Lagrange equations of motion for this free Lagrangian will yield the homogeneous Maxwell equations [4 (p74)]:

$$\partial_\nu F^{\mu\nu} = 0 \quad (1.2.64)$$

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \quad (1.2.65)$$

Or in terms of A^μ

$$\square A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0 \quad (1.2.66)$$

where $\square = \partial^\mu \partial_\mu$

Note that the fields strength tensor is proportional to the commutator of two covariant derivatives

$$F^{\mu\nu} = \frac{i}{g}[D^\mu, D^\nu] \quad (1.2.67)$$

We can show this by working out the commutator

$$[D^\mu, D^\nu] = [\partial^\mu - igA^\mu, \partial^\nu - igA^\nu] = -ig(\partial^\mu A^\nu - \partial^\nu A^\mu) = -igF^{\mu\nu} \quad (1.2.68)$$

Note that the commutator of 2 covariant derivatives is not a differential operator, but just a multiplicative one.

This commutator acting on the spinor field ψ , will transform under local U(1) just like ψ itself:

$$[D^\mu, D^\nu]\psi \rightarrow G[D^\mu, D^\nu]\psi \quad (1.2.69)$$

This means that the field strength tensor will transform as

$$F^{\mu\nu} \rightarrow F^{\mu\nu'} = GF^{\mu\nu}G^\dagger = F^{\mu\nu} \quad (1.2.70)$$

Where we used that G is an element of an abelian group. This means that the following transformation law holds:

$$F^{\mu\nu} F_{\mu\nu} \rightarrow F^{\mu\nu} F_{\mu\nu} \quad (1.2.71)$$

Which means that our free Lagrangian for the gauge field is invariant under local U(1) symmetry. Our full Lagrangian at this point is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + g\bar{\psi}\not{A}\psi \quad (1.2.72)$$

$\mathcal{L}_0 = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}}$ is the free Lagrangian, which contains both spinor and gauge fields but separated. $\mathcal{L}_1 = g\bar{\psi}\not{A}\psi$ is a Lagrangian term that couples the spinor and gauge fields, and will be interpreted as an interaction term.

We started with a field that was invariant under global U(1) symmetry, but not under local U(1). We then adjusted our Lagrangian by replacing the partial derivative by a covariant derivative, which introduces a new field. This made our new Lagrangian invariant under local U(1) symmetry, but our Lagrangian is not free anymore since it contains a term that couples the Dirac field to the new gauge field. While the Dirac field is a fermion field of spin 1/2, the gauge field will be a boson field of spin 1.

It is exactly such a Lagrangian that gives rise to quantum electrodynamics (QED). In this case A^μ is the photon field, and the coupling constant $g = -e = -|e|$, with $-e$ the electric charge of the electron.

1.2.5 Local phase invariance for a non-abelian gauge group (SU(N))

We will now try to do the same for local SU(N) transformations. The main difference with U(1) is that SU(N) is a non-abelian group, while U(1) is an abelian group.

We start with the free Dirac Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\not{\partial} - m)\psi \quad (1.2.1)$$

where the fields ψ and $\bar{\psi}$ are N component spinor fields like in section 1.2.3. We will check if it is invariant under local SU(N) transformations. We will transform the fields as follows:

$$\psi \rightarrow \psi' = G\psi = e^{i\alpha^a T^a} \psi \quad (1.2.73)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger = \bar{\psi}e^{-i\alpha^a T^a} \quad (1.2.74)$$

With $G = e^{i\alpha^a T^a}$ and $G^\dagger = G^{-1} = e^{-i\alpha^a T^a}$, with $\alpha = \alpha(x)$, and with T^a in the fundamental representation.

If we plug this in our free Dirac Lagrangian, we get

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = \bar{\psi}G^+(i\cancel{\partial} - m)(G\psi) = i\bar{\psi}G^+\cancel{\partial}(G\psi) - m\bar{\psi}\psi \quad (1.2.75)$$

The mass term $-m\bar{\psi}\psi$ transforms into itself again and is invariant. For the kinetic term we get:

$$i\bar{\psi}\cancel{\partial}\psi \rightarrow i\bar{\psi}G^+\cancel{\partial}(G\psi) = i\bar{\psi}\cancel{\partial}\psi + i\bar{\psi}G^+(\cancel{\partial}G)\psi \neq i\bar{\psi}\cancel{\partial}\psi \quad (1.2.76)$$

Since G is an element of a non-abelian group, we cannot simply apply the chain rule to $\partial_\mu G = \partial_\mu(e^{i\alpha^a T^a})$

The kinetic term $i\bar{\psi}\cancel{\partial}\psi$ is not invariant under local SU(N), which means that the Lagrangian is not invariant either:

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}'_{\text{Dirac}} = i\bar{\psi}\cancel{\partial}\psi + i\bar{\psi}G^+(\cancel{\partial}G)\psi - m\bar{\psi}\psi = \mathcal{L}_{\text{Dirac}} + i\bar{\psi}G^+(\cancel{\partial}G)\psi \quad (1.2.77)$$

The free Dirac Lagrangian is not invariant under local SU(N) symmetry, but we can use the same trick as we did for local U(1) symmetry, where we adjust the Lagrangian and force it to be invariant. Again we define the covariant derivative:

$$D_\mu := \partial_\mu + V_\mu \quad (1.2.78)$$

The vector V_μ will not be a differential operator, but a multiplicative one. The partial derivative of the field transforms as

$$\partial_\mu\psi \rightarrow \partial_\mu(G\psi) \neq G\partial_\mu\psi \quad (1.2.79)$$

Again we will demand that the covariant derivative transforms in the same way as ψ

$$D_\mu\psi \rightarrow D'_\mu(G\psi) = GD_\mu\psi \quad (1.2.80)$$

If we work out $D'_\mu(G\psi) = GD_\mu\psi$, we will get

$$V_\mu \rightarrow V'_\mu = GV_\mu G^\dagger - (\partial_\mu G)G^\dagger \quad (1.2.81)$$

G and V_μ do not necessarily commute since G is an element of a non-abelian group. In order for this to work, V_μ will not have to be purely imaginary, but will have to be some linear combination of the different iT^a instead.

We will define V_μ as

$$V_\mu = -igA_\mu = -igA_\mu^a T^a \quad (1.2.82)$$

where $A_\mu = A_\mu^a T^a$ is a convenient notation, and g the coupling constant which is a real number. The covariant derivative can now be written as

$$D_\mu = \partial_\mu - igA_\mu \quad (1.2.83)$$

Just like in the case of U(1), we replace the free Dirac Lagrangian by a similar Lagrangian in which we replace the partial derivatives by covariant derivatives.

$$\mathcal{L} = \bar{\psi}(i\cancel{D} - m)\psi = \mathcal{L}_{\text{Dirac}} + g\bar{\psi}\cancel{A}\psi \quad (1.2.84)$$

The fields and Lagrangian will now transform under local SU(N) as

$$\psi \rightarrow \psi' = G\psi \quad (1.2.85)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}G^\dagger \quad (1.2.86)$$

$$A_\mu \rightarrow A'_\mu = GA_\mu G^\dagger - \frac{i}{g}(\partial_\mu G)G^\dagger \quad (1.2.87)$$

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} \quad (1.2.88)$$

Note that $(\partial_\mu G)G^\dagger = -G\partial_\mu G^\dagger$, by working out $\partial_\mu(GG^\dagger) = \partial_\mu \mathbf{1} = 0$

The new Lagrangian is now invariant under local SU(N) symmetry and is equal to the free Dirac Lagrangian plus an extra term that contains new fields A_μ^a . For every generator of the group, there is a corresponding gauge field. This means we have $N^2 - 1$ gauge fields. Just like with U(1), these gauge fields will be bosonic spin 1 fields. Again we will need to add a new Lagrangian term for the gauge fields. We will take A_μ^a to be massless field, like we did for U(1). The new Lagrangian term will be a kinetic term for the gauge field. Because we are dealing with non-abelian vector fields, this kinetic term will not simply be the same as in the case of U(1).

In this case, we have for the field strength tensor:

$$F^{\mu\nu} = \frac{i}{g}[D^\mu, D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu] \quad (1.2.98)$$

We can also write this as

$$F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + gf^{abc}A^{\mu b}A^{\nu c} \quad (1.2.90)$$

with $F^{\mu\nu} = F^{\mu\nu a}T^a$, and where we used $[T^b, T^c] = if^{abc}T^a$.

Note that just like with U(1), the commutator of 2 covariant derivatives is not a differential operator, but just a multiplicative one.

This commutator acting on the spinor field ψ , will transform under local SU(N) just like ψ itself:

$$[D^\mu, D^\nu]\psi \rightarrow G[D^\mu, D^\nu]\psi \quad (1.2.91)$$

This means that the field strength tensor will transform as

$$F^{\mu\nu} \rightarrow F^{\mu\nu'} = GF^{\mu\nu}G^\dagger \quad (1.2.92)$$

And from this we see that the following transformation law holds:

$$F^{\mu\nu}F_{\mu\nu} \rightarrow GF^{\mu\nu}F_{\mu\nu}G^\dagger \quad (1.2.93)$$

This is not invariant. Instead we will take the trace, which will turn out to be invariant.

$$\text{Tr}(F^{\mu\nu}F_{\mu\nu}) = F^{\mu\nu a}F_{\mu\nu}^b \text{Tr}(T^a T^b) = \frac{1}{2} F^{\mu\nu a}F_{\mu\nu}^a \quad (1.2.94)$$

Where we used the trace invariant $\text{Tr}(T^a T^b) = T_F \delta^{ab} = \frac{1}{2} \delta^{ab}$

By working out $\text{Tr}(F^{\mu\nu}F_{\mu\nu}) \rightarrow \text{Tr}(F^{\mu\nu'}F_{\mu\nu}')$, and using the cyclic property of the trace, we can easily show it is invariant:

$$\text{Tr}(F^{\mu\nu'}F_{\mu\nu}') = \text{Tr}(GF^{\mu\nu}F_{\mu\nu}G^\dagger) = \text{Tr}(G^\dagger GF^{\mu\nu}F_{\mu\nu}) = \text{Tr}(F^{\mu\nu}F_{\mu\nu}) \quad (1.2.95)$$

The kinetic Lagrangian term for the gauge fields will be

$$\mathcal{L}_G = -\frac{1}{2} \text{Tr}(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{4} F^{\mu\nu a}F_{\mu\nu}^a \quad (1.2.96)$$

This term is not a free Lagrangian, because it contains terms with more than 2 fields coupled to each other. At this point, our full Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F^{\mu\nu a}F_{\mu\nu}^a = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_G + g\bar{\psi}\not{A}\psi \quad (1.2.97)$$

This Lagrangian is known as the Yang-Mills Lagrangian. We can also write our Lagrangian as follows:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \quad (1.2.98)$$

With the free Lagrangian term

$$\mathcal{L}_0 = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \quad (1.2.99)$$

And the interaction term:

$$\mathcal{L}_I = g\bar{\psi}\not{A}\psi + gf^{abc}A^\mu A^\nu \partial^\mu A^\nu - \frac{1}{4} g^2 f^{abc}f^{ade}A^\mu A^\nu A_\mu^d A_\nu^e \quad (1.2.100)$$

We will now come back to $A_\mu \rightarrow A'_\mu = GA_\mu G^\dagger - \frac{i}{g}(\partial_\mu G)G^\dagger$

For small α^a we can use the expansion

$$G = e^{i\alpha^a T^a} = \mathbf{1} + i\alpha^a T^a + \mathcal{O}(\alpha^2) \approx \mathbf{1} + i\alpha^a T^a \quad (1.2.101)$$

$$G^\dagger = e^{-i\alpha^a T^a} = \mathbf{1} - i\alpha^a T^a + \mathcal{O}(\alpha^2) \approx \mathbf{1} - i\alpha^a T^a \quad (1.2.102)$$

Wherewith $\mathcal{O}(\alpha^2)$ we mean second order in α^a or higher (“alpha squared”, not the 2nd component of α). By plugging these expressions in the gauge transformation law for the gauge field [5], we get

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \alpha^a T^a - f^{abc} \alpha^a A_\mu^b T^c = A_\mu^{a'} T^a \quad (1.2.103)$$

$$A_\mu^a \rightarrow A_\mu^{a'} = A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a - f^{abc} \alpha^b A_\mu^c \quad (1.2.104)$$

And for δA_μ^a we get

$$\delta A_\mu^a = A_\mu^{a'} - A_\mu^a = \frac{1}{g} \partial_\mu \alpha^a - f^{abc} \alpha^b A_\mu^c = \frac{1}{g} (\delta^{ab} \partial_\mu - g f^{abc} A_\mu^c) \alpha^b \quad (1.2.105)$$

Consider the covariant derivative in the adjoint representation (instead of fundamental representation):

$$D_\mu = \mathbf{1}_{N \times N} \partial_\mu - ig A_\mu^a T_A^a \quad (1.2.106)$$

Recall the adjoint representation $(T_A^a)^{bc} = -if^{abc}$

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c \quad (1.2.107)$$

Which is exactly what's in the brackets in the expression for δA_μ^a

$$\delta A_\mu^a = \frac{1}{g} D_\mu^{ab} \alpha^b = \frac{1}{g} (\delta^{ab} \partial_\mu - g f^{abc} A_\mu^c) \alpha^b \quad (1.2.108)$$

Which means that in the adjoint representation we can write

$$A_\mu^a \rightarrow A_\mu^{a'} = A_\mu^a + \frac{1}{g} D_\mu^{ab} \alpha^b \quad (1.2.109)$$

If we compare this to the transformation in case of U(1), where we had

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \alpha \quad (1.2.110)$$

We notice this is of a very similar form but with ∂_μ replaced by D_μ^{ab} , and contracting it with α^b .

Extra remarks:

- The Dirac field ψ transforms according to the fundamental representation of the SU(N) group. This is related to ψ being a combination of N spinor fields. Such a spinor particle will carry 1 “colour” charge that can take on N values.
- The Dirac field $\bar{\psi}$ transforms according to the anti-fundamental representation of the SU(N) group. This is because $\bar{\psi}$ is also a combination of N spinor fields, and is the adjoint field for ψ , resulting it to be anti-fundamental. Such a spinor particle will carry 1 “colour” charge that can take on N values, and that are the opposite of the values for the colour charge of ψ .
- The gauge field A^μ transforms according to the adjoint representation of the SU(N) group. Because of this, the field A^μ carries 1 colour charge that can take on $N^2 - 1$ values. Each value corresponds to a field $A^{\mu a}$.

1.2.6 For SU(3) (QCD)

In case of QCD, we have local SU(3) symmetry, which means 8 different gauge fields, called gluons. The spinor fields are the quark fields, which are 3 component spinors. There are also 6 different quark flavours, which have to be summed over in the QCD Lagrangian.

$$\mathcal{L} = \sum_f (\bar{\psi}_f (i\not{D} - m_f)\psi_f) - \frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a \quad (1.2.111)$$

With the spinor fields:

$$\psi_f = \begin{pmatrix} \psi_{fr} \\ \psi_{fg} \\ \psi_{fb} \end{pmatrix} \quad (1.2.112)$$

$$\bar{\psi}_f = (\bar{\psi}_{fr} \quad \bar{\psi}_{fg} \quad \bar{\psi}_{fb}) \quad (1.2.113)$$

The field ψ_{fi} corresponds to a quark field of flavour f and a fundamental SU(3) colour (fundamental representation). The flavour f can be u, d, s, c, b, t (up, down, strange, charm, bottom, top). The colour can be r, g, b (red, green, blue). (The name “colour” was chosen because of some similarities with actual rgb colours, given that there are three different QCD colours for quarks.) Each of the ψ_{fi} is a spinor with 4 components for the gamma matrices.

1.3 Quantisation

In this section we will fully quantise the Lagrangian, turning our “classical” field theory into a quantum field theory. There are two ways to quantise the Lagrangian, canonical quantisation and quantisation by path integrals (functional integral). The easiest way to quantise the QCD Lagrangian is by path integrals, but canonical quantisation has an advantage when it comes to interpreting the Lagrangian and interactions. First, we will briefly cover the basics of canonical quantisation just as a means to easier understand what the fields and the terms in the Lagrangian mean. We will also briefly cover Feynman diagrams. Then we will quantise the U(1) gauge theory via path integrals, after which we will do the same for the SU(N) gauge theory.

1.3.1 Basics of canonical quantisation

Similar to how one would go from classical mechanics to quantum mechanics by promoting physical quantities (position, momentum, etc) to operators, we can go from classical field theory to quantum field theory by promoting the fields to field operators. This form of quantisation is called canonical quantisation or second quantisation.

If we want to quantise the free Dirac field, we start with the classical Lagrangian, Eq (1.2.1). (For a detailed construction, see [5] (p52-63) and [4] (p61-69))

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\not{\partial} - m)\psi \quad (1.3.1)$$

We will promote $\psi(x)$ and $\bar{\psi}(x)$ to field operators, satisfying the equal-time anticommutation relations:

$$\{\psi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} = i\delta_{ab}\delta(\mathbf{x} - \mathbf{x}') \quad (1.3.2)$$

$$\{\psi_a(t, \mathbf{x}), \psi_b(t, \mathbf{x}')\} = \{\pi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} = 0 \quad (1.3.3)$$

Where the indices a and b denote spinor components, and $\pi(x)$ is the canonical momentum conjugate of $\psi(x)$, which we found in section 1.2.1, Eq (1.2.13).

$$\pi_a = i\psi_a^\dagger \quad (1.3.4)$$

Thus we can rewrite the equal-time anticommutation relations as

$$\{\psi_a(t, \mathbf{x}), \psi_b^\dagger(t, \mathbf{x}')\} = \delta_{ab}\delta(\mathbf{x} - \mathbf{x}') \quad (1.3.5)$$

$$\{\psi_a(t, \mathbf{x}), \psi_b(t, \mathbf{x}')\} = \{\psi_a^\dagger(t, \mathbf{x}), \psi_b^\dagger(t, \mathbf{x}')\} = 0 \quad (1.3.6)$$

Remember that $\bar{\psi} = \psi^\dagger\gamma^0$. We use anticommutation relations because according to the spin-statistics theorem, fermions will have wavefunctions that are antisymmetric under permutation, and thus fermions must be anticommuting fields.

Remind that the Hamiltonian is given by

$$H_{\text{Dirac}} = \int d^3x \mathcal{H}_{\text{Dirac}}(x) = \int d^3x \bar{\psi} \left(-i\gamma^j \frac{\partial}{\partial x^j} + m \right) \psi \quad (1.3.7)$$

We can Fourier transform the fields by

$$\psi(t, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\mathbf{x}} \psi(t, \mathbf{p}) \quad (1.3.8)$$

And our Hamiltonian will be the Hamiltonian for an infinite amount of simple harmonic oscillators. We will have one simple harmonic oscillator for every momentum \mathbf{p} . The fields can be decomposed in an infinite amount of creation and annihilation operators:

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a_{\mathbf{p}}^s u^s(p) e^{-ipx} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ipx}) \quad (1.3.9)$$

$$\bar{\psi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (b_{\mathbf{p}}^s \bar{v}^s(p) e^{-ipx} + a_{\mathbf{p}}^{s\dagger} \bar{u}^s(p) e^{ipx}) \quad (1.3.10)$$

Where $u^s(p)$ and $v^s(p)$ are the four independent (and constant) solutions for the Dirac equation (with index s the spin state, which can be 1 or 2), and $E_{\mathbf{p}}$ the energy. $a_{\mathbf{p}}^s$ and $b_{\mathbf{p}}^s$ are annihilation operators, while $a_{\mathbf{p}}^{s\dagger}$ and $b_{\mathbf{p}}^{s\dagger}$ are creation operators. These operators have the anticommutation relations:

$$\{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} = \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta(\mathbf{p} - \mathbf{q}) \quad (1.3.11)$$

And all other anticommutation relations are equal to 0.

An annihilation operator acting on the vacuum state will result in zero:

$$a_{\mathbf{p}}^s |0\rangle = b_{\mathbf{p}}^s |0\rangle = 0 \quad (1.3.12)$$

A creation operator acting on the vacuum state, will result in a state proportional to a one-particle state, and we will define the one-particle state as follows:

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{s\dagger} |0\rangle \quad (1.3.13)$$

And the same for the operator $b_{\mathbf{p}}^{s\dagger}$.

$a_{\mathbf{p}}^s$ will annihilate a field excitation with spin state s and momentum \mathbf{p} , while $a_{\mathbf{p}}^{s\dagger}$ will create a field excitation with spin state s and momentum \mathbf{p} . The same happens for $b_{\mathbf{p}}^s$ and $b_{\mathbf{p}}^{s\dagger}$ but for a different type of field excitation. We will interpret field excitations as particles. For the Dirac field, there are two different types of particles, one corresponding to $a_{\mathbf{p}}^s$ and $a_{\mathbf{p}}^{s\dagger}$, and one corresponding to $b_{\mathbf{p}}^s$ and $b_{\mathbf{p}}^{s\dagger}$. We will call the

particle corresponding to the “a” operators a fermion, and the particle corresponding to the “b” operators as an antifermion.

We can now also interpret what the field operators ψ and $\bar{\psi}$ mean in the Hamiltonian (or Lagrangian). ψ will annihilate (“eat”) a fermion or will create an antifermion. $\bar{\psi}$ will annihilate an antifermion or will create a fermion.

We can do a similar thing for a free photon (U(1) gauge vector field), by quantising the free Maxwell Lagrangian, Eq (1.2.61). (For a detailed construction, see [4 (p77-84)])

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1.3.14)$$

Where we will promote A^μ to field operators. The construction is very similar to that for the quantised Dirac field, but with commutation relations instead of anticommutation relations. The reason for this is that according to the spin-statistics theorem, bosons will have wavefunctions that are symmetric under permutation, and thus bosons must be commuting fields. The field operator for the photon will be

$$A_\mu(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{r=0}^3 (a_{\mathbf{p}}^r \epsilon_\mu^r(p) e^{-ipx} + a_{\mathbf{p}}^{r\dagger} \epsilon_\mu^{r*}(p) e^{ipx}) \quad (1.3.15)$$

With ϵ_μ^r the polarisation vector. In this case there is only one type of particle, the photon. The field operator A^μ will annihilate or create a photon.

An important object in QFT is the propagator (Feynman propagator), which in the case of a free field Lagrangian is the probability amplitude for a particle to travel from one point to another in spacetime (or in other words, the propagation amplitude). It is proportional to the time-ordered two-point correlation function for the field under consideration.

For the Dirac field, the propagator is

$$\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle = S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} S_F(p) e^{-ip(x-x')} \quad (1.3.16)$$

With T the time-ordering operator

$$T\psi(x)\bar{\psi}(x') = \theta(t-t')\psi(x)\bar{\psi}(x') - \theta(t'-t)\bar{\psi}(x')\psi(x) \quad (1.3.17)$$

With $\theta(t)$ the Heaviside function, which is equal to 1 if t is larger than 0, and is equal to 0 if t is smaller than 0.

For the interpretation, let us consider $t > t'$. In that case $\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle$ will create a fermion at x' and annihilate it again at x . It represents a fermion traveling from x' to

x . In case of $t < t'$, $\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle$ will create an antifermion at x and annihilate it again at x' .

The Fourier transformed propagator $S_F(p)$ will be used more often than the one in position space. It corresponds to a free fermion with four-momentum p , or to a free antifermion with opposite three-momentum.

Dirac propagator:
(fermion propagator) \xrightarrow{p} $= S_F(p)$

For the photon field we have

$$\langle 0|TA^\mu(x)A^\nu(x')|0\rangle = D_F^{\mu\nu}(x-x') = \int \frac{d^4p}{(2\pi)^4} D_F^{\mu\nu}(p)e^{-ip(x-x')} \quad (1.3.18)$$

$D_F^{\mu\nu}(p)$ corresponds to a free photon with four-momentum p .

Photon propagator: \xrightarrow{p} $= D_F^{\mu\nu}(p)$

1.3.2 Introduction on Feynman diagrams and rules

When we consider higher-order correlation functions, we can describe interaction processes if we are dealing with an interacting quantum field theory. We can decompose the higher-order correlation functions into products of two-point correlation functions via the Wick theorem. From this we can derive the Feynman rules for the theory, which is a method to describe any process in the theory by drawing a Feynman diagram for it, and then translating the diagram into a product of propagators. After integration, this will give us the probability amplitude for the process.

In a Feynman diagram there will be lines, which represent particles. Typically a line represents a particle that has a four-momentum and for which an arrow is drawn that indicates this momentum's direction. In the case of bosons or massless particles, the arrow is often omitted. While canonical quantisation mostly uses the Hamiltonian, we can still find out a lot of the quantum theory by looking at the Lagrangian. In an interacting field theory, we will have a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \quad (1.3.19)$$

Where \mathcal{L}_0 is the free field Lagrangian, and \mathcal{L}_1 the interaction Lagrangian (interaction term). While \mathcal{L}_0 only contains terms of second order of the fields, \mathcal{L}_1 will typically consist of terms that are of a higher order of the fields. In the Hamiltonian description this will correspond to

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \quad (1.3.20)$$

If we have a \mathcal{L}_1 that is a product of three fields, it will be called a three-point interaction, because it is an interaction between three particles. In Feynman diagrams, such an interaction is drawn as a vertex that connects three lines, one line for each particle involved in the interaction. This is also called a three-point vertex. If we have a \mathcal{L}_1 that is a product of four fields, the same will be true but for four particles instead of three. It will be called a four-point interaction, or four-point vertex. This generalises to any type of vertex. Technically we could also have a two-point interaction between two different fields. In elementary particle physics, this would mean "mixing" (for example neutrino mixing, which results in neutrino oscillations).

When we have a quantum process, we start from an initial state $|i\rangle$ and evolve to a final state $|f\rangle$. Both initial and final state can consist of multiple particles. The probability amplitude for state $|i\rangle$ to evolve to state $|f\rangle$ is given by the S-matrix

$$\langle f|S|i\rangle = S_{fi} \quad (1.3.21)$$

The transition probability that $|i\rangle$ has evolved into $|f\rangle$, is given by

$$|S_{fi}|^2 = |\langle f|S|i\rangle|^2 \quad (1.3.22)$$

The conservation of probability is expressed as

$$\sum_f |S_{fi}|^2 = 1 \quad (1.3.23)$$

We also have orthonormality and completeness of the states:

$$\langle f|i\rangle = \delta_{fi} \quad (1.3.24)$$

$$\sum_f |f\rangle\langle f| = \mathbf{1} \quad (1.3.25)$$

We can show that the S-matrix is a unitary matrix:

$$\sum_f |S_{fi}|^2 = \sum_f \langle f|S|i\rangle^* \langle f|S|i\rangle = \sum_f \langle i|S^\dagger|f\rangle \langle f|S|i\rangle = \langle i|S^\dagger S|i\rangle = 1 \quad (1.3.26)$$

Which means

$$S^\dagger S = \mathbf{1} \quad (1.3.27)$$

The S-matrix can be obtained from the interaction Hamiltonian H_1 (the actual interaction Hamiltonian, not the interaction Hamiltonian density) in an iterative way [4 (p93)]:

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \dots H_I(t_n) \quad (1.3.28)$$

By using the time-ordered product we can let all integrations run over the whole infinite domain. By changing the integrations over time into integrations over spacetime, we can write this as a similar expression but with the interaction Hamiltonian density \mathcal{H}_I instead [4 (p93)]:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} d^4x_1 d^4x_2 \dots d^4x_n T\{\mathcal{H}_I(x_1)\mathcal{H}_I(x_2) \dots \mathcal{H}_I(x_n)\} \quad (1.3.29)$$

We can also write an expression involving the Lagrangian density:

$$S = \exp\left(i \int d^4x \mathcal{L}_I\right) \quad (1.3.30)$$

If no interaction happens between the particles, then S will be the identity operator. This scenario corresponds to the $n = 0$ term in Eq (1.3.29). So we can define

$$S = \mathbf{1} + iT \quad (1.3.31)$$

With T the T-matrix, which contains the interactions. (Not to be confused with the time-ordering operator.)

Since the S-matrix must preserve momentum conservation, both S and T should contain a momentum conserving factor

$$\delta\left(\sum k_{\text{initial}} - \sum k_{\text{final}}\right) \quad (1.3.32)$$

Where $\sum k_{\text{initial}}$ is the sum of the momenta of all particles in the initial state $|i\rangle$, and $\sum k_{\text{final}}$ is the sum of the momenta of all particles in the final state $|f\rangle$. We can define:

$$\langle f|iT|i\rangle = (2\pi)^4 \delta\left(\sum k_{\text{initial}} - \sum k_{\text{final}}\right) i\mathcal{M}(i \rightarrow f) \quad (1.3.33)$$

Where \mathcal{M} is called the invariant matrix element, or also called Feynman amplitude. \mathcal{M} contains all the physics from \mathcal{H}_I , while the other factors contain all the physics that doesn't depend on \mathcal{H}_I . An important example is when we have an initial state of two particles with momenta k_1 and k_2 , and a final states of particles with momenta p_1, p_2, \dots , then we can define [5] (p104):

$$\langle p_1 p_2 \dots |iT|k_1 k_2\rangle = (2\pi)^4 \delta^{(4)}\left(k_1 + k_2 - \sum_f p_f\right) i\mathcal{M}(k_1 k_2 \rightarrow p_f) \quad (1.3.34)$$

For calculating S , T and \mathcal{M} , we can use Feynman diagrams. We can draw a Feynman diagram for a quantum process, typically consisting of lines and vertices.

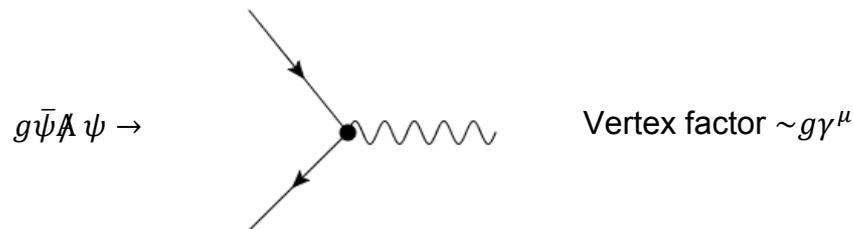
For every line we have a propagator, and for every vertex a vertex factor. For calculating S , T and \mathcal{M} , we will get a product of these propagators and vertex factors. For S we only need to take the diagrams into account that are “connected” and amputated. “Connected” means no vacuum bubbles (parts that are completely disconnected from the initial and final lines), but lines that go from initial to final particle without being connected to other parts are fine. Amputated means that if you get a disconnected external leg by cutting 1 line, you remove it. The identity matrix in Eq (1.3.31) corresponds to initial lines going to final lines without being connected to each other, or in other words, no interaction happened. This means that for iT and $i\mathcal{M}$ we only need diagrams that are fully connected (no vacuum bubbles, and all external lines are connected to each other) and amputated. A more detailed explanation (by an example) about relating the matrices with Feynman diagrams can be found in [5] (p108-115).

Important about vertices in Feynman diagrams is that momentum is conserved at each vertex, which will constrain some of the internal momenta. Any loop momenta that remain undetermined, must be integrated over. Any fermion loop receives an extra factor of -1, and additionally, there could also be extra symmetry factors. Feynman rules will depend on the specific quantum field theory that is used.

If we take the classical Lagrangian for QED (U(1)):

$$\mathcal{L}_{\text{QED,classical}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + g\bar{\psi}\not{A}\psi \quad (1.3.35)$$

We have $\mathcal{L}_0 = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}}$ and we can use the fermion and photon propagators defined earlier. We have an interaction term $\mathcal{L}_I = g\bar{\psi}\not{A}\psi = g\bar{\psi}\gamma^\mu A_\mu\psi$. This is a three-point interaction term and will result in three-point vertices in the Feynman diagrams. The vertex will connect a Dirac fermion, antifermion, and a photon.



If we take the classical Lagrangian for QCD (SU(N)), with one quark flavour for simplicity:

$$\mathcal{L}_{\text{QCD,classical}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a \quad (1.3.36)$$

With the free Lagrangian term:

$$\mathcal{L}_0 = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) \quad (1.3.37)$$

The free quark propagator will be defined as

$$\langle 0|T\psi_i(x)\bar{\psi}_j(x')|0\rangle = S_{Fij}(x-x') = \int \frac{d^4p}{(2\pi)^4} S_{Fij}(p)e^{-ip(x-x')} \quad (1.3.38)$$

With i and j the quark colour indices, which run from 1 to N .

A quark can only carry one quark colour at a time, and quarks with different colour don't "mix" (their quark colours won't oscillate), because of the lack of a quark colour mixing term in the Lagrangian. We can conclude

$$S_{Fij}(p) = \delta_{ij}S_F(p) \quad (1.3.39)$$

Quark propagator: $j \xrightarrow{p} i = S_{Fij}(p) = \delta_{ij}S_F(p)$

The free gluon propagator will be defined as

$$\langle 0|TA^{\mu a}(x)A^{\nu b}(x')|0\rangle = D_F^{\mu\nu ab}(x-x') = \int \frac{d^4p}{(2\pi)^4} D_F^{\mu\nu ab}(p)e^{-ip(x-x')} \quad (1.3.40)$$

Where a and b the "generator indices", running from 1 to $N^2 - 1$. A gluon can only carry one gluon colour at a time. The generators form an orthonormal basis, which means that a gluon's colour won't oscillate. We can conclude

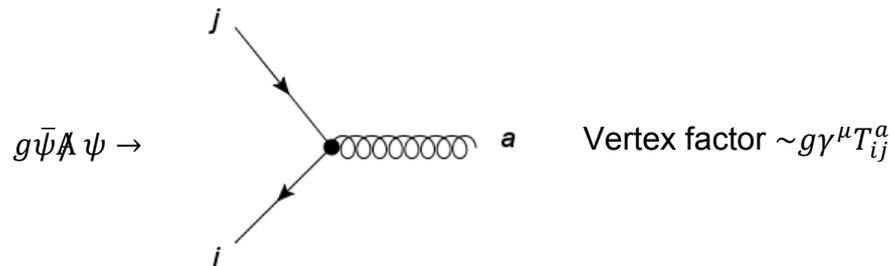
$$D_F^{\mu\nu ab}(p) = \delta^{ab}D_F^{\mu\nu}(p) \quad (1.3.41)$$

gluon propagator: $b \xrightarrow{p} a = D_F^{\mu\nu ab}(p) = \delta^{ab}D_F^{\mu\nu}(p)$

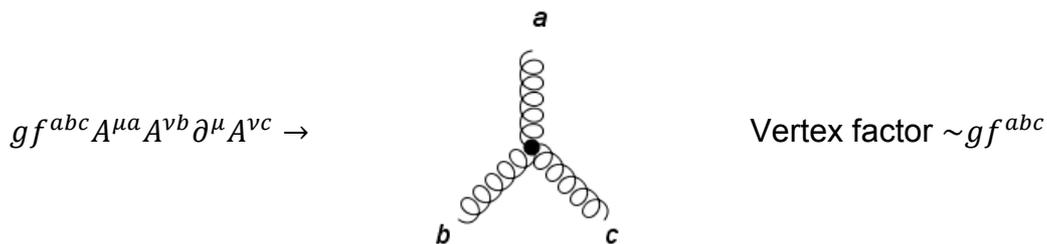
The interaction term:

$$\mathcal{L}_1 = g\bar{\psi}\not{A}\psi + gf^{abc}A^{\mu a}A^{\nu b}\partial^\mu A^{\nu c} - \frac{1}{4}g^2f^{abc}f^{ade}A^{\mu b}A^{\nu c}A_\mu^dA_\nu^e \quad (1.3.42)$$

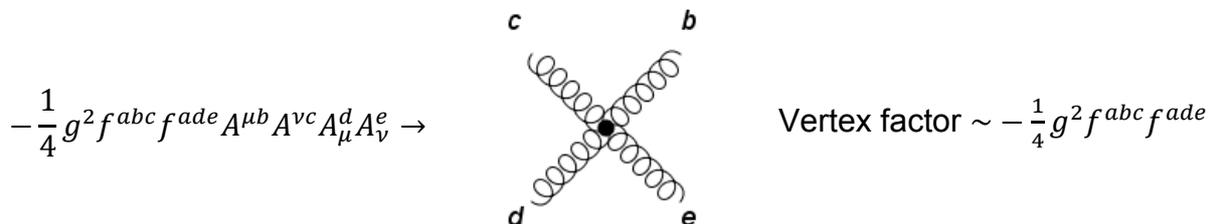
The term $g\bar{\psi}\not{A}\psi = g\bar{\psi}\gamma^\mu A_\mu^a T^a \psi$ are three-point interactions similar to the one we have in QED. In this case we have a three-point interaction for every generator T^a , between a quark, antiquark, and gluon.



The term $gf^{abc}A^{\mu a}A^{\nu b}\partial^\mu A^{\nu c}$ are three-point interactions between gluons



The term $-\frac{1}{4}g^2 f^{abc} f^{ade} A^{\mu b} A^{\nu c} A_\mu^d A_\nu^e$ are four-point interactions between gluons



An external line of a diagram, in any QFT, will correspond to the polarisation and group space orientation.

1.3.3 Basics of quantisation by functional integral (path integral)

Instead of quantising a field by promoting the fields to field operators, as we do in canonical quantisation, we can also take a different route and use functional integration to quantise a field theory. In the case of functional integration, the fields do not become field operators. While canonical quantisation uses the Hamiltonian as its fundamental object for the quantum description, the functional method will use the Lagrangian as its fundamental object instead. Because it uses the Lagrangian, all symmetries of the Lagrangian will be preserved (aside for some subtle exceptions

called “anomalies”, which we will not treat in this thesis). The functional method will also result in a quantum version of Noether’s theorem, which relates a symmetry of the Lagrangian to conservation laws.

Generally for a Lagrangian \mathcal{L} , with fields φ_i , we will define the generating functional as

$$Z[J] = \int \mathcal{D}\varphi \exp\left(i \int d^4x [\mathcal{L} + \sigma]\right) \quad (1.3.43)$$

Where φ means the collection of the fields φ_i , J is a collection of source fields J_i , and σ is a source term added to the Lagrangian:

$$\sigma = J_i \varphi_i \quad (1.3.44)$$

We can also write the generating functional as

$$Z[J_1, J_2, \dots] = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \dots \exp\left(i \int d^4x [\mathcal{L} + J_i \varphi_i]\right) \quad (1.3.45)$$

The meaning of $\mathcal{D}\phi$ for a single field ϕ is

$$\mathcal{D}\phi = \prod_j d\phi(x_j) \quad (1.3.46)$$

Where x_j is the position of the j -th point on a “square” lattice (in 4D). The integral $\int \mathcal{D}\varphi(\dots)$ is a functional integral (path integral) and will integrate over all possible field configurations.

Note that the action S (not to be confused with S-matrix) for the theory is given by

$$S[\varphi] = S[\varphi_1, \varphi_2, \dots] = \int d^4x \mathcal{L} \quad (1.3.47)$$

So we can also write the generating functional as

$$Z[J] = \int \mathcal{D}\varphi \exp\left(iS[\varphi] + i \int d^4x \sigma\right) \quad (1.3.48)$$

For a physical observable operator $O(\varphi)$ we have

$$\langle 0|TO(\varphi)|0\rangle = \frac{\int \mathcal{D}\varphi O(\varphi) \exp(iS[\varphi])}{\int \mathcal{D}\varphi \exp(iS[\varphi])} \quad (1.3.49)$$

We can relate $\langle 0|TO(\varphi)|0\rangle$ to the generating functional by taking functional derivatives of the generating functional with respect to the source fields. In case of a theory with only one field ϕ we have

$$\langle 0|TO(\phi)|0\rangle = \langle 0|T\phi(x_1)\phi(x_2) \dots |0\rangle = \left(\frac{1}{Z[J]} \left(-i \frac{\delta}{\delta J(x_1)}\right) \left(-i \frac{\delta}{\delta J(x_2)}\right) \dots Z[J]\right) \Big|_{J=0}$$

(1.3.50)

Where J is only one source field, and with $\frac{\partial}{\partial J}$ the functional derivative.

Taking one functional derivative of the generating functional will yield

$$-i \frac{\delta Z[J]}{\delta J(x_1)} = \int \mathcal{D}\phi \phi(x_1) \exp\left(iS[\phi] + i \int d^4x J\phi\right) \quad (1.3.51)$$

Taking $J = 0$ will give us $\int \mathcal{D}\phi \phi(x_1) \exp(iS[\phi])$

Taking higher multiple functional derivatives will result in

$$\left(-i \frac{\delta}{\delta J(x_1)}\right) \left(-i \frac{\delta}{\delta J(x_2)}\right) \dots Z[J] = \int \mathcal{D}\phi \phi(x_1) \phi(x_2) \dots \exp\left(iS[\phi] + i \int d^4x J\phi\right) \quad (1.3.52)$$

Taking $J = 0$ will give us $\int \mathcal{D}\phi \phi(x_1) \phi(x_2) \dots \exp(iS[\phi])$

Taking the functional derivatives corresponding to $O(\phi) = \phi(x_1) \phi(x_2) \dots$, and taking $J = 0$ afterwards, will yield $\int \mathcal{D}\phi O(\phi) \exp(iS[\phi])$. We now see it is indeed true that

$$\begin{aligned} \langle 0|TO(\phi)|0\rangle &= \langle 0|T\phi(x_1)\phi(x_2) \dots |0\rangle = \left(\frac{1}{Z[J]} \left(-i \frac{\delta}{\delta J(x_1)}\right) \left(-i \frac{\delta}{\delta J(x_2)}\right) \dots Z[J]\right) \Big|_{J=0} \\ &= \frac{\int \mathcal{D}\phi O(\phi) \exp(iS[\phi])}{\int \mathcal{D}\phi \exp(iS[\phi])} \end{aligned} \quad (1.3.53)$$

We will define

$$Z_0 = Z[0,0, \dots] = \int \mathcal{D}\varphi \exp\left(iS[\varphi] + i \int d^4x \sigma\right) \Big|_{J_i=0} = \int \mathcal{D}\varphi \exp(iS[\varphi]) \quad (1.3.54)$$

where again φ can be a single field or a collection of multiple fields. We can rewrite our earlier general expression as

$$\langle 0|TO(\varphi)|0\rangle = \frac{\int \mathcal{D}\varphi O(\varphi) \exp(iS[\varphi])}{\int \mathcal{D}\varphi \exp(iS[\varphi])} = \frac{1}{Z_0} \int \mathcal{D}\varphi O(\varphi) \exp(iS[\varphi]) \quad (1.3.55)$$

With this functional method, we can calculate all the observables.

A very important operator will be the two-point correlation function

$$\langle 0|T\phi(x_1)\tilde{\phi}(x_2)|0\rangle = \frac{1}{Z_0} \int \mathcal{D}\phi \phi(x_1)\tilde{\phi}(x_2) \exp(iS[\phi]) \quad (1.3.56)$$

Where ϕ and $\tilde{\phi}$ are two of the fields in φ , with $\tilde{\phi}$ the ‘‘adjoint’’ of ϕ . It is possible that $\tilde{\phi}$ is the same field as ϕ if this field has no antiparticle. This correlation function is the

propagator we described in 1.3.1. Typically we can express all $\langle 0|TO(\phi)|0\rangle$ in integrals involving propagators, by means of the Wick theorem, which results in the Feynman rules for the quantum theory. (which we will not derive in this thesis). An extra note of importance is that $Z[J]$ is analogous to the partition function of statistical mechanics. This allows us to describe the statistical mechanics of a macroscopic system for which the microscopic physics are governed by our quantum field theory.

1.3.4 Quantisation by functional integral for a Dirac field

We will now fully quantise the free Dirac field by a functional integral. Quantising a field by functional integrals comes down to calculating correlation functions by functional integrals. We will only do this for the two-point correlation function here, since for higher order correlation functions we can use Feynman rules if we know the two-point correlation function. We will describe the Dirac field by a Grassmann field, which is a field whose values are Grassmann numbers. Grassmann numbers are anticommuting numbers. For any two Grassmann numbers θ and η , the following is true:

$$\theta\eta = -\eta\theta \quad (1.3.57)$$

And thus for any Grassmann number θ we see that

$$\theta^2 = 0 \quad (1.3.58)$$

We will now consider $\psi(x)$ to be a Grassmann field, which we can decompose in orthonormal basis functions

$$\psi(x) = \sum_i \psi_i \phi_i(x) \quad (1.3.59)$$

Where $\phi_i(x)$ are complex number (c-number) functions, and the coefficients ψ_i are Grassmann numbers.

The correlation function is given by

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(iS[\bar{\psi}, \psi])\psi(x_1)\bar{\psi}(x_2)}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(iS[\bar{\psi}, \psi])} \quad (1.3.60)$$

Where $S[\bar{\psi}, \psi]$ is the action, and $\mathcal{D}\psi$ is defined as

$$\mathcal{D}\psi = \prod_i d\psi(x_i) \quad (1.3.61)$$

Where x_i is the position of the i-th point on a "square" lattice (in 4D). The integral $\int \mathcal{D}\bar{\psi}\mathcal{D}\psi(\dots)$ is a functional integral (path integral) and will integrate over all possible field configurations.

The action $S[\bar{\psi}, \psi]$ is given by

$$S[\bar{\psi}, \psi] = \int d^4x \mathcal{L}_{\text{Dirac}} = \int d^4x \bar{\psi}(i\cancel{\partial} - m)\psi \quad (1.3.62)$$

We will define the generating functional for the Dirac field as

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i \int d^4x [\mathcal{L}_{\text{Dirac}} + \bar{\eta}\psi + \bar{\psi}\eta]\right) \quad (1.3.63)$$

Where $\eta(x)$ is a Grassmann source field.

For a Grassmann field, if we have

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i \int d^4x d^4x' [\bar{\psi}(x)K(x, x')\psi(x')] + i \int d^4x [\bar{\eta}\psi + \bar{\psi}\eta]\right) \quad (1.3.64)$$

Then it is true that [4 (p291-292 but with different convention)]

$$Z[\bar{\eta}, \eta] = Z_0 \exp\left(i \int d^4x d^4x' [\bar{\eta}K^{-1}\eta]\right) \quad (1.3.65)$$

With $Z_0 = Z[0,0]$

From our definition of the generating functional we get

$$Z_0 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS[\bar{\psi}, \psi]} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i \int d^4x \mathcal{L}_{\text{Dirac}}\right) \quad (1.3.66)$$

For K we have the following equality:

$$\int d^4x' K(x, x')K^{-1}(x', x'') = -\delta(x - x'') \quad (1.3.67)$$

which defines the inverse of K .

In the case of the Dirac field, we have

$$\int d^4x d^4x' [\bar{\psi}(x)K(x, x')\psi(x')] = \int d^4x \mathcal{L}_{\text{Dirac}} \quad (1.3.68)$$

Where we make use of $\mathcal{L}_{\text{Dirac}} = \bar{\psi}(x)(i\cancel{\partial} - m)\psi(x) = \int d^4x' [\bar{\psi}(x')K(x', x)\psi(x)]$

We can now shift one of the fields by using the delta function:

$$\int d^4x \bar{\psi}(x)(i\cancel{\partial} - m)\psi(x) = \int d^4x d^4x' \left[\bar{\psi}(x')\delta(x - x') \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) \right] \quad (1.3.69)$$

Now we can see that

$$K(x', x) = \delta(x - x') \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) = \delta(x - x')(i\cancel{\partial}_x - m) \quad (1.3.70)$$

By the use of Fourier transform [5] (p302) we find that

$$\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle = S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-x')}}{\not{p} - m + i\varepsilon} \quad (1.3.71)$$

Where we recognise the Fourier transform for the propagator

$$S_F(p) = \frac{i}{\not{p} - m + i\varepsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} \quad (1.3.72)$$

And we have $K^{-1}(x, x') = iS_F(x - x')$

And for the generating functional we now have

$$Z[\bar{\eta}, \eta] = Z_0 \exp\left(-\int d^4x d^4x' [\bar{\eta}(x)S_F(x-x')\eta(x')]\right) \quad (1.3.73)$$

For $Z_0 = Z[0,0]$, we have

$$Z[\bar{\eta}, \eta] = Z_0 \exp\left(-\int d^4x d^4x' [\bar{\eta}(x)S_F(x-x')\eta(x')]\right) \quad (1.3.74)$$

$$S[\bar{\psi}, \psi] = \int d^4x \mathcal{L}_{\text{Dirac}} = \int d^4x \bar{\psi}(i\not{\partial} - m)\psi \quad (1.3.75)$$

We can obtain all correlation functions as functional derivatives of Z . For example the two-point correlation function (which is just the propagator):

$$\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle = \frac{1}{Z_0} \left(\left(-i \frac{\delta}{\delta \bar{\eta}(x)} \right) \left(i \frac{\delta}{\delta \eta(x')} \right) Z[\bar{\eta}, \eta] \right) \Bigg|_{\bar{\eta}, \eta=0} = S_F(x-x') \quad (1.3.76)$$

Where $\frac{\delta}{\delta \eta}$ are functional derivatives.

1.3.5 Quantisation by functional integral for an abelian gauge field (U(1))

Now we will quantise the free abelian U(1) field by functional integrals, which will be free photon field from quantum electrodynamics (QED) in case we take the coupling constant to be equal to the electron charge.

The two-point correlation function is given by

$$\langle 0|TA^\mu(x_1)A^\nu(x_2)|0\rangle = \frac{\int \mathcal{D}A \exp(iS[A])A^\mu(x_1)A^\nu(x_2)}{\int \mathcal{D}A \exp(iS[A])} \quad (1.3.77)$$

With $S[A]$ the action:

$$S[A] = \int d^4x \mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4} \int d^4x (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \quad (1.3.78)$$

In analogy to the Dirac field, we define the generating functional as

$$Z[J] = \int \mathcal{D}A \exp\left(i \int d^4x [\mathcal{L}_{\text{Maxwell}} + J^\mu A_\mu]\right) \quad (1.3.79)$$

$$Z_0 = Z[0] = \int \mathcal{D}A \exp(iS[A]) = \int \mathcal{D}A \exp\left(i \int d^4x \mathcal{L}_{\text{Maxwell}}\right) \quad (1.3.80)$$

We can rewrite the action as

$$\begin{aligned} S[A] &= -\frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} \int d^4x (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \frac{1}{2} \int d^4x \partial_\nu A_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2} \int d^4x A_\mu(x) (\square g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu(x) \end{aligned} \quad (1.3.81)$$

After Fourier transformation we get

$$S[A] = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A_\mu(p) (-p^2 g^{\mu\nu} + p^\mu p^\nu) A_\nu(p) \quad (1.3.82)$$

We will introduce the projector operators:

$$P_\perp^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}; \quad P_{//}^{\mu\nu} = \frac{p^\mu p^\nu}{p^2} \quad (1.3.83)$$

$$P_\perp^{\mu\nu} p_\nu = 0; \quad P_{//}^{\mu\nu} p_\nu = p^\mu \quad (1.3.84)$$

This allows us to rewrite the action

$$S[A] = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A_\mu(p) (-p^2 P_\perp^{\mu\nu}) A_\nu(p) \quad (1.3.85)$$

For longitudinal A^μ we have $A^\mu(p) = \alpha p^\mu$, with $\alpha = \alpha(p)$ an arbitrary scalar function of the momentum p . This means that for longitudinal $A^\mu(p)$ we get

$$-p^2 P_\perp^{\mu\nu} A_\nu(p) = -p^2 P_\perp^{\mu\nu} \alpha p_\nu = 0 \quad (1.3.86)$$

So for the longitudinal polarisation of the gauge field, we have that the action will equal 0. This means that when we take the functional integral Eq (1.3.80) we will run into the following problem:

$$Z_0 = \int \mathcal{D}A \rightarrow \infty \quad (1.3.87)$$

When we try to quantise our free abelian vector field, we run into trouble for the longitudinal polarisation of the field. Note that this is related to the U(1) gauge transformation Eq (1.2.59):

$$A^\mu \rightarrow A^\mu + \frac{1}{g} \partial^\mu \alpha \quad (1.3.88)$$

The longitudinal A^μ will introduce a “gauge volume” factor in the integral.

Another way we can see this is by Green’s method. For the abelian vector field, we have the conserved current

$$\partial_\mu F^{\mu\nu} = (\square g^{\mu\nu} - \partial^\mu \partial^\nu) A_\mu = j^\nu \quad (1.3.89)$$

If we solve this by Green’s method we get

$$(\square g^{\mu\nu} - \partial^\mu \partial^\nu) D_{\nu\lambda} = i \delta_\lambda^\mu \delta^{(4)}(x) \quad (1.3.90)$$

Or after Fourier transformation:

$$(-p^2 g^{\mu\nu} + p^\mu p^\nu) D_{\nu\lambda}(p) = Q^{\mu\nu} D_{\nu\lambda}(p) = i \delta_\lambda^\mu \quad (1.3.91)$$

Where $D^{\mu\nu}$ is the propagator of the vector field, and where we defined

$$Q^{\mu\nu} = -p^2 g^{\mu\nu} + p^\mu p^\nu \quad (1.3.92)$$

The solution for the propagator would be

$$D^{\mu\nu}(p) = i(Q^{-1})^{\mu\nu} \quad (1.3.93)$$

With Q^{-1} the inverse of Q . But Q^{-1} does not exist, because it has a zero-mode, and thus a zero eigenvalue:

$$Q^{\mu\nu} \alpha(p) p_\nu = 0 \quad (1.3.94)$$

Where $\alpha(p) p_\nu$ is an eigenvector with eigenvalue 0. This eigenvector is precisely the longitudinal polarization of A^μ .

The longitudinal mode of the abelian vector field seems to prevent us from quantising the theory, which is related to the U(1) gauge freedom in A^μ . We will need to find a way to “fix the gauge” and only count the transversal modes of A^μ in Eq (1.3.80).

We will do this via the Faddeev-Popov method:

We choose a gauge fixing condition.

$$\partial_\mu A^\mu = 0 \quad (1.3.95)$$

This gauge is called the Lorenz gauge (Ludvig Lorenz, not to be confused with Hendrik Lorentz). This can be implemented in the functional integral by a trick [5] (p295)

Let us consider a general gauge fixing condition $G(A) = 0$, which in our choice would be the Lorenz gauge $\partial_\mu A^\mu = 0$. In order to constrain the functional integral to only take the A^μ into account that satisfies the gauge fixing condition $G(A) = 0$, we will insert a functional delta function $\delta(G(A, \alpha))$, with $\alpha = \alpha(x)$ an extra scalar function, and perform an extra integration over α . For this functional delta function, the following equation holds [5] (p295)

$$1 = \int \mathcal{D}\alpha \delta(G(A, \alpha)) \det\left(\frac{\delta G(A, \alpha)}{\delta \alpha}\right) \quad (1.3.96)$$

with $\frac{\delta}{\delta \alpha}$ the functional derivative with respect to α . We will take the scalar function α equal to the α from the gauge transformation, and we can rewrite G as

$$G(A, \alpha) = G(A(\alpha)) \quad (1.3.97)$$

with $A^\mu(\alpha)$ the gauge transformed vector field

$$A^\mu(\alpha) = A^\mu(\alpha, x) := A^\mu(x) + \frac{1}{g} \partial^\mu \alpha(x) \quad (1.3.98)$$

We can now insert the functional integral Eq (1.3.96), since it equals 1 anyway, in the functional integral Eq (1.3.80) for Z_0

$$\begin{aligned} Z_0 &= \int \mathcal{D}A e^{iS[A]} \int \mathcal{D}\alpha \delta(G(A(\alpha))) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \\ &= \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(G(A(\alpha))) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \end{aligned} \quad (1.3.99)$$

In the Lorenz gauge $G(A) = \partial_\mu A^\mu$ we have

$$G(A(\alpha)) = \partial_\mu A^\mu(\alpha) = \partial_\mu \left(A^\mu + \frac{1}{g} \partial^\mu \alpha \right) = \partial_\mu A^\mu + \frac{1}{g} \square \alpha \quad (1.3.100)$$

If we work out the functional derivative in the determinant, we see that

$$\frac{\delta G(A(\alpha))}{\delta \alpha} = \frac{1}{g} \square \quad (1.3.101)$$

Thus $\det\left(\frac{1}{g} \square\right)$ is independent of A^μ and α , and can be treated as a “constant” in the functional integral, and can be taken outside the functional integral [5] (p295)

$$Z_0 = \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(G(A(\alpha))) \quad (1.3.102)$$

Note that we have the following equalities:

$$\mathcal{D}(A(\alpha)) = \mathcal{D}A \quad (1.3.103)$$

$$S[A(\alpha)] = S[A] \quad (1.3.104)$$

Now we can easily change the integration variable from A to $A(\alpha)$, and we get

$$Z_0 = \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}(A(\alpha)) e^{iS[A(\alpha)]} \delta(G(A(\alpha))) \quad (1.3.105)$$

Since $A(\alpha)$ in the integral is just a dummy variable, we can rename it to A again

$$Z_0 = \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(G(A)) \quad (1.3.106)$$

We note that there is no α dependence anymore in $\int \mathcal{D}A e^{iS[A]} \delta(G(A))$, so we can write

$$Z_0 = \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A e^{iS[A]} \delta(G(A)) \quad (1.3.107)$$

The functional integral $\int \mathcal{D}A e^{iS[A]} \delta(G(A))$ will result in $\int \mathcal{D}A e^{iS[A]}$ with only the A that satisfies the gauge fixing condition $G(A) = 0$, which in our case is $\partial_\mu A^\mu = 0$

We will now do the same for a general class of gauge fixing conditions, the generalised Lorenz gauge:

$$G(A) = \partial_\mu A^\mu - \omega \quad (1.3.108)$$

With $\omega = \omega(x)$ a scalar function. Note that in this case, we have

$$G(A(\alpha)) = \partial_\mu A^\mu(\alpha) - \omega = \partial_\mu A^\mu + \frac{1}{g} \square \alpha - \omega \quad (1.3.109)$$

And for the determinant of the functional derivative we have the same as before:

$$\det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) = \det\left(\frac{1}{g} \square\right) \quad (1.3.110)$$

So for Z_0 we have the same as Eq (1.3.107), which will now become

$$Z_0 = \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A e^{iS[A]} \delta(\partial_\mu A^\mu - \omega) \quad (1.3.111)$$

This holds for any $\omega(x)$, so we can replace the expression for Z_0 by a properly normalised linear combination involving different $\omega(x)$. Then we can integrate over all $\omega(x)$ with a Gaussian weighting function centered on $\omega = 0$, Ref [5] (p296)

$$Z_0 = N(\xi) \int \mathcal{D}\omega \exp\left(-i \int d^4x \frac{\omega^2}{2\xi}\right) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A e^{iS[A]} \delta(\partial_\mu A^\mu - \omega) \quad (1.3.112)$$

with ξ an arbitrary constant and $N(\xi)$ a normalisation factor. We use the $\delta(\partial_\mu A^\mu - \omega)$ to perform the $\mathcal{D}\omega$ integral, and we get

$$\begin{aligned} Z_0 &= N(\xi) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A e^{iS[A]} \exp\left(-i \int d^4x \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right) \\ &= N(\xi) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A \exp\left(i \int d^4x \left[\mathcal{L}_{\text{Maxwell}} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right]\right) \end{aligned} \quad (1.3.113)$$

$N(\xi) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right)$ is an infinite “gauge-volume” factor.

$\int \mathcal{D}A \exp\left(i \int d^4x \left[\mathcal{L}_{\text{Maxwell}} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right]\right)$ is a “gauge-fixed” functional integral.

If an operator $O(A)$ is a gauge invariant, then we can find in a similar way that the following expression holds [5] (p296):

$$\begin{aligned} &\int \mathcal{D}A O(A) \exp(iS[A]) \quad (1.3.114) \\ &= N(\xi) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A O(A) e^{iS[A]} \exp\left(-i \int d^4x \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right) \\ &= N(\xi) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A O(A) \exp\left(i \int d^4x \left[\mathcal{L}_{\text{Maxwell}} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right]\right) \end{aligned}$$

We notice that we get the same gauge-volume factor in front of the gauge-fixed functional integral as with Z_0 . This means that for the correlation function we have

$$\langle 0|T O(A)|0\rangle = \frac{\int \mathcal{D}A O(A) \exp(iS[A])}{\int \mathcal{D}A \exp(iS[A])} = \frac{\int \mathcal{D}A O(A) \exp\left(i \int d^4x \left[\mathcal{L}_{\text{Maxwell}} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right]\right)}{\int \mathcal{D}A \exp\left(i \int d^4x \left[\mathcal{L}_{\text{Maxwell}} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right]\right)} \quad (1.3.115)$$

This means that for any physical observable $O(A)$ the infinite gauge volume is factored out and cancels out. The expression on the right hand side of the equation

looks like the expression in the middle of the equation, but with an extra term added to the Lagrangian. This gauge-fixed Lagrangian will be the full Lagrangian needed for quantisation of the free abelian (U(1)) gauge field

$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{GF}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad (1.3.116)$$

ξ is the gauge parameter, and since this is an arbitrary parameter, it must not appear in physical observables.

We will now calculate the propagator for the free abelian vector boson, we will refer to as the photon propagator. We have

$$Z_0 = C_\infty \int \mathcal{D}A \exp\left(i \int d^4x \mathcal{L}\right) = C_\infty \int \mathcal{D}A \exp\left(i \int d^4x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]\right) \quad (1.3.117)$$

With C_∞ the infinite gauge volume. If we rework the action:

$$\begin{aligned} S[A] &= \int d^4x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right] \\ &= \frac{1}{2} \int d^4x A_\mu(x) \left(\square g^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu\right) A_\nu(x) \end{aligned} \quad (1.3.118)$$

And after Fourier transforming we get

$$\begin{aligned} S[A] &= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A_\mu(p) \left(-p^2 g^{\mu\nu} + p^\mu p^\nu - \frac{1}{\xi} p^\mu p^\nu\right) A_\nu(p) \\ &= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A_\mu(p) Q^{\mu\nu} A_\nu(p) \end{aligned} \quad (1.3.119)$$

With $Q^{\mu\nu} = -p^2 \left(P_\perp^{\mu\nu} + \frac{1}{\xi} P_{//}^{\mu\nu}\right) = -p^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu$

$Q^{\mu\nu}$ has no longer a zero-mode.

For longitudinal $A^\mu(p) = \alpha(p)p^\mu$, we now have

$$Q^{\mu\nu} A_\nu = -p^2 \left(P_\perp^{\mu\nu} + \frac{1}{\xi} P_{//}^{\mu\nu}\right) \alpha p_\nu = -\frac{1}{\xi} \alpha(p) p^2 p^\mu \quad (1.3.120)$$

Where we used

$$P_\perp^{\mu\nu} p_\nu = 0; \quad P_{//}^{\mu\nu} p_\nu = p^\mu \quad (1.3.121)$$

For the inverse Q^{-1} we will use the method of the Green's function

$$i\delta_\mu^\lambda = Q_{\mu\nu}D^{\nu\lambda}(p) = \left(-p^2g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)p_\mu p_\nu\right)D^{\nu\lambda}(p) \quad (1.3.122)$$

Due to the lack of a zero-mode for Q , Q^{-1} will exist, and we get for the propagator [5] (p297)

$$D_F^{\mu\nu}(p) = i(Q^{-1})^{\mu\nu} = -\frac{i}{p^2}\left(P_\perp^{\mu\nu} + \xi P_{//}^{\mu\nu}\right) = -\frac{i}{p^2}\left(g^{\mu\nu} - (1 - \xi)\frac{p^\mu p^\nu}{p^2}\right) \quad (1.3.123)$$

In the complex momentum plane, this has poles in $p^0 = \pm\sqrt{p^2}$. In order to take the correct integrals that involve the photon propagator, we include an infinitesimal $i\varepsilon$

$$D_F^{\mu\nu}(p) = -\frac{i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - (1 - \xi)\frac{p^\mu p^\nu}{p^2}\right) \quad (1.3.124)$$

Or in position space:

$$\langle 0|TA^\mu(x)A^\nu(x')|0\rangle = D_F^{\mu\nu}(x - x') = -\int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-x')}}{p^2 + i\varepsilon}\left(g^{\mu\nu} - (1 - \xi)\frac{p^\mu p^\nu}{p^2}\right) \quad (1.3.125)$$

While we still have a ξ in our propagator, there will be no ξ dependence in physical observables. This results from the conservation of physical currents.

Note that the gauge parameter ξ is a Lagrange multiplier for the constraint field by gauge fixing

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{1}{2\xi}G(\dots) \quad (1.3.126)$$

When working with photon propagators, usually an explicit choice of gauge parameter is made. There are 2 main important choices of gauge parameter:

- $\xi = 0$ is called the Landau gauge, in which the propagator is proportional to the transversal projector operator $P_\perp^{\mu\nu}$.
- $\xi = 1$ is called the Feynman gauge, in which the propagator is proportional to the metric $g^{\mu\nu}$.

Summary:

When we tried to quantise the abelian gauge field (the photon field), we ran into problems. Due to the longitudinal modes of the photon field, we had:

- $Z_0 \rightarrow \infty$ because of an infinite gauge volume factor.
- The longitudinal modes were zero-modes for $Q^{\mu\nu}$, which meant Q^{-1} didn't exist, resulting in the propagator $D^{\mu\nu}(p) = i(Q^{-1})^{\mu\nu}$ to not exist either.

The longitudinal mode of the abelian gauge field seemed to prevent us from quantising the theory, which is related to the U(1) gauge freedom in

$$A^\mu \rightarrow A^\mu + \frac{1}{g} \partial^\mu \alpha \quad (1.3.88)$$

Via the Faddeev-Popov method, we made our functional integrals obey a generalised Lorenz gauge fixing condition $\partial_\mu A^\mu(x) - \omega(x) = 0$. Thanks to this, we could factor the infinite gauge volume out of the functional integrals, and cancel them out in physical observables. This resulted in adding a new term, the gauge fixing term, in our Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{GF}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \quad (1.3.116)$$

With ξ the gauge parameter, which is arbitrary and does not appear in physical observables. This new Lagrangian for the abelian gauge field results in the existence of Q^{-1} , because Q no longer has any zero-modes anymore. This means we can calculate our photon propagator:

$$D_F^{\mu\nu}(p) = -\frac{i}{p^2 + i\varepsilon} \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right) \quad (1.3.124)$$

By performing the Faddeev-Popov method, we are able to cancel the unphysical longitudinal modes of the gauge field by gauge fixing, and only take the physical transversal modes into account.

1.3.6 Full quantised Lagrangian for QED

Now that we quantised the free Dirac field and the free abelian gauge field, we can quantise the field theory that couples them via U(1) gauge invariance. For the classical field theory that couples the Dirac field and the U(1) gauge field, we had Eq (1.2.72):

$$\mathcal{L}_{\text{classical}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + g\bar{\psi}\not{A}\psi = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (1.3.127)$$

Where $\mathcal{L}_0 = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}}$ is the free Lagrangian, and $\mathcal{L}_I = g\bar{\psi}\not{A}\psi$ is the interaction term that couples the Dirac field and gauge field.

Quantising this theory by functional integrals comes down to adding the gauge fixing term to the classical Lagrangian, like we did at Eq (1.3.116), and using the propagators for the free fields in combination with the Feynman rules for this theory.

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_I + \mathcal{L}_{\text{GF}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \quad (1.3.128)$$

We follow the same Feynman rules as in [5] (p801-802), but for a general gauge parameter.

For the (free) fermion propagator we had

$$\begin{array}{c} p \\ \longrightarrow \end{array} \quad S_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \quad (1.3.72)$$

For the (free) photon propagator we had

$$\begin{array}{c} p \longrightarrow \\ \nu \text{ wavy } \mu \end{array} \quad D_F^{\mu\nu}(p) = -\frac{i}{p^2 + i\epsilon} \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right) \quad (1.3.124)$$

The interaction term $\mathcal{L}_1 = g\bar{\psi}\not{A}\psi = g\bar{\psi}\gamma^\mu A_\mu\psi$ gives rise to a three-point interaction and will result in three-point vertices in the Feynman diagrams. The vertex will connect a Dirac fermion, antifermion, and a photon.

$$g\bar{\psi}\not{A}\psi \rightarrow \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \begin{array}{c} \text{wavy } \mu \end{array} \quad \text{Vertex factor} = ig\gamma^\mu \quad (1.3.129)$$

Additionally there are also rules for the external particles, which are just the polarisation spinors/vectors:

For an initial, respectively final, fermion: $u^s(p)$, $\bar{u}^s(p)$

For an initial, respectively final, antifermion: $\bar{v}^s(p)$, $v^s(p)$

For an initial, respectively final, photon: $\epsilon_\mu^s(p)$, $\epsilon_\mu^{s*}(p)$

Where $s(= 1,2)$ labels the polarisations. For physical external photons, only transverse polarisations are allowed.

Remember that if we take the coupling constant $g = -e$, we have the QED theory for our universe. In that case, the vertex factor will be $-ie\gamma^\mu$.

We now have all the building blocks to describe higher order processes in QED.

1.3.7 Quantisation by functional integral for a non-abelian gauge field (SU(N))

Now we will quantise the non-abelian SU(N) field by functional integrals, which will be gluon fields from quantum chromodynamics (QCD) in case we take $N = 3$ and the coupling constant g equal to g_s from the Strong Interaction. Most of the definitions and many of the equations will be of the same form as with the quantisation of the abelian gauge field, with the main differences that we now have a symmetry group that is non-abelian (non-commutative), that our Lagrangian is not a free field Lagrangian, and that the gauge transformation law for the gauge field is

$$A_\mu \rightarrow A'_\mu = GA_\mu G^\dagger - \frac{i}{g}(\partial_\mu G)G^\dagger \quad (1.3.130)$$

With $G = e^{i\alpha^a T^a}$

The free Lagrangian for the gauge fields is (from Eq (1.2.99)):

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) = -\frac{1}{4}G^{\mu\nu a}G_{\mu\nu}^a \quad (1.3.131)$$

With $G^{\mu\nu a} := \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}$

But the full classical gluon-only Lagrangian is Eq (1.2.96):

$$\mathcal{L}_G = -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a \quad (1.3.132)$$

With $F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + gf^{abc}A^{\mu b}A^{\nu c}$

The two-point correlation function is given by

$$\langle 0|TA^{\mu a}(x_1)A^{\nu b}(x_2)|0\rangle = \frac{\int \mathcal{D}A A^{\mu a}(x_1)A^{\nu b}(x_2)\exp(iS[A])}{\int \mathcal{D}A \exp(iS[A])} \quad (1.3.133)$$

With $S[A]$ the action:

$$\begin{aligned} S[A] &= \int d^4x \mathcal{L}_G = -\frac{1}{4} \int d^4x F^{\mu\nu a}F_{\mu\nu}^a \\ &= -\frac{1}{4} \int d^4x (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + gf^{abc}A^{\mu b}A^{\nu c})(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{ade}A_\mu^d A_\nu^e) \end{aligned} \quad (1.3.134)$$

We define the generating functional as

$$Z[J] = \int \mathcal{D}A \exp\left(i \int d^4x [\mathcal{L}_G + J^{\mu a}A_\mu^a]\right) \quad (1.3.135)$$

$$Z_0 = Z[0] = \int \mathcal{D}A \exp(iS[A]) = \int \mathcal{D}A \exp\left(i \int d^4x \mathcal{L}_G\right) \quad (1.3.136)$$

Now when we try to quantise this theory, we will just like with the abelian case run into the same trouble due to the longitudinal modes of the gauge fields.

We will apply the Faddeev-Popov method to the non-abelian case as well. For Z_0 we have like in Eq (1.3.99)

$$Z_0 = \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS[A]} \delta(G(A(\alpha))) \det\left(\frac{\delta G(A(\alpha))}{\delta\alpha}\right) \quad (1.3.137)$$

But this time our gauge transformation law for the gauge field is (Eq (1.2.87)):

$$A_\mu \rightarrow A'_\mu = GA_\mu G^\dagger - \frac{i}{g}(\partial_\mu G)G^\dagger \quad (1.3.138)$$

Recall that for small α^a we had (Eq (1.2.101-104, 109)):

$$G = e^{i\alpha^a T^a} \approx \mathbf{1} + i\alpha^a T^a \quad (1.3.139)$$

$$G^\dagger = e^{-i\alpha^a T^a} \approx \mathbf{1} - i\alpha^a T^a \quad (1.3.140)$$

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{g}\partial_\mu\alpha^a T^a - f^{abc}\alpha^a A_\mu^b T^c = A_\mu^{a'} T^a \quad (1.3.141)$$

$$A_\mu^a \rightarrow A_\mu^{a'} = A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha^b A_\mu^c = A_\mu^a + \frac{1}{g}D_\mu^{ab}\alpha^b \quad (1.3.142)$$

With the covariant derivative in the adjoint representation (Eq 1.2.106):

$$D_\mu = \mathbf{1}_{N \times N} \partial_\mu - igA_\mu^a T_A^a \quad (1.3.143)$$

With $(T_A^a)^{bc} = -if^{abc}$

We see that in the transformation law for A_μ^a , the term $-f^{abc}\alpha^b A_\mu^c$ is purely non-abelian. This means $\det\left(\frac{\delta G(A(\alpha))}{\delta\alpha}\right)$ will no longer be independent of A_μ^a , since for the Lorenz gauge condition

$$\begin{aligned} G(A(\alpha)) &= \partial_\mu A^{\mu a}(\alpha) = \partial_\mu \left(A^{\mu a} + \frac{1}{g}\partial^\mu\alpha^a - f^{abc}\alpha^b A^{\mu c} \right) \\ &= \partial_\mu A^{\mu a} + \frac{1}{g}\square\alpha^a - f^{abc}\partial_\mu\alpha^b A^{\mu c} \end{aligned} \quad (1.3.144)$$

Which indeed means that the determinant of the functional derivative will be dependent on A_μ^a

$$\det\left(\frac{\delta G(A(\alpha))}{\delta\alpha}\right) = \det\left(\frac{1}{g}\partial^\mu D_\mu\right) \quad (1.3.145)$$

And $D_\mu^{ab} = \delta^{ab}\partial_\mu - gf^{abc}A_\mu^c$

This means the determinant can no longer be pulled out of the functional integral over $\mathcal{D}A$. The same happens for a generalised Lorenz gauge condition

$$G(A(\alpha)) = \partial_\mu A^{\mu a}(\alpha) - \omega^a \quad (1.3.146)$$

With $\omega^a = \omega^a(x)$. So for Z_0 we have from Eq (1.3.137):

$$Z_0 = N(\xi) \int \mathcal{D}\alpha \int \mathcal{D}A \exp\left(i \int d^4x \left[-\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a - \frac{1}{2\xi} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a}\right]\right) \det\left(\frac{\delta G(A(\alpha))}{\delta \alpha}\right) \quad (1.3.147)$$

The fact that the determinant can't be pulled out the integrals, makes this different from the abelian case. We will solve this by another trick from Faddeev and Popov, where we represent the determinant as a functional integral over new field c and \bar{c} , which are called Faddeev-Popov ghost fields. The main properties of these fields are:

- They belong to the adjoint representation of SU(N).
- They belong to the spin 0 representation of the Lorentz group, in other words, they are scalar fields.
- They are anticommuting and thus obey Fermi-Dirac statistics.

The last two properties seem to contradict. A spin 0 field is a scalar boson field, but a boson should be commuting, and thus obey Bose-Einstein statistics. This field, on the other hand, is anticommuting, which means it obeys the statistics for a fermion. But a fermion has half-integer spin, not spin 0. These new fields having the wrong relation between spin and statistics hints that they will not be physical particles.

The expression for the determinant is [5] (p514)

$$\det\left(\frac{1}{g} \partial^\mu D_\mu\right) = \int \mathcal{D}c \int \mathcal{D}\bar{c} \exp\left(-i \int d^4x [\bar{c}^a \partial_\mu D^{\mu ab} c^b]\right) \quad (1.3.148)$$

Plugging this in Eq (1.3.147) yields

$$Z_0 = N(\xi) \int \mathcal{D}\alpha \int \mathcal{D}c \int \mathcal{D}\bar{c} \int \mathcal{D}A \exp\left(i \int d^4x \left[\mathcal{L}_G - \frac{1}{2\xi} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} - \bar{c}^a \partial_\mu D^{\mu ab} c^b\right]\right) \quad (1.3.149)$$

This trick adds yet another term to the Lagrangian, the ghost term

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{GF} + \mathcal{L}_{ghost} = -\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a - \frac{1}{2\xi} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} - \bar{c}^a \partial_\mu D^{\mu ab} c^b \quad (1.3.150)$$

We can rewrite the ghost term

$$\mathcal{L}_{ghost} = -\bar{c}^a \partial_\mu D^{\mu ab} c^b = -\bar{c}^a \square c^a - g f^{abc} \bar{c}^a \partial_\mu A^{\mu b} \partial_\mu c^c = \mathcal{L}_{ghost,0} - g f^{abc} \bar{c}^a \partial_\mu A^{\mu b} \partial_\mu c^c$$

(1.3.151)

With $\mathcal{L}_{\text{ghost},0} = -\bar{c}^a \square c^a$ a free field Lagrangian for the ghost field, while the other term is an interaction term between ghost fields and the gauge field.

For a physical observable we have

$$\langle 0|TO(A)|0\rangle = \frac{\int \mathcal{D}A O(A)\exp(iS[A])}{\int \mathcal{D}A \exp(iS[A])} = \frac{\int \mathcal{D}A O(A)\exp(i\int d^4x[\mathcal{L}_G + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}}])}{\int \mathcal{D}A \exp(i\int d^4x[\mathcal{L}_G + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}}])} \quad (1.3.152)$$

This means that the unphysical factors that were in front of the integrals, cancel out in physical observables. The gauge parameter ξ will also disappear from physical observables, like last time, due to conserved physical currents.

The gauge field propagator will be very similar to the one from the U(1) case

$$D_F^{\mu\nu ab}(p) = -\frac{i\delta^{ab}}{p^2 + i\varepsilon} \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right) \quad (1.3.153)$$

But we also need a propagator for the ghost field. The ghost term $-\bar{c}^a \square c^a$ gives the ghost propagator:

$$G_F^{ab}(p) = \frac{i\delta^{ab}}{p^2 + i\varepsilon} \quad (1.3.154)$$

Now, what is the physical interpretation of the ghost fields? Just like the gauge fixing term, they are there to cancel unphysical longitudinal gauge field modes. If we go back to the gauge transformation for the gauge field:

$$A_\mu^a \rightarrow A_\mu^{a'} = A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a - f^{abc} \alpha^b A_\mu^c \quad (1.3.142)$$

The term $\frac{1}{g} \partial_\mu \alpha^a$ is a longitudinal shift, which is $\frac{1}{g} \alpha^a p_\mu$ in momentum space. The longitudinal shift re-interacts with the gauge fields via the $-f^{abc} \alpha^b A_\mu^c$ term. This $-f^{abc} \alpha^b A_\mu^c$ term can also be interpreted as a colour gauge rotation.

When we quantise the theory, the longitudinal degrees of freedom can propagate. This means we need extra fields to cancel these unphysical degrees of freedom. This will prevent these unphysical degrees of freedom from propagating.

There will be a very interesting class of gauge choices for non-abelian gauge field theory, “physical” non-covariant gauges. With physical we mean the theory will be ghost-free, while with non-covariant we mean that manifest Lorentz covariance is lost (which means it is not easy to see if something is Lorentz invariant just by looking at the tensorial form of the expressions). In such a gauge we will fix a preferred

direction n^μ in spacetime, with n^μ a constant vector, and set the gauge fixing condition $n^\mu A_\mu^a = 0$. In this gauge, the ghosts do not appear because the gauge transformation of the gauge fixing condition contains no gauge field.

$$n^\mu A_\mu^a \rightarrow n^\mu A_\mu^{a'} = n^\mu A_\mu^a + n^\mu \frac{1}{g} \partial_\mu \alpha^a - n^\mu f^{abc} \alpha^b A_\mu^c \quad (1.3.155)$$

And due to the gauge fixing condition, the first and last terms are each equal to 0, which results in

$$n^\mu A_\mu^a \rightarrow n^\mu A_\mu^{a'} = n^\mu \frac{1}{g} \partial_\mu \alpha^a \quad (1.3.156)$$

This contains no A_μ^a , so we will be able to bring the determinant outside of the functional integrals, and this means that there will be no ghosts. In other words, non-abelian gauge field theories in the “physical” gauges resemble the abelian (U(1)) gauge field theory. This also shows again that the ghost fields are unphysical fields, since we can find a gauge fixing condition for which there are no ghosts in our theory. The gauge fixing term in a “physical” gauge will be

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} n^\mu A_\mu^a n^\nu A_\nu^a = -\frac{1}{2\xi} (n \cdot A)^2 \quad (1.3.157)$$

While this seems to make things easier, there will still be a price to pay. The gluon propagator (non-abelian gauge field propagator) will be a lot more complicated in “physical” gauges than in covariant gauges, and the propagator will also depend on n^μ . In case $\xi = 0$ we would get

$$D_F^{\mu\nu ab}(n, p) = -\frac{i\delta^{ab}}{p^2 + i\varepsilon} \left(g^{\mu\nu} - \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} + n^2 \frac{p^\mu p^\nu}{(p \cdot n)^2} \right) \quad (1.3.158)$$

When we will calculate the gluon-to-quark splitting function at the end of the thesis, we will use such a “physical” gauge.

1.3.8 Full quantised Lagrangian for QCD

Now that we quantised the Dirac field and the non-abelian gauge field, we can quantise the field theory that couples them via SU(N) gauge invariance. For the classical field theory that couples the Dirac field and the SU(N) gauge fields, we had the Yang-Mills Lagrangian

$$\mathcal{L}_{\text{classical}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_G + g \bar{\psi} \not{A} \psi = \bar{\psi} (i\not{D} - m) \psi - \frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a \quad (1.3.159)$$

With $F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + g f^{abc} A^{\mu b} A^{\nu c}$

And $D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igA_\mu^a T^a$

ψ and $\bar{\psi}$ are the quark fields, and A_μ^a are the gluon fields.

Remember that $\mathcal{L}_G = -\frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a$

If we take the quark flavours into account, our classical Lagrangian will be

$$\mathcal{L}_{\text{classical}} = \sum_f (\bar{\psi}_f (i\mathcal{D} - m_f)\psi_f) - \frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a \quad (1.3.160)$$

With the quark fields:

$$\psi_f = \begin{pmatrix} \psi_{f1} \\ \psi_{f2} \\ \vdots \end{pmatrix} \quad (1.3.161)$$

$$\bar{\psi}_f = (\bar{\psi}_{f1} \quad \bar{\psi}_{f2} \quad \dots) \quad (1.3.162)$$

Quantising this theory by functional integrals comes down to adding the gauge fixing term and the ghost term to the classical Lagrangian and using the propagators for the free fields in combination with the Feynman rules for this theory.

$$\mathcal{L}_{\text{QCD}} = \sum_f (\bar{\psi}_f (i\mathcal{D} - m_f)\psi_f) - \frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a - \frac{1}{2\xi} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} - \bar{c}^a \partial_\mu D^{\mu ab} c^b \quad (1.3.163)$$

With $\mathcal{D} = \gamma^\mu D_\mu$ in the fundamental representation, $D^{\mu ab}$ in the adjoint representation.

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - gf^{abc} A_\mu^c \quad (1.3.164)$$

We can also write the Lagrangian as

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_0 + \mathcal{L}_1 \quad (1.3.165)$$

With the free field Lagrangian

$$\mathcal{L}_0 = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost},0} \quad (1.3.166)$$

And with the interaction Lagrangian

$$\mathcal{L}_1 = g\bar{\psi}_f \not{A} \psi_f + \mathcal{L}_G - \mathcal{L}_{\text{Maxwell}} - gf^{abc} \bar{c}^a \partial_\mu A^{\mu b} \partial_\mu c^c \quad (1.3.167)$$

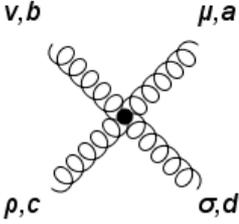
Where $\mathcal{L}_G - \mathcal{L}_{\text{Maxwell}}$ are the higher order terms resulting from $-\frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a$

We can rewrite this as

$$\mathcal{L}_1 = g\bar{\psi}_f \not{A} \psi_f + gf^{abc} A^{\mu a} A^{\nu b} \partial^\mu A^{\nu c} - \frac{1}{4}g^2 f^{abc} f^{ade} A^{\mu b} A^{\nu c} A_\mu^d A_\nu^e - gf^{abc} \bar{c}^a \partial_\mu A^{\mu b} \partial_\mu c^c$$

(1.3.173)

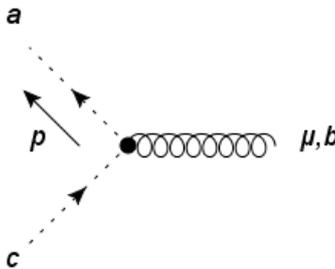
The term $-\frac{1}{4}g^2 f^{abc} f^{ade} A^{\mu b} A^{\nu c} A_{\mu}^d A_{\nu}^e$ are four-point interactions between gluons. We will refer to this vertex as the “4-boson vertex”, or “4-gluon” vertex.

$$-\frac{1}{4}g^2 f^{abc} f^{ade} A^{\mu b} A^{\nu c} A_{\mu}^d A_{\nu}^e \rightarrow$$


$$= -ig^2 [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})]$$

(1.3.174)

The term $-gf^{abc} \bar{c}^a \partial_{\mu} A^{\mu b} \partial_{\mu} c^c$ are three-point interactions that arise from the Faddeev-Popov gauge fixing method for QCD. The vertex will connect a ghost, antighost, and a gluon. We will refer to this vertex as the “ghost vertex”.

$$-gf^{abc} \bar{c}^a \partial_{\mu} A^{\mu b} \partial_{\mu} c^c \rightarrow$$


$$= -gf^{abc} p^{\mu}$$

(1.3.175)

There rules for the external particles are again just the polarisation spinors/vectors like in QED, but now also with an orientation in the group space:

For an initial, respectively final, quark: $u^{is}(p)$, $\bar{u}^{is}(p)$

For an initial, respectively final, antifermion: $\bar{v}^{is}(p)$, $v^{is}(p)$

For an initial, respectively final, gluon: $\epsilon_{\mu}^{as}(p)$, $\epsilon_{\mu}^{as*}(p)$

Where $s(= 1,2)$ labels the polarisations, $i(= 1, \dots, N)$ labels the quark colours and $a(= 1, \dots, N^2 - 1)$ labels the gluon colours (generators). For gluons only transverse polarisations are allowed as physical external states.

Remember that if we take $SU(N) = SU(3)$, and the coupling constant $g = g_s$ from the Strong Interaction, we have the QCD theory for our universe. In that case, we have $T^a = \frac{1}{2} \lambda^a$ with λ^a the Gell-Mann matrices (see ref [5]). The quark colours i are r, g, b .

With these basic building blocks, we can describe higher order processes. By using the Feynman rules for QCD, we can calculate the Feynman amplitude \mathcal{M} of some process.

Feynman rules:

1. Draw Feynman diagram.
2. For each line: write down the propagator.
3. For each vertex: write down the vertex factor.
4. For each external line: write down the factor for an external line.
5. Impose momentum conservation at each vertex.
6. Integrate over every undetermined (loop) momentum.
7. For every quark loop: take the trace and multiply with a factor -1.
8. For every ghost loop: multiply with -1.
9. Divide by symmetry factor.

(For the derivation of the Feynman rules we refer to [5] (p801-803, p506-515))

We now have a fully quantised QCD theory for a general SU(N) gauge symmetry.

1.3.9 Symmetries and quantum version of Noether's theorem

Without going into the details of the calculation, we will now briefly discuss the role of symmetries, and in particular global and local unitary symmetries (U(N) and SU(N)), in quantum field theory. In the classical field theory, continuous symmetries give rise to conservation laws via Noether's theorem. Noether's theorem relates exactly one conserved current to each generator of the symmetry group. As we discussed before, in case of a unitary symmetry group (U(N) or SU(N)) these were the colour charge currents, resulting in colour charge conservation. Since the functional integrals preserve symmetries, quantising the Lagrangian by functional integrals means that any symmetry that leaves the classical Lagrangian invariant, will do so for the quantum Lagrangian as well. This means that for quantised gauge field theories, we will still have the global and local unitary symmetries. This results in a quantum version of Noether's theorem called the Ward-Takahashi identities.

In the case of QED, the global U(1) symmetry will result in the Ward-Takahashi identity [5] (p311)

$$i\partial_\mu \langle 0 | T j^\mu(x) \psi(x_1) \bar{\psi}(x_2) | 0 \rangle = -ig\delta(x - x_1) \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle + ig\delta(x - x_2) \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle \quad (1.3.176)$$

$$(1.3.176)$$

Which translates to the amplitudes

$$-ik_\mu \mathcal{M}^\mu(k, p, q) = -ig\mathcal{M}_0(p, q - k) + ig\mathcal{M}_0(p + k, q) \quad (1.3.177)$$

This is the Ward-Takahashi identity for two external fermions.

For a process that involves two external fermions and an external photon as well, this will reduce to the Ward identity:

$$\mathcal{M}^{\mu\nu} k_\nu = 0 \quad (1.3.178)$$

With k^μ the momentum of the external photon.

As an example, consider the $e\gamma \rightarrow e\gamma$ scattering process in QED. On the tree level we have:

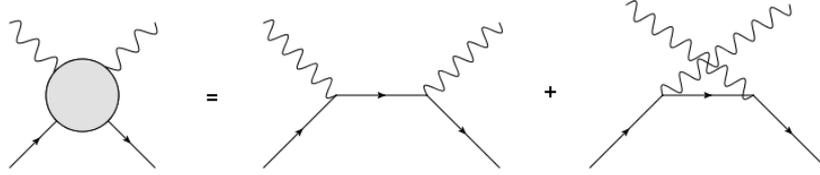


Figure 1.4: $e\gamma \rightarrow e\gamma$ scattering process on the tree level.

We can write the amplitudes for it as

$$\mathcal{M}^{\mu\nu} = \mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu} \quad (1.3.179)$$

The Ward identity is $\mathcal{M}^{\mu\nu} k_\nu = 0$, with k the momentum of one of the external photons. But for the amplitude of one of the individual tree level processes, the contracted product with k_ν shouldn't necessarily be 0. So we could have $\mathcal{M}_1^{\mu\nu} k_\nu \neq 0$ and $\mathcal{M}_2^{\mu\nu} k_\nu \neq 0$, but the sum of them should always be 0

$$\mathcal{M}^{\mu\nu} k_\nu = \mathcal{M}_1^{\mu\nu} k_\nu + \mathcal{M}_2^{\mu\nu} k_\nu = 0 \quad (1.3.180)$$

For the gauge transformed photon we have $A^\mu(k) \rightarrow A^\mu(k) + \alpha(k)k^\mu$, and $\alpha(k)k^\mu$ are the unphysical, longitudinal modes of the photon. Because of the Ward (Ward-Takahashi) identity, the longitudinal polarization mode of the photon disappears from this amplitude.

In case of QCD (SU(N)), the global SU(N) symmetry will result in the Ward-Takahashi identity for a non-abelian gauge theory, but in case of QCD, we call this the Slavnov-Taylor identity. For a process that involves two external quarks and an external gluon as well, we have Eq (1.3.178)

$$\mathcal{M}^{\mu\nu} k_\nu = 0 \quad (1.3.178)$$

For example consider the $qg \rightarrow qg$ scattering process in QCD. On the tree level we have:

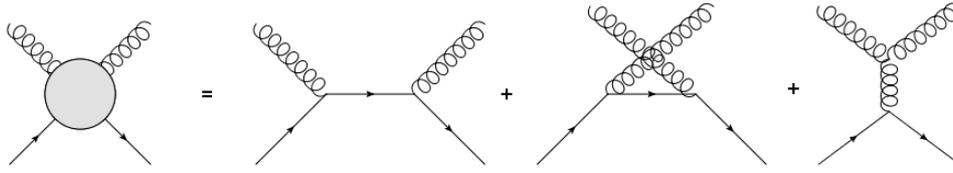


Figure 1.5: $qg \rightarrow qg$ scattering process on the tree level.

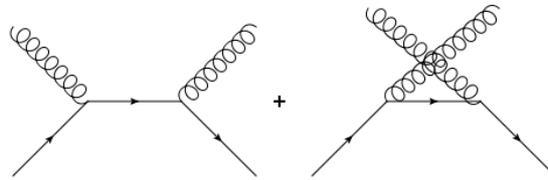
We can write the amplitudes for it as

$$\mathcal{M}^{\mu\nu ab} = \mathcal{M}_1^{\mu\nu ab} + \mathcal{M}_2^{\mu\nu ab} + \mathcal{M}_3^{\mu\nu ab} \quad (1.3.181)$$

The Slavnov-Taylor identity is $\mathcal{M}^{\mu\nu ab} k_\nu = 0$, with k the momentum of one of the external gluons. But for the amplitude of one of the individual tree level processes, the contracted product with k_ν shouldn't necessarily be 0. So we could have $\mathcal{M}_1^{\mu\nu ab} k_\nu \neq 0$, $\mathcal{M}_2^{\mu\nu ab} k_\nu \neq 0$, and $\mathcal{M}_3^{\mu\nu ab} k_\nu \neq 0$, but the sum of them should always be 0

$$\mathcal{M}^{\mu\nu ab} k_\nu = \mathcal{M}_1^{\mu\nu ab} k_\nu + \mathcal{M}_2^{\mu\nu ab} k_\nu + \mathcal{M}_3^{\mu\nu ab} k_\nu = 0 \quad (1.3.182)$$

Note that unlike for QED, for QCD the following sum does not result in 0 after the contracted product with k_ν



$$(\mathcal{M}_1^{\mu\nu ab} + \mathcal{M}_2^{\mu\nu ab}) k_\nu \neq 0 \quad (1.3.183)$$

Just like in QED, the longitudinal polarization modes of gluons will disappear from the amplitude $\mathcal{M}^{\mu\nu ab}$ because of the Slavnov-Taylor identity.

1.4 Renormalisation

While we have a fully quantised theory, and the quantised Lagrangian is good for dealing with processes at the tree level (no loops), we will run into trouble when loops arise in processes. Since we are usually applying perturbation theory to quantum field theories, loops will arise in most of those theories at higher orders. They will result in corrections to the tree level processes, called loop corrections. There will be two main effects of these loop corrections. First of all, it is possible that a certain process involving loops will give rise to integrals that diverge. This could endanger

our quantum field theory if we are not able to keep observables from diverging. And second, for observable parameters that appear in the Lagrangian, the loop corrections to tree level processes might result in a difference between these parameters their values in the Lagrangian and their values obtained by calculating via perturbation theory. Because of these two effects, our quantum field theory will need renormalisation in order to deal with loop corrections. This section will go over the basics of renormalisation. Some more detailed calculations and applications will appear in the next chapter, in section 2.1.

1.4.1 UV power counting and renormalizability

Feynman diagrams that contain loops could give rise to integrals that diverge. The divergences could come from high-momentum regions of the momentum space that is been integrated over. Consider a Feynman diagram containing a loop integral of the form

$$\int d^4k \frac{N(k)}{M(k)} \quad (1.4.1)$$

In the case of one loop, the superficial degree of divergence will be

$$D = 4 + (\text{powers of } k \text{ in } N) - (\text{powers of } k \text{ in } M) \quad (1.4.2)$$

If $D \geq 0$ we will most likely have a UV (“Ultraviolet”) divergence.

We will now characterise a quantum field theory by its “renormalisability”. The first type of QFTs that we will consider, are those with only a finite number of UV divergent amplitudes. This includes QED and QCD.

QED has 3 UV divergent amplitudes [4 (p199)]:

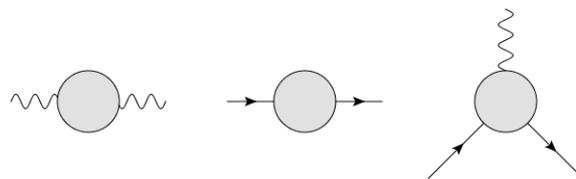


Figure 1.6: UV divergent amplitudes for QED. From left to right: photon self-energy, fermion self-energy, “fermion vertex” (fermion-photon vertex).

QCD, on the other hand, has 7 UV divergent amplitudes [4 (p323)]:

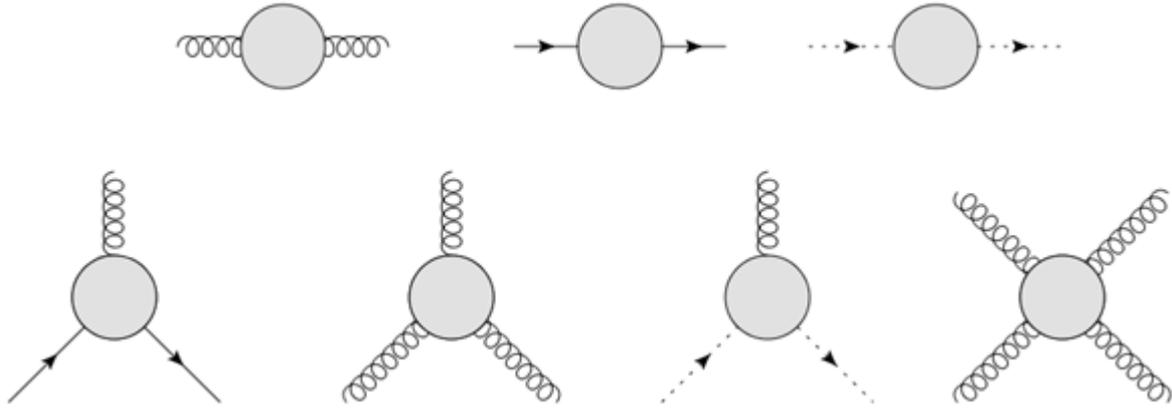


Figure 1.7: UV divergent amplitudes for QCD. Top left to right: gluon self-energy, quark self-energy, ghost self-energy. Bottom left to right: “quark vertex” (quark-gluon vertex), 3-gluon vertex, “ghost vertex” (ghost-gluon vertex), 4-gluon vertex.

A first way to characterise a theory as a renormalisable theory is that the number of UV divergent amplitudes is finite. Still, it is possible that a theory has an infinite amount of UV divergent amplitudes, but is still renormalisable. This can happen if those amplitudes are divergent because they contain one of the finite amounts of primitively divergent amplitudes.

Renormalisability its main point is that all the UV divergences can be absorbed into rescalings of the parameters and fields of the theory. This will be done via the renormalisation scheme:

1. Compute all the UV divergent amplitudes via the regularization method. This involves taking a UV cut off, and dimensional analysis.
2. Assign renormalisation conditions which will absorb all the divergences via rescalings of fields and parameters in the Lagrangian:

$$\varphi \rightarrow \varphi_0 = Z\varphi \quad (1.4.3)$$

With φ_0 the bare field, φ the renormalised field, and Z the renormalisation constant. And we define

$$Z = 1 + \delta \quad (1.4.4)$$

With δ the counter term, which can possibly be infinite. In the Lagrangian we replace our original φ by φ_0 , and after this we can write our Lagrangian as

$$\mathcal{L} = \mathcal{L}_R + \delta\mathcal{L} \quad (1.4.5)$$

With \mathcal{L}_R the renormalised Lagrangian, and $\delta\mathcal{L}$ the counter term Lagrangian.

3. All two-point correlation functions will now be finite, and unambiguously calculable.

If we can do this scheme for a quantum field theory, it means that the theory is renormalisable. If this is impossible to do for a quantum field theory, then it means that the theory is non-renormalisable.

We distinguish between the following type of quantum field theories [5] (p321):

- Super-renormalisable theory: A finite amount of diagrams that are superficially divergent.
- Renormalisable theory: A finite amount of amplitudes that are superficially divergent, but divergences appear at all orders in the perturbation theory.
- Non-renormalisable theory: All amplitudes are divergent at a sufficiently high order.

The type of renormalisability is related to the mass dimension of the coupling constants in the theory, in the following way [5] (p322):

- Super-renormalisable theory: The coupling constant has positive mass dimension.
- Renormalisable theory: The coupling constant is dimensionless.
- Non-renormalisable theory: The coupling constant has negative mass dimension.

1.4.2 Rescaling and counterterms

We will now carry out the rescaling in our QCD Lagrangian. By rescaling the fields and parameters, we will get counterterms in the Lagrangian. Our quantised QCD Lagrangian is Eq (1.3.163):

$$\mathcal{L}_{\text{QCD}} = \sum_f (\bar{\psi}_f (i\not{D} - m_f)\psi_f) - \frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a - \frac{1}{2\xi} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} - \bar{c}^a \partial_\mu D^{\mu ab} c^b \quad (1.4.6)$$

We have 6 different quantities: ψ, A, c, m, g, ξ

Now we rescale the fields and parameters, with the following convention:

For the free fields:

$$A \rightarrow A_0 = \sqrt{Z_g} A; \quad \psi \rightarrow \psi_0 = \sqrt{Z_q} \psi; \quad c \rightarrow c_0 = \sqrt{Z_c} c \quad (1.4.7)$$

For the mass and gauge parameter:

$$m \rightarrow m_0 = \frac{Z_m}{Z_q} m; \quad \xi \rightarrow \xi_0 = \frac{Z_g}{Z_\xi} \xi \quad (1.4.8)$$

For the interaction couplings, we have 4 equations.

For the coupling from quark-gluon vertex:

$$g \rightarrow g_0 = \frac{Z_{gq}}{Z_q \sqrt{Z_g}} g \quad (1.4.9)$$

For the coupling from ghost-gluon vertex:

$$g \rightarrow g_0 = \frac{Z_{gc}}{Z_c \sqrt{Z_g}} g \quad (1.4.10)$$

For the coupling from 3-gluon vertex:

$$g \rightarrow g_0 = \frac{Z_{3g}}{Z_g^{3/2}} g \quad (1.4.11)$$

For the coupling from 4-gluon vertex:

$$g^2 \rightarrow g_0^2 = \frac{Z_{4g}}{Z_g^2} g^2 \quad (1.4.12)$$

The Z_g is related to the gluon self-energy, Z_q is related to the quark self-energy, Z_c is related to the ghost self-energy, Z_{gq} is related to the quark-gluon vertex, Z_{gc} is related to the ghost-gluon vertex, Z_{3g} is related to the 3-gluon vertex, and Z_{4g} is related to the 4-gluon vertex. One renormalisation constant for each UV divergent amplitude. So in our case, we have 7 of them.

For each renormalisation constant, there will be a counterterm in the Lagrangian.

$$\forall K: Z_K = 1 + \delta_K \quad (1.4.13)$$

If we plug the renormalised (rescaled) fields and parameters in our Lagrangian, we can write it as Eq (1.4.5):

$$\mathcal{L} = \mathcal{L}_R + \delta\mathcal{L}$$

With \mathcal{L}_R the original Lagrangian, but in terms of the renormalised fields and parameters. $\delta\mathcal{L}$ is the counterterm Lagrangian.

In case of QED there are only 5 parameters, and their rescalings are analogous to those of QCD.

$$A \rightarrow A_0 = \sqrt{Z_\gamma} A; \quad \psi \rightarrow \psi_0 = \sqrt{Z_e} \psi \quad (1.4.14)$$

$$m \rightarrow m_0 = \frac{Z_m}{Z_e} m; \quad \xi \rightarrow \xi_0 = \frac{Z_\gamma}{Z_\xi} \xi \quad (1.4.15)$$

$$e \rightarrow e_0 = \frac{Z_{\gamma e}}{Z_e \sqrt{Z_\gamma}} e \quad (1.4.16)$$

1.4.3 Relations among counterterms

Gauge invariance will result in constraints on the counterterms. Or in other words, gauge invariance will result in relations among the UV divergences.

In QCD we have 6 quantities (ψ, A, c, m, g, ξ) and 9 counterterms. This means there will be 3 relations. Because of gauge invariance, all the g 's will be the same, and all the g_0 's will be the same. This means:

$$\frac{Z_{gq}}{Z_q\sqrt{Z_g}} = \frac{Z_{gc}}{Z_c\sqrt{Z_g}} = \frac{Z_{3g}}{Z_g^{3/2}} = \frac{\sqrt{Z_{4g}}}{Z_g} \quad (1.4.17)$$

These are the 3 relations among UV divergences. We can simplify this a bit more:

$$\frac{Z_{gq}}{Z_q} = \frac{Z_{gc}}{Z_c} = \frac{Z_{3g}}{Z_g} = \sqrt{\frac{Z_{4g}}{Z_g}} \quad (1.4.18)$$

These equations result from Slavnov-Taylor identities.

If we plug in the counterterms, we get:

$$\frac{1 + \delta_{gq}}{1 + \delta_q} = \frac{1 + \delta_{gc}}{1 + \delta_c} = \frac{1 + \delta_{3g}}{1 + \delta_g} = \sqrt{\frac{1 + \delta_{4g}}{1 + \delta_g}} \quad (1.4.19)$$

In case of QED there will only be one relation, which results from the Ward-Takahashi identity [5] (p243):

$$Z_{\gamma e} = Z_e \quad (1.4.20)$$

And for counterterms:

$$\delta_{\gamma e} = \delta_e \quad (1.4.21)$$

Due to Eq (1.4.20), Eq (1.4.16) simplifies to

$$e = \sqrt{Z_\gamma} e_0 \quad (1.4.22)$$

1.4.4 General effects from renormalisation

By renormalising a theory by rescaling parameters and fields, the bare parameters and bare fields in the Lagrangian are no longer the parameters and fields that one would observe. Even more, the bare parameters and bare fields can even be infinitely large. The parameters and fields that would be observed, are now the renormalised (rescaled) parameters and fields, since they are essentially the parameters and fields that you get after applying all higher order corrections of the

quantum field theory. The renormalised parameters and fields are what we call “dressed” (dressed with a “cloud” of higher order corrections). The possible infinities in the bare parameters and fields can be separated from the renormalised parameters and fields according to Eq (1.4.5). The counterterm Lagrangian $\delta\mathcal{L}$ contains everything that causes a difference between renormalised and bare parameters/fields.

Another important effect of renormalisation is that the renormalised parameters and fields will be scale dependent. While the bare parameters and fields are independent of energy scale, the higher order corrections will cause the renormalised ones to be dependent on the energy scale of the process that we look at. In other words, the masses, coupling “constants”, fields, and other parameters that we observe, now depend on the energy scale at which we observe them. We call this effect “running parameters” and “running fields”. This effect isn’t obvious from what we have done in this section, but will be shown in the next chapter (in section 2.1).

2 Application of QCD to high-energy collisions

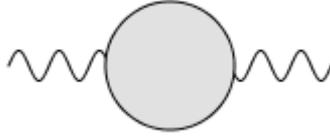
This chapter explores QCD and renormalisation, and its dependence on energy scale, with the goal to explain and understand collisions at high energy. First we will look at the effects of renormalisation of gauge theories, and find that the coupling constant becomes energy scale dependent. For QCD specifically, this “running coupling” will lead to asymptotic freedom. After that we will discuss the parton model and hard scattering, and how we can apply factorisation to some classes of hard scattering processes. For factorisation we will mostly focus on the example of Deep Inelastic Scattering (DIS), which will be of key importance for further chapters. This chapter is based on the books [4], [5], [9], and the course [10].

2.1 Asymptotic freedom

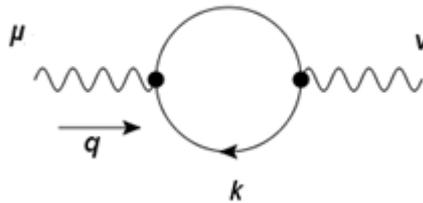
Non-Abelian gauge theories like QCD can have a surprising property that is very different from QED. Unlike QED which becomes stronger at shorter distances, QCD becomes weaker at shorter distances. This means that for the limit of the distance going to zero (or the limit of the energy scale going to infinity), quarks don't feel the Strong Interaction anymore and become free particles. This property is called “asymptotic freedom”. In this section we will first see how the renormalisation of a gauge theory results in the coupling constant being dependent on the energy scale it is measured at. This effect is called the “running coupling”. Then we will look at the renormalisation group and the beta function, which encodes the dependence of the coupling on the energy scale. We will show that in the case of QCD this leads to asymptotic freedom and a Landau pole. Finally we will also give a possible explanation of how the Landau pole of QCD could explain colour confinement.

2.1.1 Running coupling constant

In this section we will treat the renormalisation of the coupling constant. The derivation for this requires us to use one of the equations Eq (1.4.9-12) and calculate the diagrams related to the involved renormalisation constants. We will choose Eq (1.4.9) and calculate Z_g , Z_q and Z_{gq} via the diagrams of the gauge field self-energy, quark self-energy, and quark-gluon vertex correction respectively. When this is done up to second order, they will just contain one-loop diagrams. For simplicity we will first look at the case of QED. We will use Eq (1.4.22), which means that for the renormalisation of the electric charge we only need to calculate Z_γ , which is done via the photon self-energy diagram:



This diagram corresponds to the renormalised photon propagator $D^{\mu\nu}$. Note that the two photon lines are not necessarily physical external lines, but are bare photon propagators $D_0^{\mu\nu}$, equal to $D_F^{\mu\nu}$ in Eq (1.3.124). The grey disk represents the sum of all possible intermediate diagrams. Up to second order in the coupling constant, we only have to take into account one electron loop:



This diagram corresponds to $D_0^{\mu\alpha}(q)ie_0^2\Pi_{\alpha\beta}(q)D_0^{\beta\nu}(q)$, with the loop itself (without the external propagators) given by $ie_0^2\Pi^{\alpha\beta}(q)$. By using the Feynman rules of QED, we can express this loop as

$$ie_0^2\Pi^{\mu\nu}(q) = \frac{e_0^2}{(2\pi)^4} \text{Tr} \int d^4k \gamma^\mu S_0(k+q) \gamma^\nu S_0(k) \quad (2.1.1)$$

With S_0 the bare fermion propagator, equal to S_F in Eq (1.3.72). The Ward identity Eq (1.3.178) tells us that

$$q_\mu \Pi^{\mu\nu}(q) = 0 \quad (2.1.2)$$

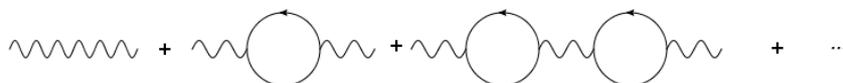
and this means that $\Pi^{\mu\nu}(q)$ must be proportional to the transversal projector. We can write the $\Pi^{\mu\nu}(q)$ as in [5] (p245):

$$i\Pi^{\mu\nu}(q) = q^2 P_\perp^{\mu\nu}(q) \Pi(q^2) \quad (2.1.3)$$

Where $P_\perp^{\mu\nu}(q)$ is the transversal projection operator:

$$P_\perp^{\mu\nu}(q) = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \quad (2.1.4)$$

We can do a Dyson resummation as follows:



Which means we can take the “resummed” photon propagator to be equal to

$$D = D_0 + D_0 \Pi D_0 + \dots \quad (2.1.5)$$

with D_0 the propagator for the free photon. We notice $D = D_0 + D_0 \Pi D$, and so we can rewrite our resummed propagator as

$$D = \frac{D_0}{1 - D_0 \Pi} = \frac{1}{\frac{1}{D_0} - \Pi} \quad (2.1.6)$$

Repeatedly applying the transversal projector in Π and using the gauge invariance, which makes longitudinal contributions vanish, the Dyson resummation gives us

$$D = D_0 \frac{1}{1 - \Pi(q^2)} \quad (2.1.7)$$

In the Feynman gauge, the free photon propagator is

$$D_F^{\mu\nu}(q) = -\frac{ig^{\mu\nu}}{q^2 + i\epsilon} \quad (2.1.8)$$

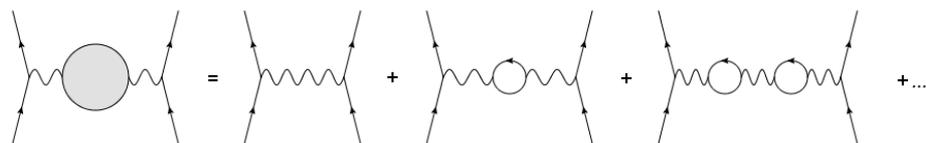
If we plug this in our equation for the “resummed” photon propagator, we get

$$D^{\mu\nu} = \frac{-ig^{\mu\nu}}{q^2(1 - \Pi(q^2))} \quad (2.1.9)$$

Where we ignored the $i\epsilon$ since we have extracted a tree level part, and the $i\epsilon$ only matters in loops. Since $\Pi(q^2)$ is regular at $q = 0$, the photon remains massless. [5] (p245-246)

This new photon propagator that we get from Dyson resummation, is the renormalised photon propagator for this specific Dyson resummation. Ofcourse this is not the fully renormalised photon propagator, because we haven’t taken multi-loop diagrams into account, nor did we take diagrams into account that connect 1-loop diagrams in a different way than what we have done. But this new photon propagator will already be more accurate to describe physics, than the free photon propagator.

We can now use this renormalised photon propagator in other diagrams. Let us use it in the following scattering process:



This represents the following expression:

$$e_0^2 D_0 \rightarrow e^2 D = e_0^2 D_0 \frac{1}{1 - \Pi(q^2)} \quad (2.1.10)$$

With e_0^2 the bare electric charge. We can rewrite this as

$$\frac{e_0^2}{q^2} \rightarrow \frac{e_0^2}{q^2} \frac{1}{1 - \Pi(q^2)} = \frac{e^2}{q^2} \quad (2.1.11)$$

And thus e is defined as

$$e^2 = \frac{e_0^2}{1 - \Pi(q^2)} \quad (2.1.12)$$

Now we will use the following expression which separates the divergent part and finite part in Π :

$$1 - \Pi(q^2) = (1 - \Pi(0))(1 - \Pi(q^2) - \Pi(0)) + \mathcal{O}(\alpha^2) \quad (2.1.13)$$

Where $(1 - \Pi(0))$ is divergent and $(1 - \Pi(q^2) - \Pi(0))$ finite. α is the fine structure constant, which is also called the coupling strength

$$\alpha = \frac{e^2}{4\pi} \quad (2.1.14)$$

By using the expression that separates the divergent part from finite part, we get

$$\frac{e_0^2}{q^2} \rightarrow \frac{e_0^2}{q^2} \frac{1}{1 - \Pi(q^2)} \approx \frac{1}{q^2} \frac{e_0^2}{1 - \Pi(0)} \frac{1}{1 - (\Pi(q^2) - \Pi(0))} = \frac{e^2(0)}{q^2} \frac{1}{1 - (\Pi(q^2) - \Pi(0))} \quad (2.1.15)$$

Where we defined

$$e^2(0) = \frac{e_0^2}{1 - \Pi(0)} \quad (2.1.16)$$

Note that $e^2 = e^2(0)$ in the approximation that we made in Eq (2.1.15). If we renormalise QED, we have $e^2 = Z_\gamma e_0^2$ from Eq (1.4.22), with e the renormalised electric charge and Z_γ the renormalisation constant. From Eq (2.1.16) we find that this renormalisation constant for the electric charge $e^2(0)$, thus now redefined as $e^2(0) = Z_\gamma e_0^2$, is equal to

$$Z_\gamma = \frac{1}{1 - \Pi(0)} \approx 1 + \Pi(0) \quad (2.1.17)$$

Where we have done Taylor expansion to get the right most part. The relation between renormalisation constant and counterterm is $Z_\gamma = 1 + \delta_\gamma$, so we have for the counterterm:

$$\delta_\gamma = \frac{2 - \Pi(0)}{1 - \Pi(0)} \approx \Pi(0) \quad (2.1.18)$$

The full calculation of $\Pi(q^2)$ is done in [4 (p211-214)], by using dimensional regularisation in $d = 4 - \varepsilon$ dimensions, and results in

$$\Pi(q^2) = -\frac{1}{12\pi^2} \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) + \dots \quad (2.1.19)$$

$$\Pi^{\mu\nu}(q) = -(q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{\mu^{-\varepsilon}}{12\pi^2} \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) + \dots \quad (2.1.20)$$

Where μ is a mass scale resulting from dimensional regularisation, and γ_E the Euler-Mascheroni constant. Then for Z_γ we have the expression [4 (p334)] (in the reference it is called Z_3):

$$Z_\gamma = 1 - \frac{e_0^2 \mu^{-\varepsilon}}{12\pi^2} \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) \quad (2.1.21)$$

And with $e^2(0) = e_0^2 Z_\gamma \mu^{-\varepsilon}$ in dimensional regularisation.

Note that, in the reference [4], in $\Pi^{\mu\nu}$ and Z_γ , the factors $e_0^2 \mu^{-\varepsilon}$ are replaced by the partially renormalised charge e_r^2 . But since the difference between e_r and $e_0 \mu^{-\varepsilon/2}$ is of order e_r^3 , we omitted this difference as an approximation. [4 (p334)]

Z_γ is a series in powers of α and can be approximated by [5] (p252):

$$Z_\gamma \approx 1 - \frac{2\alpha}{3\pi\varepsilon} + \dots \quad (2.1.22)$$

And this means for the counterterm:

$$\delta_\gamma \approx -\frac{2\alpha}{3\pi\varepsilon} + \dots \quad (2.1.23)$$

From renormalisation we have these effects in the approximations that we made:

- $e^2(0) = Z_\gamma e_0^2 = e_0^2 / (1 - \Pi(0)) \quad (2.1.24)$

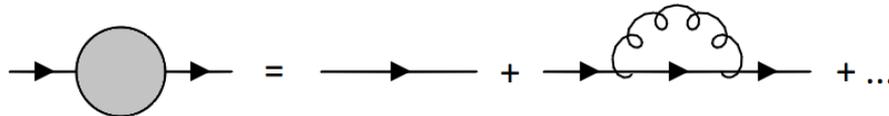
- $\delta_\gamma \approx \Pi(0) \approx -2\alpha / (3\pi\varepsilon) \quad (2.1.25)$

This shows that the charge that is observed in the scattering process is not the bare charge e_0^2 , but a renormalised charge. If we don't do the approximation that sets $e^2 = e^2(0)$, but take Eq (2.1.12), then another effect that we have from renormalisation is a finite observable dependence of the electric charge on q^2 . The electric charge e , and thus also the coupling strength $\alpha = e^2 / (4\pi)$, are no longer constant but depend on the transferred momentum q^2 in the scattering process. They are now a "running charge" and "running coupling".

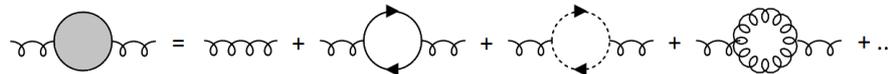
Now we will try to do the same for QCD. For the rescaling of the coupling constant we have from Eq (1.4.9):

$$g = \frac{Z_q \sqrt{Z_g}}{Z_{gq}} g_0 \quad (2.1.26)$$

Z_q corresponds to the quark self-energy:

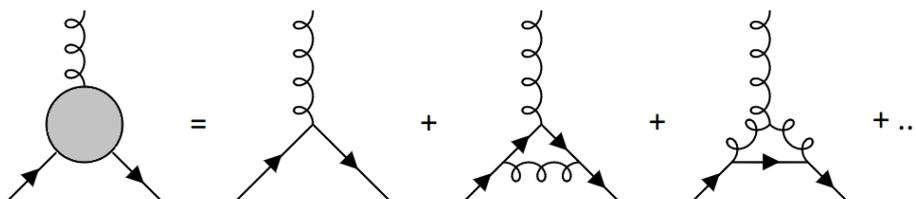


Z_g corresponds to the gluon self-energy. The calculation for this will be similar to that of the Z_γ and the photon self-energy, but there will be more diagrams because not only do we have a quark loop, but also a gluon loop and a ghost loop.



Other 2nd order diagrams for Z_q and Z_g vanish by looking at the colour, vertex and Lorentz factors when applying Feynman rules. [4 (p345-346)] The calculation for the diagram with the quark loop is done in almost the same way as the photon self-energy, but with an extra factor $T_F \delta_{ab}$ in Eq (2.1.3), with T_F the trace invariant (for SU(N) we have $T_F = 1/2$).

Z_{gq} corresponds to the quark-gluon vertex:



We refer to [4 (p346-351)] for results for each of the diagrams.

The total result for the gluon self-energy, in dimensional regularisation, is given by [4 (p347-348)]

$$\Pi_{ab}^{\mu\nu}(q) = -\delta_{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{\mu^{-\epsilon}}{16\pi^2} \left(\frac{2n_f}{3} - 5 \right) \left(\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) \right) + \dots \quad (2.1.27)$$

Where a and b are the colours of the incoming and outgoing gluon, and n_f the number of quark flavours. This result is similar to the photon self-energy Eq (2.1.20), but with orientation in the group space and replacement of

$$\frac{e_0^2}{12\pi^2} \rightarrow \frac{g_0^2}{16\pi^2} \left(\frac{2n_f}{3} - 5 \right) \quad (2.1.28)$$

With g_0 the bare coupling constant of QCD. And similar to α in QED, we can define the coupling strength of QCD (strong fine structure constant):

$$\alpha_s := \frac{g^2}{4\pi} \quad (2.1.29)$$

The renormalisation constant Z_g turns out to be similar to Z_γ , Eq (2.1.21), and is given by [4 (p349)] (in the reference it is called Z_3):

$$Z_g = 1 - \frac{g_0^2 \mu^{-\varepsilon}}{16\pi^2} (2n_f - 5) \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) \quad (2.1.30)$$

The result for the quark self-energy, in dimensional regularisation, is given by [4 (p331,346)]

$$\Sigma_{ij}(p) = \delta_{ij} \frac{\mu^{-\varepsilon}}{12\pi^2} (\not{p} - 4m_0) \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) + \dots \quad (2.1.31)$$

Where i and j are the colours of the incoming and outgoing quark, p the momentum of the quark. The renormalisation constant Z_q is given by [4 (p347)] (in the reference it is called Z_2):

$$Z_q = 1 - \frac{g_0^2 \mu^{-\varepsilon}}{12\pi^2} \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) \quad (2.1.32)$$

The result for the quark-gluon vertex, in dimensional regularisation, is given by [4 (p349)]

$$\Gamma_{ij}^{\mu a}(p, p') = g_0 \left(\gamma^\mu T_{ij}^a + g_0^2 \Lambda_{ij}^{\mu a}(p, p') \right) \quad (2.1.33)$$

With j and p the colour and momentum of the incoming quark, i and p' the colour and momentum of the outgoing quark, and a the colour of the gluon leg. $\Lambda_{ij}^{\mu a}$ is given by [4 (p350)]

$$\Lambda_{ij}^{\mu a}(p, p') = \gamma^\mu T_{ij}^a \frac{13\mu^{-\varepsilon}}{48\pi^2} \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) + \dots \quad (2.1.34)$$

The renormalisation constant Z_{gq} is given by [4 (p350)] (in the reference it is called Z_1):

$$Z_{gq} = 1 - \frac{13g_0^2 \mu^{-\varepsilon}}{48\pi^2} \left(\frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) \right) \quad (2.1.35)$$

Note that, in the reference [4], in the loop corrections and renormalisation constants, the factors $g_0^2 \mu^{-\varepsilon}$ are replaced by the partially renormalised coupling constant g_r^2 . But since the difference between g_r and $g_0 \mu^{-\varepsilon/2}$ is of order g_r^3 , we omitted this difference as an approximation.

From these renormalisation constants we can calculate the renormalised coupling g via Eq (2.1.26), which in dimensional regularisation becomes [4 (p351)]:

$$g = g_0 \mu^{-\varepsilon/2} \frac{Z_q \sqrt{Z_g}}{Z_{gq}} \quad (2.1.36)$$

Making the substitution of Eq (2.1.30), Eq (2.1.32) and Eq (2.1.35) results in [4 (p351-352)]:

$$g = g_0 \mu^{-\varepsilon/2} \left[1 + \frac{g^2}{32\pi^2} \left(11 - \frac{2n_f}{3} \right) \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) \right) + \mathcal{O}(g^4) \right] \quad (2.1.37)$$

Just like in the QED case, the coupling constant that is observed in a scattering process is not the bare coupling g_0^2 , but a renormalised coupling g . Also, the coupling constant g will have a finite observable dependence on the transferred momentum q^2 in the scattering process. The q^2 dependence is not evident from Eq (2.1.37), but is hidden as a μ dependence. The q^2 dependence can be retrieved in a similar way as for QED, see Eq (2.1.24) and Eq (2.1.a). In general we can say that the renormalised coupling depends on a renormalisation scale μ , which is on the same order as the momentum scale of the considered process.

2.1.2 Coupling strength α_s , the β function, and breaking of scale invariance

If we take Eq (2.1.37) and go back to 4D spacetime by $\varepsilon \rightarrow 0$, the following equation will hold [4] (p351-352):

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = \frac{-\beta_0 g^3}{16\pi^2} \quad (2.1.38)$$

With $\beta_0 = 11 - \frac{2n_f}{3}$, and we neglected terms of order g^5 . Solving this equation will determine the μ -dependence of the renormalised coupling. Notice that β_0 will be positive if n_f is less than 17. This means that g will decrease if μ increases.

If we substitute the strong coupling strength Eq (2.1.29):

$$\alpha_s(\mu) = \frac{g^2}{4\pi} \quad (2.1.29)$$

Then for $\beta(g)$ we get

$$\beta(g) = \mu \sqrt{\frac{\pi}{\alpha_s(\mu)}} \frac{\partial \alpha_s(\mu)}{\partial \mu} = \frac{-\beta_0}{2\sqrt{\pi}} (\alpha_s(\mu))^{3/2} \quad (2.1.39)$$

This leads to

$$\frac{d\alpha_s(\mu)}{\alpha_s^2(\mu)} = \frac{-\beta_0}{2\pi\mu} d\mu \quad (2.1.40)$$

And integrating this will yield

$$\frac{1}{\alpha_s(\mu)} = \frac{\beta_0}{2\pi} \ln(\mu/\Lambda) \quad (2.1.41)$$

With Λ a scale parameter. Eq (2.1.41) results in an expression for the coupling strength in function of μ

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln(\mu/\Lambda)} = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} \quad (2.1.42)$$

Where the last step is done because that is the form that is usually used. Λ is a scale parameter that characterises the scale $\mu \approx \Lambda$ at which $\alpha_s(\mu)$ becomes large, for decreasing μ , [4] (p352). Λ is a physical scale which must be determined from experiment.

There is also a different way we can get an expression for $\alpha_s(\mu)$. From Eq (2.4.24) We can also get [4] (p351-352):

$$\mu \frac{\partial (\alpha_s(\mu))^{-1}}{\partial \mu} = \frac{\beta_0}{2\pi} \quad (2.1.43)$$

This results in the following expression, [4] (p351-352)

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + (\beta_0/4\pi)\alpha_s(\mu_0)\ln(\mu^2/\mu_0^2)} \quad (2.1.44)$$

μ_0 is another scale parameter. But while Λ is a physical scale at which the coupling becomes strong, μ_0 is an arbitrarily chosen reference scale. Notice how all α_s that appear in the right hand side of Eq (2.1.44) have μ_0 as their argument. We could first measure α_s at a reference scale μ_0 , and then use Eq (2.1.44) to calculate the value of α_s at different scale μ . So while in Eq (2.1.42) Λ was the experimentally measured parameter, in Eq (2.1.44) it is $\alpha_s(\mu_0)$ that is the experimentally measured parameter. Per convention the Z boson mass is usually chosen for μ_0 , which is around 91 GeV. [4] (p351-352), [11]

Notice that we can approximate Eq (2.1.44) as

$$\alpha_s(\mu) = \alpha_s(\mu_0) + \mathcal{O}(\alpha_s^2(\mu_0)) \quad (2.1.45)$$

If we would want to know what the coupling strength would be at the a varying momentum scale $\mu^2 = q^2$, with q^2 the invariant mass of the considered process, we can take make a substitution in Eq (2.1.44):

$$\alpha_s(q) = \frac{\alpha_s(\mu_0)}{1 + (\beta_0/4\pi)\alpha_s(\mu_0)\ln(q^2/\mu_0^2)} \quad (2.1.46)$$

2.1.3 Renormalisation schemes

We originally started with a Lagrangian with scale invariant coupling. When we calculate loops, we will encounter UV divergences, and we perform regularisation and renormalisation to deal with this. Regularisation will introduce a dimensionfull mass scale like μ or Λ , either as a cut-off frequency in the integrals, or as a way to keep the correct mass dimension (in dimensional regularisation). When we renormalise, the renormalised parameters will be μ dependent. However, the dependence on μ will disappear from physical dimensionless observables, despite that scale invariance will still be broken because the coupling depends on the energy scale μ . Our renormalised theory cannot give an absolute normalisation of the coupling strength and other Lagrangian parameters, but can predict the rate of change with respect to the energy scale μ the process occurs at.

The exact expressions of the renormalisation constants and counter terms is not unique, but depends on a choice. This choice will affect how and where exactly the infinities go in the bare parameters and renormalisation constants. A specific choise is called a renormalisation scheme. For example the $2/\varepsilon - \gamma_E + \ln(4\pi)$ that appear in the renormalisation constants in section 2.1.1 is such a choice.

In the expressions of running α_s in section 2.1.2, we introduced new scales. Here is also a choice we can make: which reference scale do we use for the running coupling to describe its dependence on energy scale μ ? We also saw that there needs to be input from experiment. We either take the reference scale as a physical scale Λ , which must be determined from experiment; or we take the reference scale as an auxillary scale μ_0 and experimentally determine α_s at a specific choice of μ_0 .

To sum up the different types of choices we make:

- Regularisation:
 - The most common choices are momentum cut-off and dimensional regularisation.
 - Momentum cut-off in divergent integrals: introduces a physical cut-off scale to make integrals finite. We lose Poincaré invariance.
 - Dimensional regularisation: go to a different spacetime dimensionality to solve integrals and take the limit. Introduces a mass scale and typically $1/\varepsilon$ poles. We do keep Poincaré invariance.

- Renormalisation scheme:
 - The exact choice of renormalisation constants and bare parameters.
 - Regularisation is not the same as renormalisation, but regularisation can be seen as a necessary part of the renormalisation scheme.
 - Renormalisation scheme will introduce a renormalisation scale, which is an unphysical scale parameter (has no effect on physical dimensionless observables).
 - The measured values of renormalised parameters are independent of renormalisation scheme, but the way the renormalised parameters depend on the bare parameters (which are unobservable) will depend on renormalisation scheme.
- Running coupling as a function of scale:
 - Introduces a reference scale.
 - Most common choices are to either take the reference scale as a physical scale determined from experiment, or take it as an arbitrary scale at which α_s is determined from experiment.
 - Either way, in order to calculate $\alpha_s(\mu)$ at some scale μ requires an experimentally determined parameter in the equation for $\alpha_s(\mu)$.
 - The measured values of observables are (theoretically), independent of the choice of reference scale. (However, the accuracy in calculations and measurements can differ, related to the accuracy of the measurement of the experimentally determined parameter that fixes the equation for $\alpha_s(\mu)$)

An example of a renormalisation scheme is the $\overline{\text{MS}}$ scheme. In this scheme, the counter terms will only contain the divergent part coming from divergent loop calculations. A different but similar scheme is the $\overline{\text{MS}}$ scheme, in which the counter terms not only contain the divergent part, but an universal constant is also added to it. In this thesis we will use the $\overline{\text{MS}}$ renormalisation scheme and dimensional regularisation. The exact form of this divergent part and additional constant is, up to an overall factor, equal to $2/\epsilon - \gamma + \ln(4\pi)$, which can be seen in the renormalisation constants Eq (2.1. 21), Eq (2.1.30), Eq (2.1.32), Eq (2.1.35).

2.1.4 Renormalisation group

The previous effects and structure of the theory can be expressed by what we call the “renormalisation group” (RG). This is the group related to scale transformations for the renormalised parameters. The main idea is that renormalised parameters are dependent on renormalisation scale μ , while physical dimensionless observables are independent of μ . The renormalisation group will allow us to characterise the scale dependence of the renormalised parameters and, given a measurement at a reference scale, allow us to calculate them at another scale. We will now give a more

formal definition of the β function by this renormalisation group. We will also redefine β_0 to get rid of the appearance of the 4π in the equations for $\alpha(\mu)$ that we saw in section 2.1.2.

We will take for $\alpha(\mu)$

$$\alpha(\mu) = \alpha_0 \left(1 + \alpha_0 \beta_0 \ln \left(\frac{\sigma^2}{\mu^2} \right) + \mathcal{O}(\alpha^2) \right) \quad (2.1.50)$$

In the perturbative region, α 's evaluated at different scales differ by powers of α itself. This means we can write

$$\alpha_0 = \alpha(\mu) \left(1 + \mathcal{O}(\alpha(\mu)) \right) \quad (2.1.51)$$

We will now compute $\frac{d\alpha}{d(\ln \mu^2)}$

α_0 is independent of μ up to higher order in α itself. This means that when we take the derivative we can treat α_0 as if it would be independent of μ . This leads to

$$\frac{d\alpha}{d(\ln \mu^2)} = -\alpha_0^2 \beta_0 + \mathcal{O}(\alpha^2) = -\alpha^2(\mu) \beta_0 + \mathcal{O}(\alpha^2) \quad (2.1.52)$$

For the last equality we have used

$$\alpha_0^2 = \alpha^2(\mu) \left(1 + \mathcal{O}(\alpha(\mu)) \right)^2 = \alpha^2(\mu) \left(1 + \mathcal{O}(\alpha(\mu)) \right) \quad (2.1.53)$$

Note that after taking the derivative with respect to $\ln(\mu^2)$, nor σ nor α_0 appear in the equation anymore.

We will generalise this to

$$\frac{d\alpha}{d(\ln \mu^2)} = \beta(\alpha(\mu)) \quad (2.1.54)$$

And β has an expansion in powers of α :

$$\beta(\alpha(\mu)) = -\beta_0 \alpha^2(\mu) - \beta_1 \alpha^3(\mu) - \beta_2 \alpha^4(\mu) + \dots \quad (2.1.55)$$

By computing the β function, we can predict the μ dependence of the coupling from theory.

We could generalise this even further by $G_0(p_i, \alpha_0) = ZG(p_i, \alpha, \mu)$, with G_0 the bare Green function, G the renormalised Green function, Z the renormalisation constant, and p_i the physical momenta. In this case, the following differential equation will hold

$$\frac{dG_0}{d(\ln \mu^2)} = \frac{d(ZG)}{d(\ln \mu^2)} = 0 \quad (2.1.56)$$

By using the product rule, and taking into account that $G(p_i, \alpha, \mu)$ dependence on μ is also via its dependence on $\alpha(\mu)$, we can reach the following equation:

$$Z \frac{\partial G}{\partial(\ln \mu^2)} + Z \frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial(\ln \mu^2)} + \frac{\partial Z}{\partial(\ln \mu^2)} G = 0 \quad (2.1.57)$$

After dividing everything by Z , we get

$$\frac{\partial G}{\partial(\ln \mu^2)} + \frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial(\ln \mu^2)} + \frac{\partial(\ln Z)}{\partial(\ln \mu^2)} G = 0 \quad (2.1.58)$$

This is what we call the Callan-Symanzik equation:

$$\left(\frac{\partial}{\partial(\ln \mu^2)} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma(\alpha) \right) G(p_i, \alpha, \mu) = 0 \quad (2.1.59)$$

Where we defined the functions β and γ as

$$\beta(\alpha) = \frac{\partial \alpha}{\partial(\ln \mu^2)} \quad (2.1.60)$$

$$\gamma(\alpha) = \frac{\partial(\ln Z)}{\partial(\ln \mu^2)} \quad (2.1.61)$$

2.1.5 Solutions of RG equation

If we solve the RG equation by using only the leading order (LO) for the beta function, Eq (2.1.55), we have to solve

$$\frac{d\alpha}{d(\ln(\mu^2))} = \beta(\alpha(\mu)) \approx -\beta_0 \alpha^2(\mu) \quad (2.1.62)$$

Solving this for $\alpha(\mu)$ leads to

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \beta_0 \alpha(\mu_0) \ln(\mu^2/\mu_0^2)} \quad (2.1.63)$$

For the abelian case (QED) we will have:

$$\beta_0 = -\frac{1}{3\pi} \quad (2.1.64)$$

And the 1-loop QED running coupling is equal to

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 - \frac{\alpha(\mu_0)}{3\pi} \ln(\mu^2/\mu_0^2)} \quad (2.1.65)$$

For the non-abelian case we will have:

$$\beta_0 = \frac{1}{12\pi}(11N_C - 4n_f T_F) \quad (2.1.66)$$

With N_C the number of colours (SU(N_C)), n_f the number of quark flavours, and T_F the trace invariant. If we have $N_C = 3$ and $T_F = 1/2$, then β_0 will be positive for $n_f \leq 16$, which is the case for QCD because as far as we know there are only 6 quark flavours. The running coupling is equal to

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{12\pi}(11N_C - 4n_f T_F)\ln(\mu^2/\mu_0^2)} \quad (2.1.67)$$

The main difference between QED and QCD for the running coupling is the sign of the μ -dependent term in the denominator. This difference will lead to two very different behaviours.

Consider a general theory with a coupling of the form Eq (2.1.63). The behaviour can differ depending on the sign of β_0

A theory with $\beta_0 < 0$ (like QED) will have its coupling become larger for larger μ . In other words, the interaction grows stronger with increasing energy scale.

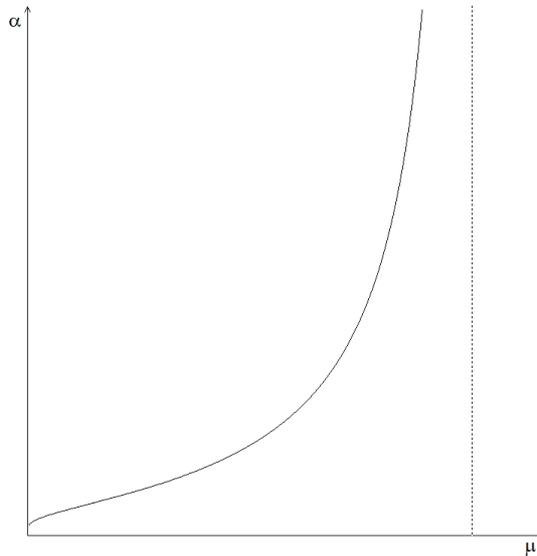


Figure 2.1: Dependence of coupling α on energy μ for a negative β_0 (Like QED) in leading order.

An increasing coupling with increasing energy means that the coupling (and thus the effective charge) will decrease with increasing distance. This effect is called screening of the effective charge.

A theory with $\beta_0 > 0$ will have its coupling become smaller for larger μ . In other words, the interaction becomes weaker with increasing energy scale.

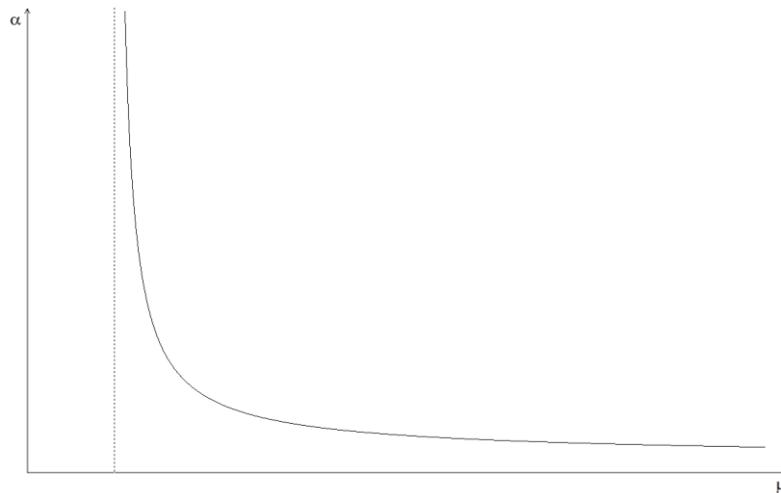


Figure 2.2: Dependence of coupling α on energy μ for a positive β_0 (Like QCD) in leading order.

A decreasing coupling with increasing energy means that the coupling will increase with increasing distance. This effect is called anti-screening of the effective coupling (or effective charge).

There is also a more qualitative way to explain screening and anti-screening via a charge and virtual pairs. For QED consider a charge in an otherwise vacuum. Due to virtual electron-positron pair creation, the vacuum will have a dielectric property that causes the effective charge to decrease at large distance. In other words, the electron-positron pairs will screen the charge. In QCD the same happens with virtual quark-antiquark pairs, they will screen a colour charge. But there are also virtual gluon pairs, which will contribute to an increasing effective charge at large distance. In other words, the gluon pairs will anti-screen the charge. Generally for a non-abelian gauge theory, the overall effect will depend on the number of fundamental colours and the number of quark flavours. For QCD the anti-screening effect of the gluon pairs wins from the screening effect of the quark-antiquark pairs.

2.1.6 Landau pole and QCD scale

In case of QCD, β_0 will be larger than 0. This means that QCD is an interaction that becomes weaker for higher energy scales. In fact, the coupling of QCD becomes so small at high enough energy scales, that quarks and gluons can be considered free particles in the limit of $\mu \rightarrow \infty$. This phenomenon is “asymptotic freedom”. For very low μ on the other hand, the coupling will diverge. Since we have been doing perturbation theory, this just means that for low μ our QCD theory becomes non-

perturbative. In other words, our perturbation theory for QCD does not apply for low energy scales. Our theory exhibits what is called a Landau pole. We have Eq (2.1.63):

$$\alpha_s(q) = \frac{\alpha_s(\mu_0)}{1 + \beta_0 \alpha_s(\mu_0) \ln(q^2/\mu_0^2)} \quad (2.1.68)$$

We see that this singularity happens at a scale $q = \Lambda_{QCD}$ when

$$1 + \beta_0 \alpha_s(\mu_0) \ln(\Lambda_{QCD}^2/\mu_0^2) = 0 \quad (2.1.69)$$

This means our Landau pole is at

$$\Lambda_{QCD}^2 = \mu_0^2 e^{-\frac{1}{\beta_0 \alpha_s(\mu_0)}} \quad (2.1.70)$$

Λ_{QCD} is invariant under what we call a renormalisation group transformation (RG transformation).

$$\mu_0 \rightarrow \mu' = \mu_0 e^{t/2} \quad (2.1.71)$$

With t real and arbitrary.

Then α_s will transform as

$$\alpha_s(\mu_0) \rightarrow \alpha_s(\mu') = \frac{\alpha_s(\mu_0)}{1 + \beta_0 \alpha_s(\mu_0) \ln(\mu'^2/\mu_0^2)} \quad (2.1.72)$$

For Λ_{QCD} this means that

$$\Lambda_{QCD}^2 \rightarrow \mu'^2 e^{-\frac{1}{\beta_0 \alpha_s(\mu'^2)}} = \mu_0^2 e^t e^{-\frac{1 + \beta_0 \alpha_s(\mu_0) \ln(e^{t/2})}{\beta_0 \alpha_s(\mu_0)}} = \mu_0^2 e^{-\frac{1}{\beta_0 \alpha_s(\mu_0)}} \quad (2.1.73)$$

So Λ_{QCD} stays the same under a RG transformation and is a physical energy scale independent from μ_0 . It is an intrinsic parameter of QCD which encodes the physics of scale invariance breaking. We started with a Lagrangian that has a scale invariant coupling, but by analysing higher order corrections we encountered divergences. After renormalisation we've gotten rid of the divergences, but we get a dimensionfull parameter Λ_{QCD} and a scale dependent coupling, and thus scale invariance is broken.

We can relate $\alpha_s(q)$ to Λ_{QCD} by substituting the μ_0^2 in the logarithm:

$$\alpha_s(q) = \frac{1}{\beta_0 \ln(q^2/\Lambda_{QCD}^2)} \quad (2.1.74)$$

Since we can relate $\alpha_s(\mu_0)$ to Λ_{QCD} , we can find Λ_{QCD} from experiment by measuring the coupling. This Λ_{QCD} scale is around 200 MeV/c², Ref [5] (p552), [9] (p34). QED also has a Landau pole, but at a very high energy, even far beyond the Plank scale.

For QCD we can apply perturbation theory if the energy scale is well above Λ_{QCD} , while for QED we can apply perturbation theory if the energy scale is well below Λ_{QED} .

By using lattice QCD, which is a technique that describes QCD on a lattice, we can make calculations in the energy range where perturbation theory doesn't apply anymore. This seems to point to the Landau pole not being a true Landau pole at all, or in other words that the coupling doesn't become infinite. In QED however, the Landau pole seems to be a true Landau pole. But since the QED Landau pole is much higher than even the Planck scale, we expect quantum gravity to be relevant at that energy scale. So it is likely that we shouldn't use QED at that energy scale, and no coupling becomes infinite.

2.1.7 Confinement

If we look at the coupling strength and the QCD scale in the position picture, we get that the coupling strength increases for increasing distance. The QCD scale can be expressed as an inverse distance, $\Lambda_{\text{QCD}} \approx 200\text{MeV} \approx 1\text{ fm}^{-1}$. This means that $1/\Lambda_{\text{QCD}}$ is a distance at which the coupling becomes very large and perturbation theory stops being valid, while we can use perturbation theory below this distance. Consider a quark and an antiquark with opposite colour charge. With lattice QCD it is possible to calculate the static quark-antiquark potential, [9] (p49). For small distances the potential is approximately of the form

$$V(r) = V_0 - \frac{\alpha_s(ar)}{r} \quad (2.1.75)$$

With a the lattice spacing and r the distance in lattice units. For large distances the potential is approximately of the form

$$V(r) = V_0 + Kr \quad (2.1.76)$$

With K a constant known as the string tension. Since K is positive, we have at large distances a linearly increasing potential. This means that for large distances the force between a quark and antiquark will be approximately constant.

If we try to separate the pair, a gluon fluxtube will form and stretch between them. It is believed that at some point, when the separation is roughly $1/\Lambda_{\text{QCD}} \approx 1\text{ fm}$, it will be more favourable to break up the flux tube and create a new quark-antiquark pair, instead of continuing to extend the gluon flux tube. This due to the linear potential at large distance between the pair. After the new quark-antiquark pair is created, the new quark will be close to the old antiquark, and the new antiquark will be close to the old quark, each with opposite colour charge. The same would happen if we would start with 3 quarks with each a different colour, and then try to separate one of the quarks from the rest. A similar thing would happen if we would try to separate a gluon

from other quarks and gluons, but by creating a new gluon pair instead of a quark-antiquark pair.

The result of this would be that quarks and gluons only appear in colour neutral bound states, which we call hadrons. This effect is called colour confinement. The most simple hadrons are mesons (valence quark-antiquark pair), baryons (3 valence quarks), antibaryons (3 valence antiquarks). More exotic hadrons that contain more than 3 valence quarks and antiquarks, and glueballs which contain only gluons but no valence quarks, could also be possible. Experimentally we indeed only directly observe hadrons, not the quarks and gluons themselves, although we do “observe” quarks and gluons in indirect ways. It are the hadrons that are detected by our detectors, not single quarks or gluons.

Using this explanation, the size of a hadron would be roughly 1 fm. From experiment we find that this is indeed about the size of the light hadrons. By using lattice QCD to study the structure of hadrons, the gluon flux tubes inside the hadrons can be found and studied, and the mass and several other properties of the hadrons can be predicted. Important to note is that while colour confinement is observed in nature, and the story above might sound satisfying, there hasn't been an analytic proof yet from the theory.

While quarks and gluons are confined into hadrons, two hadrons can still interact with each other via the strong interaction, even at energies (distances) beyond the QCD scale. The lower energy strong interaction between hadrons is responsible for binding protons and neutrons together into nuclei. The strong interaction between hadrons in this regime cannot be described by QCD currently. This is merely due to the inability to apply QCD outside the perturbative regime. However, there are models outside of QCD that try to describe strong interactions between hadrons in nuclear physics.

2.2 Parton model

In this section we will discuss the parton model and hard scattering processes. Hard scattering processes are scattering processes via the Strong Interactions that involve momentum transfers that are large compared to the QCD scale $\Lambda_{\text{QCD}} \approx 200 \text{ MeV}$. By this convention, processes that only involve momentum transfers that are small compared to Λ_{QCD} can be called soft processes. The parton model was the model proposed by Feynman to explain high energy scattering experiments involving hadrons. The model came at a time when the exact nature of the Strong Interaction was not yet fully known or verified, nor were any asymptotically free quantum field theories in 4D known yet at that time. Even though the original model is incomplete, it is still very usefull for high energy scattering. When the quark model, asymptotic freedom and Bjorken scaling were verified/observed, partons could be understood as

quarks and gluons, and QCD became the theory of the Strong Interaction. Now the parton model can be understood and modified from QCD. In this section we will already assume and apply QCD for the parton model.

In this section we will also give some important definitions and theorems that are often used in high energy scattering, and that we will use further on, like Mandelstam variables, cross section and the optical theorem.

2.2.1 Overview parton model

From asymptotic freedom of QCD we know that for scattering processes with momentum transfer $Q \gg \Lambda_{\text{QCD}}$, our coupling strength will be $\alpha_s(Q) \ll 1$, and quarks and gluons will be asymptotically free particles, or at most “weakly” interacting particles. (To avoid confusion: With “weakly” interacting we mean here that the Strong Interaction its coupling strength is very small in this region, and thus the colour carrying particles will almost not feel forces that are governed by Strong Interactions. We don’t mean “the” Weak Interactions, which is an entirely different force.) This means that in hard scattering processes we can think of hadrons as being made out of “weakly” interacting partons. (“Parton” is the collective name of particles that are constituents of hadrons, or in other words, quarks and gluons). In this region, perturbation methods apply to QCD. While the partons inside a hadron can be considered “weakly” interacting in this regime, they will still be confined to colour neutral hadrons. This hints that we would have to treat dynamics at short distances differently from those at long distances. There could also be other interactions involved in hard scattering processes, like QED or Weak Interactions.

Another important thing to note is that theoretical predictions are in terms of coloured fields, while experimental measurements are in terms of colourless hadrons. We need principles or tricks in order to make the connection between theory and experiment.

Because of these reasons above, we will separate dynamics at short distances and dynamics at long distances. The exact way this will be done, depends on the scattering process under consideration. Some examples are “infrared safety” and “factorisation”. In this thesis we will discuss factorisation, which is important for the splitting functions.

2.2.2 Mandelstam variables

When we are dealing with scattering processes, it is useful to use Mandelstam variables. Consider two ingoing and two outgoing external particles:

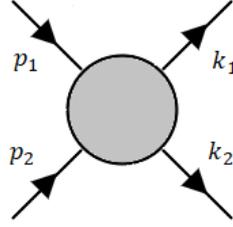


Figure 2.3: Two incoming particles with momenta p_1 and p_2 interact and result in two outgoing particles with momenta k_1 and k_2 .

We define the Mandelstam variables as

$$\begin{aligned}
 \hat{s} &= (p_1 + p_2)^2 = (k_1 + k_2)^2 \\
 \hat{t} &= (k_1 - p_1)^2 = (k_2 - p_2)^2 \\
 \hat{u} &= (k_2 - p_1)^2 = (k_1 - p_2)^2
 \end{aligned}
 \tag{2.2.1}$$

By this definition, \hat{s} is the square of the center-of-mass (com) energy, and is also known as the invariant mass. \hat{t} is the square of the four-momentum transfer. If we sum the three variables we get the sum of the square masses of the particles:

$$\hat{s} + \hat{t} + \hat{u} = \sum_{i=1}^4 m_i^2
 \tag{2.2.2}$$

In case of 4 massless particles, we would get:

$$\begin{aligned}
 \hat{s} &= 2p_1 \cdot p_2 = 2k_1 \cdot k_2 \\
 \hat{t} &= -2p_1 \cdot k_1 = -2p_2 \cdot k_2
 \end{aligned}
 \tag{2.2.3}$$

$$\begin{aligned}
 \hat{u} &= -2p_1 \cdot k_2 = -2p_2 \cdot k_1 \\
 \hat{s} + \hat{t} + \hat{u} &= 0
 \end{aligned}
 \tag{2.2.4}$$

This is also approximately true in the ultrarelativistic limit (or massless limit) for massive particles, since that means setting $p_1^2 = m_1^2 \approx 0$, $p_2^2 = m_2^2 \approx 0$, etc.

If the scattering process only contains one internal line, then there are only 3 possible diagrams for Fig. 2.3. We name these after the 3 Mandelstam variables:

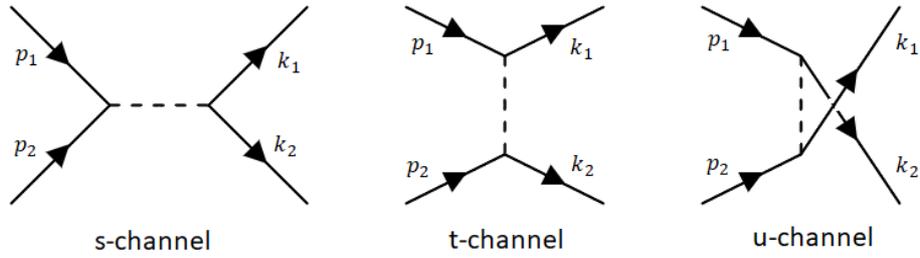


Figure 2.4: The 3 basic “channels” for 2-body to 2-body scattering.

Mandelstam variables are useful for expressing amplitudes and cross sections for 2-body to 2-body scattering. An example is $e^+e^- \rightarrow \mu^+\mu^-$. At 2nd order in QED we only have the s-channel:

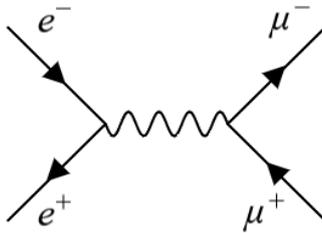


Figure 2.5: $e^+e^- \rightarrow \mu^+\mu^-$ in QED.

If we work in the massless limit, the spin-summed square Feynman amplitude will be [5] (p156-157)

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2e^4}{\hat{s}^2} (\hat{t}^2 + \hat{u}^2) \quad (2.2.5)$$

Another example is $e^-\mu^- \rightarrow e^-\mu^-$, which is just Fig. 2.5 turned on its side and the Mandelstam variables renamed. So at 2nd order in QED we only have the t-channel:

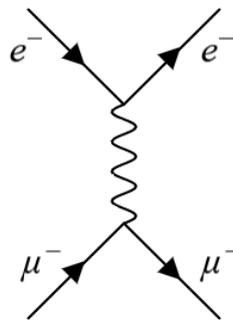


Figure 2.6: $e^-\mu^- \rightarrow e^-\mu^-$ in QED.

If we work in the massless limit, the spin-summed square Feynman amplitude will be [5] (p156-157)

$$\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = \frac{2e^4}{\hat{t}^2} (\hat{s}^2 + \hat{u}^2) \quad (2.2.6)$$

2.2.3 Cross section

In particle colliders we typically collide 2 beams (with well defined momenta), or a beam with a target, and observe what comes out. We will define the cross section as a quantity that characterises the likelihood that a final state particle is different from the initial state particle it originated from, or in other words the likelihood that scattering took place. We will also define it in such a way that it allows us to compare different experiments with different beam sizes and intensities. We will assume that the particle density is uniform inside the beam. This will be a good assumption if we assume that the range of the interaction between particles and the width of the particle wavepackets are both small compared to the beam width [5] (p100). Consider a target at rest of particle type 1, length l_1 and particle density ρ_1 . At this target we aim a beam of particle type 2, length l_2 and particle density ρ_2 .

We expect the number of scattering events $n_{scatter}$ to be proportional to the lengths, densities, and the cross-sectional area A between the two bundles of particles. We define the cross section σ as the proportionality factor

$$\sigma = \frac{n_{scatter}}{l_1 \rho_1 l_2 \rho_2 A} \quad (2.2.7)$$

We could have also taken the bundle of particle type 2 at rest, or worked in a frame where both bundles are moving, it wouldn't change this definition of cross section. If N_1 and N_2 are the total numbers of particles of particle type 1 and 2, then we can also express the cross section as

$$\sigma = \frac{n_{scatter} A}{N_1 N_2} \quad (2.2.8)$$

For a real particle collider, the densities aren't uniform in the beams and are typically higher near the center. In that case we would have to integrate the density over the beam area (given that the density is uniform along the length of the beam).

$$n_{scatter} = \sigma l_1 l_2 \int d^2x \rho_1(x) \rho_2(x) \quad (2.2.9)$$

In the expressions above we didn't take the momenta of the outgoing particles into account. If we do take this into account then we define the differential cross section $d\sigma/(d^3p_1 \dots d^3p_n)$. In case we are left with 2 final state particles, we can due to

momentum conservation just use 2 angles, and the differential cross section will be $d\sigma/d\Omega$ with $d\Omega$ the solid angle.

The cross section is an important quantity that relates theory to experiment. It can be measured via measuring the number of scattering events, and can be calculated from theory via the invariant matrix \mathcal{M} , [5] (p106-107):

$$d\sigma = \frac{1}{4E_1E_2|v_1 - v_2|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f} \right) (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum p_f) |\mathcal{M}(p_1 p_2 \rightarrow \{p_f\})|^2 \quad (2.2.10)$$

Where $|v_1 - v_2|$ is the relative velocity between the bundles of particles, and f runs over the final state particles. $|\mathcal{M}|^2$ contains all the dynamics, while the other factors contain the kinematics.

In case we only have 2 final state particles, we can express the differential cross section in the center of mass frame (COM) as [5] (p106-107):

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{COM}} = \frac{1}{4E_1E_2|v_1 - v_2|} \frac{|\mathbf{k}_1|}{(2\pi)^2 4\sqrt{s}} |\mathcal{M}(p_1 p_2 \rightarrow k_1 k_2)|^2 \quad (2.2.11)$$

In the case were all four particles have the same mass, this becomes

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{COM}} = \frac{|\mathcal{M}(p_1 p_2 \rightarrow k_1 k_2)|^2}{64\pi^2 \hat{s}} \quad (2.2.12)$$

2.2.4 Optical theorem

Generally when 2 particles scatter, there are many possible finite states, including final states with more than 2 outgoing particles.

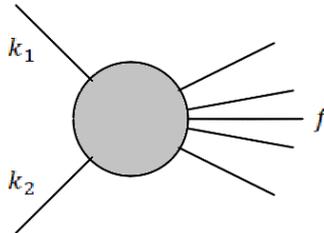


Figure 2.7: two particles with momenta k_1 and k_2 scatter to a final state f .

Recall that the probability amplitude is given by the S-matrix $S_{fi} = \langle f|S|i\rangle$, and that we have conservation of probability:

$$\sum_f |S_{fi}|^2 = 1 \quad (2.2.13)$$

Which means the S-matrix is a unitary matrix:

$$S^\dagger S = \mathbf{1} \quad (2.2.14)$$

If we substitute the T-matrix, $S = \mathbf{1} + iT$, then we get

$$-i(T - T^\dagger) = T^\dagger T \quad (2.2.15)$$

If we express this by using indices for states, like T_{fi} , we get

$$-i(T_{ba} - T_{ab}^*) = \sum_f T_{bf} T_{af}^* \quad (2.2.16)$$

We will now take the forward amplitude, which means setting the final state equal to the initial state, and thus $b = a$

$$-i(T_{aa} - T_{aa}^*) = \sum_f T_{af} T_{af}^* = \sum_f |T_{af}|^2 \quad (2.2.17)$$

Note that we can write the left-hand side as an imaginary part:

$$\text{Im}(T_{aa}) = -\frac{i}{2}(T_{aa} - T_{aa}^*) \quad (2.2.18)$$

And so we finally have

$$\text{Im}(T_{aa}) = \frac{1}{2} \sum_f |T_{af}|^2 \quad (2.2.19)$$

By using Eq (1.3.34) for $\langle p_1 p_2 | iT | k_1 k_2 \rangle$ we can also get an equation for the Feynman amplitude \mathcal{M} , [5] (p231)

$$-i(\mathcal{M}(a \rightarrow b) - \mathcal{M}^*(b \rightarrow a)) = \sum_f \int d\Pi_f \mathcal{M}^*(b \rightarrow f) \mathcal{M}(a \rightarrow f) \quad (2.2.20)$$

Where $\sum_f \int d\Pi_f$ is a sum over the sets of final state particles and their momenta:

$$\sum_f \int d\Pi_f A = \sum_n \left(\prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3} \frac{A}{2E_i} \right) \quad (2.2.21)$$

By applying forward scattering $b = a$, we get

$$2\text{Im} \mathcal{M}(a \rightarrow a) = \sum_f \int d\Pi_f \mathcal{M}^*(a \rightarrow f) \mathcal{M}(a \rightarrow f) = \sum_f \int d\Pi_f |\mathcal{M}(a \rightarrow f)|^2 \quad (2.2.22)$$

By using the definition of cross section, we get the following equation [5] (p231):

$$\sigma_{tot}(a \rightarrow \text{anything}) = \frac{1}{2E_{COM}|\mathbf{p}|_{COM}} \text{Im } \mathcal{M}(a \rightarrow a) \quad (2.2.23)$$

Where E_{COM} is the total energy in the center-of-mass frame, and \mathbf{p}_{COM} the momentum of either particle in the center-of-mass frame.

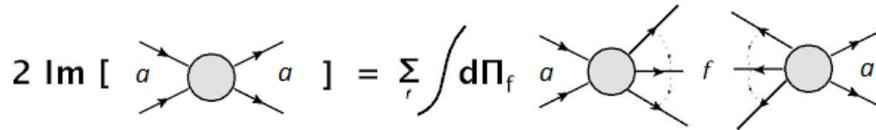


Figure 2.8: The optical theorem represented by Feynman diagrams. Figure from [12]

This is the essence of the Optical Theorem, which states that we can relate the imaginary part of a forward scattering amplitude to a sum over the contributions of all possible intermediate particles. This can be generalised to one-particle states or higher multiple-particle states as well.

This also means that if we have a Feynman diagram that has mirror symmetry, with the mirror axis cutting through internal lines, we could also apply the optical theorem.

By applying perturbation theory on both sides of the equation, we can relate diagrams with the Optical Theorem that are of the same order in the coupling strengths. Even more, with the Optical Theorem we can relate a diagram that has mirror symmetry, with the mirror axis cutting through internal lines, to half the diagram (cut along the mirror axis) [5] (p232-236). As an example we give:

$$2\text{Im} \left(\left(\text{diagram with mirror axis} \right) \right) = \int d\Pi \left| \left(\text{cut diagram} \right) \right|^2$$

Figure 2.9: Example of applying optical theorem on a single diagram, by cutting along the mirror axis.

This results in the Cutkosky cutting rules. These rules and their derivation can be found in the references. [5] (p232-236), [13]

2.2.5 Deep inelastic scattering and structure functions

Deep inelastic scattering (DIS) is scattering between a high energy lepton and a hadron target, in which the hadron absorbs some of the kinetic energy of the lepton, and possibly even breaks up into multiple hadrons. In order to avoid taking Weak Interactions into account, we will only consider charged leptons. We define the momenta as follows:

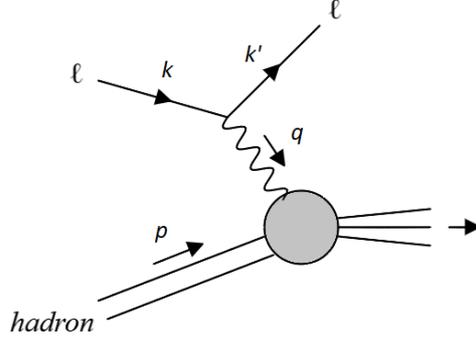


Figure 2.10: Deep inelastic lepton-hadron scattering. After the collision the hadron has typically changed into another hadron or a collection of multiple hadrons.

The hadron has momentum p^μ , the lepton has momentum k^μ and is left with momentum k'^μ after scattering. The momentum transfer q^μ is carried by a virtual photon. We will define a space-like vector Q^μ

$$Q^2 = -q^2 \quad (2.2.24)$$

With $\sqrt{Q^2} \gg \Lambda_{\text{QCD}} \approx 1 \text{ fm}^{-1}$. In Mandelstam variables, Q^2 equals \hat{t} . The mass M of the hadron is given by

$$M^2 = p^2 \quad (2.2.25)$$

We will also define

$$x = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2M(E - E')} \quad (2.2.26)$$

$$y = \frac{p \cdot q}{p \cdot k} = 1 - E'/E \quad (2.2.27)$$

Where E and E' are the energies of the lepton in the restframe of the hadron. x is called the Bjorken scaling variable. y is the relative loss in energy for the lepton. The meaning of x , which is the ratio between the transferred momentum squared and twice the scalar product of hadron momentum and transferred momentum, becomes more intuitive if we go to high momenta. At high momenta, x will approximately be the fraction of the hadron's momentum that the struck parton carries. When $x = 1$ we have no longer inelastic scattering, but elastic scattering instead. If we only consider the situation with $Q^2 \ll M_Z^2$ then the exchange particle will be a photon (and not a Z boson), and the differential scattering cross section will be [9] (p87-88):

$$\frac{d^2\sigma_{l-h}}{dx dy} = \frac{8\pi\alpha^2 ME}{Q^4} \left((1 + (1 - y)^2)x F_1 + (1 - y)(F_2 - 2xF_1) - (M/2E)xy F_2 \right) \quad (2.2.28)$$

With $F_i = F_i(x, Q^2)$ the structure functions for lepton-hadron scattering. The structure functions parametrise the hadron target as "seen" by the virtual photon. We now take

the Bjorken limit, which is defined as the limit for $Q^2 \rightarrow \infty$, $p \cdot q \rightarrow \infty$ and with x fixed. In this limit we observe that the structure functions only depend on the dimensionless variable x .

$$F_i(x, Q^2) \rightarrow F_i(x) \quad (2.2.29)$$

This approximate scaling law is called Bjorken scaling.

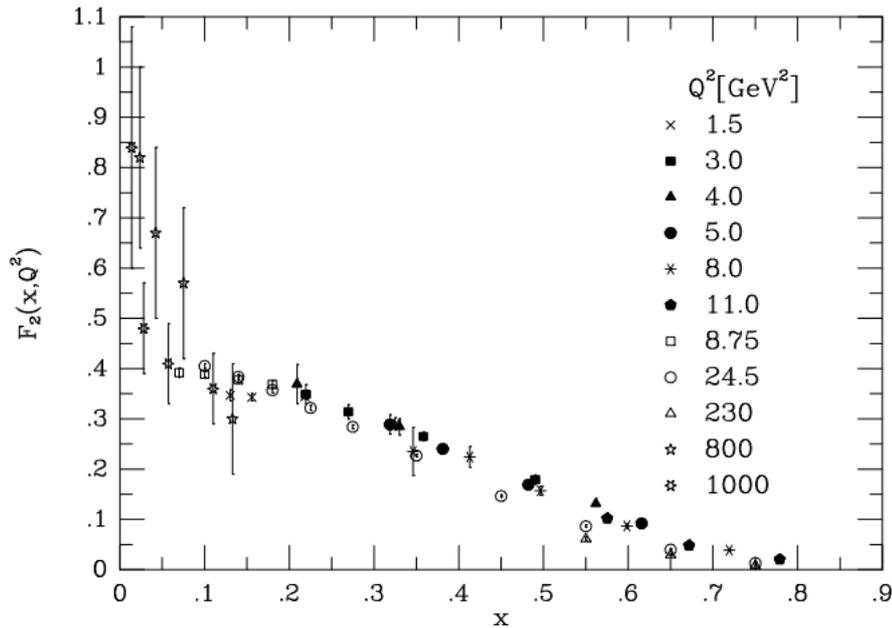


Figure 2.11: Structure function F_2 from the experiment collaborations SLAC-MIT, BCDMS, H1 and ZEUS. Note how in most of the range of x , there is little variation in F_2 for a very large range of Q^2 . [9] (p87-88)

Bjorken scaling means that when a hadron is struck via a virtual exchange photon, and the transferred momentum is very large, that the hadron's structure looks the same no matter how hard it is struck. It also implies that the exchange photon interacts with pointlike constituents, [9] (p87-88), [5] (p478-479). Because if this is not the case, then the structure functions would depend on Q/Q_0 , with Q_0 some scale related to the size of the constituents [9] (p87-88). Bjorken scaling is a remarkable feature in that can be seen in most of the x range and for a very large range of Q^2 , but a small scaling violation can be seen. The naïve parton model predicts exact Bjorken scaling, but QCD gives rise to a scaling violation by logarithms of Q^2 . This scaling violation can be calculated from perturbative QCD.

2.3 Factorisation

In this section we will discuss the factorisation method. Factorising a hard scattering process means we separate the long-time effects (which are non-perturbative for QCD) from the short-time effects (which are perturbative for QCD). If we have a process with hard momentum scales of the order of Q (The “hard scale”) and soft momentum scales of the order of the hadron mass M (of the order of Λ_{QCD}), then we can try to express the cross section in a factorised form

$$\sigma(Q, M) = C(Q, \text{partonmomenta} > \mu) \otimes f(\text{partonmomenta} < \mu, M) \quad (2.3.1)$$

Where C is a factor that only depends on hard momenta, and f is a factor that only depends on soft momenta, and μ is an auxiliary mass scale we have to introduce to define the separation of hard and soft. Here, $C \otimes f$ means a convolution.

If such a formula is valid, then this will be the benefits:

- Q dependence will be computable in perturbation theory.
- Although f is not computable, its dependence on μ will be computable (because it has to cancel against C).
- From the μ dependence we can learn about the Q dependence. (Renormalisation Group)

In this section we will use deep inelastic scattering (DIS) and Drell-Yan as examples for factorisation.

2.3.1 Infinite momentum frame

For hard scattering processes, the relative velocities between incoming initial particles is extremely high. If at least one of the initial particles is a hadron, we could consider the scattering process in an “infinite momentum frame”. This is a reference frame in which an incoming hadron has a momentum p that goes in the limit to infinity. We can justify this if p is much larger than the momentum transfer and hadron mass ($p \gg Q, M$). This means that the other particle, lets say an electron for example (could be a lepton, quark, etc), sees a time-dilated and Lorentz-contracted hadron. Lorentz contraction means that the hadron will look like a thin layer of partons. This means the time it takes for the electron to traverse through the hadron, $\Delta t_{\text{scatter}}$, will be very small. Due to time dilation, the hadrons’ internal interactions will be time-dilated, and thus the typical timescale for interactions between the hadron’s partons τ_{parton} will be large. This means that

$$\Delta t_{\text{scatter}} \ll \tau_{\text{parton}} \quad (2.3.2)$$

In other words, the electron sees a hadronic state with a definite number of partons, each of which has a definite momentum characterised by a fraction ξ of p .

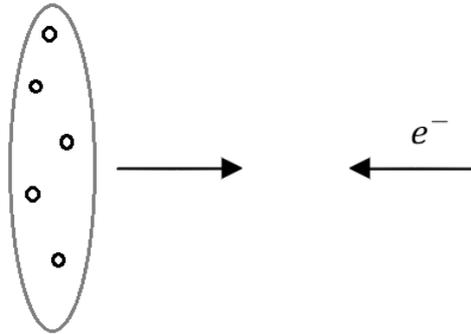


Figure 2.12: Hadron-electron scattering in the infinite momentum frame. The hadron is Lorentz contracted to the point that the electron sees a thin “layer” of distributed partons. The hadron’s internal interactions between partons are time dilated to the point that they take much longer than the time the electron needs to traverse the hadron.

If the virtual exchange particle travels only a small distance ($\Delta l \ll 1$ fm), and the density of the partons is not very high, then the exchange particle will interact with only 1 parton. This would mean the interaction between the exchange particle and a parton will not interfere with interactions among partons that occur at time-dilated scales. However it is important to realise that while this means that short-time dynamics and long-time dynamics do not interfere, the overall scattering process will still be sensitive to long-time interactions. This because the partons have existed in the hadron since a long time before scattering takes place, and thus the parton wavefunctions has had a long time to develop.

2.3.2 Factorisation in deep inelastic scattering

Based on section 2.2.5 we will now factorise deep inelastic lepton-hadron scattering. The typical example for DIS is electron-proton scattering. Most of the equations in this section will apply to other leptons and hadrons as well. We will use the same naming convention as in Fig. 2.10. We will use the “infinite momentum frame”, and we will apply the reasoning of the previous section 2.3.1 to factorise DIS. Note that for $Q \gg \Lambda_{\text{QCD}}$ the virtual photon will be highly off mass shell, and thus it will only travel a small distance Δl_γ

$$\Delta l_\gamma \sim \frac{1}{|Q|} \ll 1 \text{ fm} \quad (2.3.3)$$

So as we said in previous section, if the density of the partons is not very high, then the photon will interact with only 1 parton. This parton will be a quark since photons

don't directly interact with gluons. Since this means that photon-quark interactions don't interfere with parton-parton interactions inside the hadron that occur at time-dilated scales, we can now fully factorise DIS as follows:

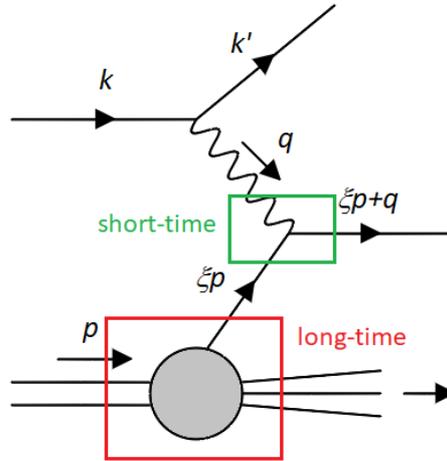


Figure 2.13: Factorised deep inelastic lepton-hadron scattering.

Recall from section 2.2.5 that

$$\begin{aligned}
 Q^2 &= -q^2 \\
 M^2 &= p^2 \\
 x &= \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2M(E - E')} \\
 y &= \frac{p \cdot q}{p \cdot k} = 1 - E'/E
 \end{aligned}
 \tag{2.3.4}$$

We will work in the lepton-hadron center-of-mass frame. We will consider the situation where the momentum of the hadron is very high, and we will consider the hadron to be a collection of loosely bound partons without mass. If this is the case then the hadron's partons will move approximately collinear with the hadron. Each of the hadron's partons will have a momentum that is approximately just a fraction of the hadron's momentum. So we can consider the parton that interacts with the virtual exchange photon to have momentum ξp , with $0 < \xi < 1$, Ref [5] (p477). Note that we can use the Mandelstam variables for the massless limit in this section. Also important to note is that again we will only consider the situation with $Q^2 \ll M_Z^2$, to avoid Weak Interactions. When Q^2 becomes of the order M_Z^2 or above, we would have to take extra contribution into account from the weak bosons. In that case we will also have neutrino-hadron DIS. The factorisation will be the same, and the structure functions will be different but still similar to the $Q^2 \ll M_Z^2$ case.

When we separate the short-time and long-time dynamics, we could calculate the process by combining probabilities rather than amplitudes. For the cross section for lepton-hadron scattering we get [10]

$$\sigma_{l-h}(x, Q) = \sum_i \int_x^1 d\xi \sigma_{l-i}\left(\frac{x}{\xi}, Q, \mu\right) f_{i/h}(\xi, \mu) \left(1 + \mathcal{O}(\Lambda_{\text{QCD}}^2/Q^2)\right) \quad (2.3.5)$$

where σ_{l-i} is the (hard) cross section for scattering of the lepton and quark i , which carries momentum ξp . The function $f_{i/h}$ is the (soft) distribution function of quark i in the hadron. By separating hard and soft contributions, we introduced an auxiliary mass scale μ as a momentum cut-off, in both σ_{l-i} and $f_{i/h}$. If we ignore the μ dependence, we will get in lowest order of perturbation a cross section that exhibits Bjorken scaling in $1/Q^2$, and we get the Bjorken-Feynman parton model (pre-QCD). [10] The QCD features of the equation lie within the μ dependence. Up to any perturbative order, small logarithmic violations of scaling will appear via the μ dependence. [10] Important to note is that the exact form of σ_{l-i} and $f_{i/h}$ in Eq (2.3.5) will depend on the factorisation scheme that is being used.

We can express the DIS cross section in terms of structure functions by applying the optical theorem as follow:

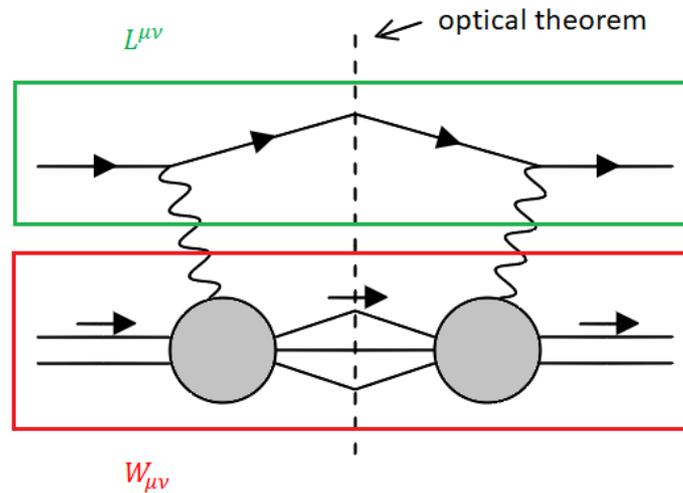


Figure 2.14: Applying optical theorem on DIS, and defining the leptonic and hadronic tensors.

Due to factorisation, and based on Lorentz and gauge invariance, we can write the cross section as [9] (p94)

$$\frac{d\sigma_{l-h}}{dx dy} \propto L^{\mu\nu} W_{\mu\nu} \quad (2.3.6)$$

With $L^{\mu\nu}$ the leptonic tensor and $W_{\mu\nu}$ the hadronic tensor. $L^{\mu\nu}$ is calculable in QED via the following diagram

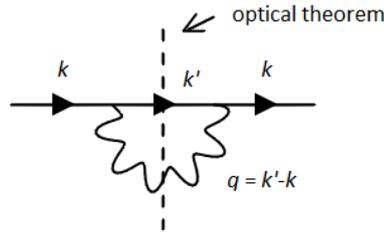


Figure 2.15: Feynman diagram for the leptonic tensor $L^{\mu\nu}$. Note that there is no integral over the loop, since all momenta are fixed.

By applying Feynman rules we get the following expression for the leptonic tensor, [9] (p94-95)

$$L_{\mu\nu} = e^2 \text{TR}(k' \gamma_\mu k \gamma_\nu) = 4e^2 (k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k \cdot k') \quad (2.3.7)$$

$W^{\mu\nu}$ can be expressed via the following diagram

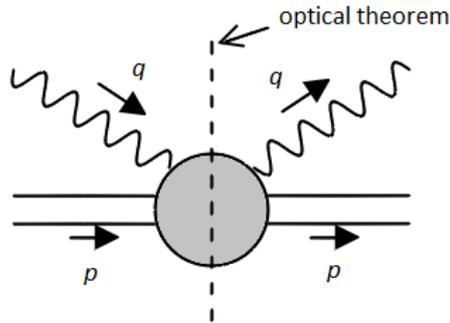


Figure 2.16: Feynman diagram for the hadronic tensor $W^{\mu\nu}$.

From this we get for $W^{\mu\nu}$, [5] (p623-625)

$$W^{\mu\nu} = i \int d^4x e^{iq \cdot x} \langle p | T \{ J^\mu(x) J^\nu(0) \} | p \rangle \quad (2.3.8)$$

With J^μ the quark electromagnetic current. Combining the leptonic and hadronic tensors, and applying the optical theorem gives us this expression for the differential cross section [5] (p623-625):

$$\frac{d\sigma_{l-h}}{dx dy} = \frac{e^2 y}{2(4\pi)^2 Q^4} L_{\mu\nu} \text{Im}(W^{\mu\nu}) \quad (2.3.9)$$

The Ward-Takahashi identity requires that

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0 \quad (2.3.10)$$

There are only 2 possible tensors made from p and q that satisfy this. We can write $W^{\mu\nu}$ as a linear combination of the two, [5] (p623-625):

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right) W_2 \quad (2.3.11)$$

The scalar functions $W_1 = W_1(x, Q^2)$ and $W_2 = W_2(x, Q^2)$ are related to the structure functions $F_1 = F_1(x, Q^2)$ and $F_2 = F_2(x, Q^2)$, Ref [9] (p94-95):

$$\begin{aligned} F_1(x, Q^2) &= W_1(x, Q^2) \\ F_2(x, Q^2) &= p \cdot q W_2(x, Q^2) \end{aligned} \quad (2.3.12)$$

We can substitute this in the equation for $W^{\mu\nu}$:

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \frac{1}{p \cdot q} \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right) F_2 \quad (2.3.13)$$

Substituting $L_{\mu\nu}$ and $W^{\mu\nu}$ in the equation for the differential cross section yields

$$\frac{d\sigma_{l-h}}{dx dy} = \frac{\alpha^2 y}{Q^4} (s^2(1-y)\text{Im}(W_2) + 2xys \text{Im}(W_1)) \quad (2.3.14)$$

With the Mandelstam variable $s = (p+k)^2 = 2p \cdot k$ in the massless limit.

We can also write the differential cross section as [10]

$$\frac{d\sigma_{l-h}}{dx dQ^2} = \frac{4\pi\alpha^2}{xQ^4} \left(\left(1-y + \frac{y^2}{2}\right) F_2(x, Q^2) - \frac{y^2}{2} F_L(x, Q^2) \right) \quad (2.3.15)$$

With $F_2 = F_L + F_T$, where F_L is the longitudinal structure function, and F_T is the transversal structure function. These are structure functions related to the transversely and longitudinally polarised virtual photons. Any of the structure functions F_n themselves obey the factorisation formula, similar to the cross section σ_{l-h} , [10]:

$$F_n(x, Q) = \sum_i \int_x^1 d\xi C_{ni}(x/\xi, \alpha_s(\mu), Q/\mu) f_{i/h}(\xi, \mu) + \mathcal{O}(\Lambda_{\text{QCD}}^2/Q^2) \quad (2.3.16)$$

With $f_{i/h}$ the parton distributions and C_{ni} process-dependent coefficient functions. The C_{ni} are computable in perturbation theory as a series in powers of α_s

$$C_{ni}(x/\xi, \alpha_s, Q/\mu) = \sum_k C_{ni}^{(k)}(x/\xi, Q/\mu) \alpha_s^k \quad (2.3.17)$$

If we set the scale μ at a value on the order of Q , potentially large logarithmic corrections in Q/μ are automatically taken into account, rather than appearing in the expansion of the coefficient functions at any order. [10]

If a quark is spin 1/2 then it cannot absorb a longitudinally polarised photon. On the other hand, if a quark would be spin 0 then it cannot absorb a transversely polarised

photon. For spin 1/2 quarks we have $2xF_1 = F_2$, which is called the Callan-Gross relation. For spin 0 quarks we would have $F_1 = 0$ and $F_2 = F_L$ in the Bjorken limit. (Ref [9], p91) Structure function measurements from experiment show that $F_L \ll F_2$ is true, so this confirms that quarks have spin 1/2. [9] (p91)

2.3.3 Hadron structure and parton distribution functions

Lets think a moment about (charged) lepton-hadron scattering in general. At the lowest energies, a hadron looks like a point particle. For some time in the past, we indeed thought the proton, neutron and other particles were elementary particles, because they act like point particles at the lowest energies. At higher energies however, this is not the case. Starting at the lowest energies, when we increase the center-of-mass energy of the lepton-hadron scattering, we will at some point reach an “intermediate” regime in which we see excited states of the hadron. When we keep increasing the center-of-mass energy of the scattering, we reach the regime in which a hadron looks like the current “typical” view of hadrons, namely a composite particle made out of valence quarks. Mesons are made out of a valence quark and valence antiquark, baryons (for example proton and neutron) are made out of 3 valence quarks, antibaryons are made out of 3 valence antiquarks. “Valence quarks” means that they are the particles providing the hadron’s charge, baryon number, spin, and other quantum numbers. At even higher energies, there will be a point where we start to see even more partons than just the valence quarks. These extra partons are gluons and sea quarks (“sea quarks” are quark-antiquark pairs). The gluons and sea quarks carry some of the hadron’s momentum, but they do not contribute to the hadron’s quantum numbers. If we keep increasing the center-of-mass energy, we will see more and more of these extra partons.

The information of which quarks are in a hadron, and their probabilities, is captured in the parton distribution functions (also called parton densities). The parton distribution function $f_{i/h}$ gives the probability to find a parton of type i in a hadron h , and this as a function of the momentum fraction ξ carried by the parton and the auxiliary scale μ . This means we can probe the inner structure of a hadron via scattering experiments. This isn’t exclusively the case for lepton-hadron scattering, but scattering to a target hadron in general. The parton distribution functions are universal for the same hadron, within the same factorisation scheme. At the energy scales typical for this chapter, we are in the regime where aside for valence quarks we also see sea quarks and gluons. For lepton-hadron scattering at these energy scales, we can include quark flavours as long as their masses are significantly lower than μ . At this point from our theory, we don’t expect to see gluons, because leptons don’t directly couple to gluons.

Lets take for example electron-proton scattering. We will consider Q to be smaller than the mass of the bottom quark, $Q^2 \ll m_b$. The quark species that we will include

will be up, down, strange and charm quarks, and their antiquarks. We expect the largest contribution to be from the up and down quark distribution functions, because those contain the valence quarks as well as sea quarks. The quark distribution functions for the other quark species only contain sea quarks. We expect the distribution functions for the anti-up and anti-down to be (almost) the same, because they have almost the same mass. From experiment however, we see an asymmetry between the anti-up and anti-down. From experiment we can determine that only about half of the proton's momentum is carried by the quarks. The other half is carried by the gluons. [9] (p92)

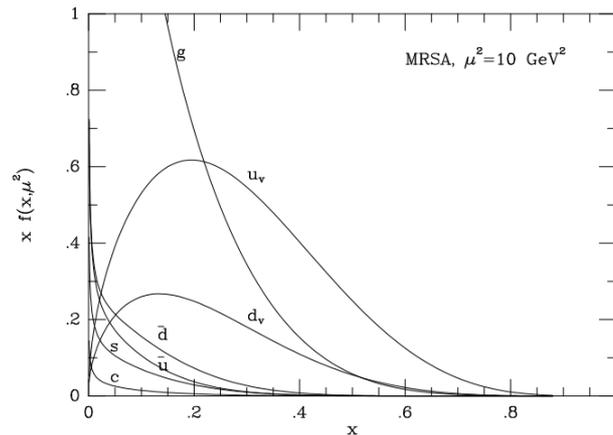


Figure 2.17: Quark and gluon distribution functions of the proton, at scale $\mu^2 = 10 \text{ GeV}^2$, fitted to DIS data. Note that u_v and d_v are the valence up and down quarks. To get the distribution for up and down quarks in general, take the sums $u = u_v + \bar{u}$ and $d = d_v + \bar{d}$. The gluon distribution is provided by DIS data at small x , [14], [9] (p93)

2.3.4 Factorisation in Drell-Yan

We will very briefly give an overview of factorising the Drell-Yan process. The Drell-Yan process is a hadron-hadron scattering process in which a quark from one hadron and an antiquark from the other hadron annihilate each other, creating a virtual photon or Z boson which decays into a lepton pair. We define the momenta as follows:

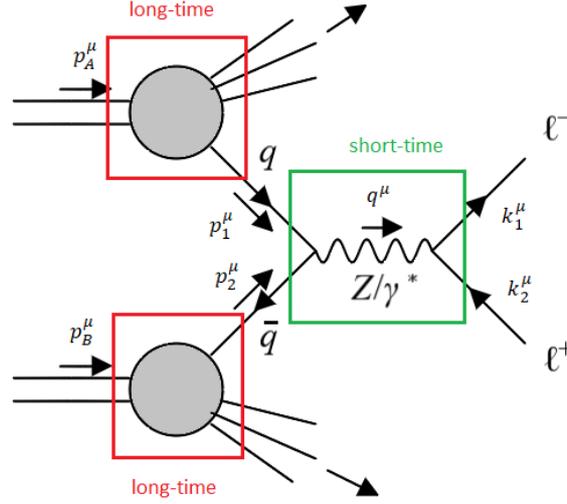


Figure 2.18: Factorised Drell-Yan lepton pair production.

The two hadrons have momenta p_A^μ and p_B^μ , the interacting quark and antiquark have momenta p_1^μ and p_2^μ , and the momentum of the virtual neutral vector boson (photon or Z boson) is q^μ . We define the time-like vector Q^μ

$$Q^2 = q^2 \quad (2.3.18)$$

We also define M as the invariant mass of the lepton pair

$$M^2 = (k_1 + k_2)^2 \quad (2.3.19)$$

By the same reasoning as with DIS, we can consider the quark and antiquark to be moving collinear with their respective hadron, due to the large momenta.

$$\begin{aligned} p_1 &= \xi_1 p_A \\ p_2 &= \xi_2 p_B \end{aligned} \quad (2.3.20)$$

By analogous reasoning as with DIS, we can factorise this in a part with long-time dynamics, and a part with short-time dynamics. The part with the long-time dynamics is the hadronic part, where de-hadronisation takes place. The timescale for this part is $\Delta t \gg 1/\sqrt{Q^2}$. The short-time part is the part where quark-antiquark annihilation and lepton pair production happen. The timescale for this part is $\Delta t \sim 1/\sqrt{Q^2} \ll 1 \text{ fm}$.

We can factorise the cross section as [9] (p301-302)

$$\sigma_{DY}(\hat{s}, Q) = \sum_{a,b} \int d\xi_1 d\xi_2 f_{a/A}(\xi_1, \mu) H(Q, \mu, \xi_1, \xi_2, \hat{s}) f_{b/B}(\xi_2, \mu) + \sum_{j=1} K_j \left(\frac{\Lambda_{\text{QCD}}}{Q} \right)^j \quad (2.3.21)$$

Where the $f_{a/A}$ and $f_{b/B}$ are the parton distribution functions for the two hadrons, which corresponds to the long timescales. H corresponds to the short timescales, or in other words it relates to the high momentum transfer, so we can use perturbative QCD for it and expand it in powers of $\alpha_s(\mu)$. The term at the end captures nonperturbative effects of long timescales, which originate from the transition between short and long scale process.

Similar to deep inelastic scattering we can also use factorisable hadron-hadron scattering, like Drell-Yan, to study the inner structure of hadrons and the renormalisation group from experiment.

3 Evolution equations in QCD

Factorisation of deep inelastic scattering (DIS) allows us to derive a set of equations for the parton distribution functions (PDF), called the DGLAP equations. They are one example of evolution equations for the PDFs. These equations also contain the splitting functions. In this chapter we will derive the DGLAP equations and discuss them as well as the DGLAP splitting functions. This chapter is based on [5], [9], [10] and [15].

3.1 DGLAP equations

In this section we will derive the DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equations [1] [2] [3], which are evolution equations for the parton distribution functions (PDF). With evolution equation we mean an equation which characterises the PDF's rate of change if we vary the auxiliary scale μ . These equations will allow us to evolve the PDFs from one energy scale to another. While deriving the DGLAP equations, we will introduce the DGLAP splitting functions (or often just called Altarelli-Parisi splitting functions, or just splitting functions). Again we will use deep inelastic scattering (DIS) as an example, but DGLAP equations can be used in other factorisable hard scattering processes as well.

3.1.1 From RG to DGLAP equations

We start from Eq (2.3.16) for the structure functions:

$$F_n(x, Q) = \sum_i \int_x^1 d\xi C_{ni}(x/\xi, \alpha_s(\mu), Q/\mu) f_i(\xi, \mu) + \mathcal{O}(\Lambda_{\text{QCD}}^2/Q^2) \quad (3.1.1)$$

And note that F_n is independent of μ . The μ dependence in the coefficients C_{ni} cancels the μ dependence in the PDFs f_i . We also have that F_n is dimensionless, which can be seen from the differential cross section expressed in structure functions.

Since we have a convolution $F = C \otimes f$, we will first diagonalise the x dependence in Eq (3.1.1) by taking N -moments for any function of x . [10] We can do this by introducing the Mellin transformation:

$$F_n(N, Q) := \int_0^1 dx x^{N-1} F_n(x, Q) \quad (3.1.2)$$

Inverse Mellin transformation:

$$F_n(x, Q) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} F_n(N, Q) \quad (3.1.3)$$

Where c is taken to be right of all singularities. For the PDF we will have the following Mellin transform:

$$f_i(N, \mu) := \int_0^1 d\xi \xi^{N-1} f_i(\xi, \mu) \quad (3.1.4)$$

$$f_i(\xi, \mu) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} \xi^{-N} f_i(N, \mu) \quad (3.1.5)$$

Again with c taken to be right of all singularities.

If we take the Mellin transformation of Eq (3.1.1), the convolution \otimes will turn into an ordinary product:

$$F_n(N, Q) = \sum_i C_{ni}(N, Q, \mu) f_i(N, \mu) \quad (3.1.6)$$

Since F_n is independent of μ , we have

$$\frac{dF_n(x, Q)}{d(\ln \mu^2)} = 0 \quad (3.1.7)$$

And we have the same for the Mellin transformed F_n

$$\frac{dF_n(N, Q)}{d(\ln \mu^2)} = 0 \quad (3.1.8)$$

Lets consider for simplicity that we have only 1 quark flavour. From equations Eq (3.1.6) and (3.1.8) we get the differential equation:

$$C_n(N, Q, \mu) \frac{df(N, \mu)}{d(\ln \mu^2)} = -f(N, \mu) \frac{dC_n(N, Q, \mu)}{d(\ln \mu^2)} \quad (3.1.9)$$

If we divide both sides by the product $C_n f$, we can rewrite it as

$$\frac{d(\ln f(N, \mu))}{d(\ln \mu^2)} = -\frac{d(\ln C_n(N, Q, \mu))}{d(\ln \mu^2)} \quad (3.1.10)$$

Since $\ln f$ is dimensionless, we can use the Callan-Symanzik equation from the Renormalisation Group (RG):

$$\left(\frac{\partial}{\partial(\ln \mu^2)} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \gamma(\alpha_s) \right) \ln f(N, \mu) = 0 \quad (3.1.11)$$

Since the partial derivative with respect to α_s equals 0, Eq (3.1.11) simplifies to the following RG equation

$$\frac{d(\ln f(N, \mu))}{d(\ln \mu^2)} = -\gamma_N(\alpha_s(\mu)) = -\frac{d(\ln C_n(N, Q, \mu))}{d(\ln \mu^2)} \quad (3.1.12)$$

Where γ_N is the anomalous dimension of the PDF. The anomalous dimension has an expansion in powers of α_s , [10]:

$$\gamma_N(\alpha_s(\mu)) = \sum_{m=1}^{\infty} b_N^{(m)} \alpha_s^m \quad (3.1.13)$$

We can solve the RG equation Eq (3.1.12), which gives us

$$f(N, \mu) = f(N, \mu_0) \exp\left(-2 \int_{\mu_0}^{\mu} \gamma_N(\alpha_s(\mu')) \frac{d\mu'}{\mu'}\right) \quad (3.1.14)$$

We can rewrite the RG equation Eq (3.1.12) as

$$\frac{df(N, \mu)}{d(\ln \mu^2)} = -\gamma_N(\alpha_s(\mu)) f(N, \mu) \quad (3.1.15)$$

If we restore all quark flavours, we would get

$$\frac{df_i(N, \mu)}{d(\ln \mu^2)} = -\sum_j \gamma_{N,ij}(\alpha_s(\mu)) f_j(N, \mu) \quad (3.1.16)$$

We now have what we refer to as an evolution equation for the PDF, but we are still in the Mellin transformed space. Lets define the inverse Mellin transform of the anomalous dimension by

$$-\gamma_{N,ij}(\alpha_s(\mu)) = \int_0^1 dz z^{N-1} P_{ij}(z, \alpha_s(\mu)) \quad (3.1.17)$$

If we now take Eq (3.1.16), substitute the inverse Mellin transform for the functions, and make a change of integration variable from, we get the DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equations [1] [2] [3] [10]

$$\frac{df_i(x, \mu)}{d(\ln \mu^2)} = \sum_j \int_x^1 \frac{d\xi}{\xi} P_{ij}(x/\xi, \alpha_s(\mu)) f_j(\xi, \mu) \quad (3.1.18)$$

The DGLAP equation, or evolution equation, describes the coupled evolution for the parton distribution functions (PDF). (Ref [5] (p590-591)). The functions P_{ij} are the DGLAP splitting functions. The splitting functions turn out to be calculable as a series in orders of α_s

$$P_{ij}(z, \alpha_s) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n P_{ij}^{(n-1)}(z) \quad (3.1.19)$$

The PDFs are not calculable from QCD because they contain the long timescale effects, but by using the DGLAP equations we can calculate their μ dependence.

3.1.2 DGLAP and higher order corrections in structure functions.

While we already derived the DGLAP equations from the Renormalisation Group and introduced the splitting functions, we will now show the link between Feynman diagrams and the DGLAP equations. We will take the example of DIS again. If we work in the “naive” parton model (no μ dependence) and if we ignore the higher order $\mathcal{O}(\Lambda_{\text{QCD}}^2/Q^2)$ terms, the factorized DIS cross section Eq (2.3.5) will be

$$\sigma_{l-h}(x, Q) = \sum_q \int_x^1 d\xi \sigma_{l-q}\left(\frac{x}{\xi}, Q\right) f_q(\xi) \quad (3.1.20)$$

Where we didn't include the gluon term because the photon only interacts with quarks (and antiquarks), and not with gluons. We recall the expression for differential cross section with the structure function [9] (p89):

$$\frac{d^2\sigma_{l-h}}{dx dQ^2} = \frac{2\pi\alpha^2}{xQ^4} (1 + (1-y)^2) F_2(x) \quad (3.1.21)$$

Where we set $F_2(x) = 2xF_1(x)$ due to Callan-Gross relation. [9] (p91)

And for the structure function we have [9] (p90)

$$F_2 = 2xF_1 = \sum_q e_q^2 x f_q(x) \quad (3.1.22)$$

If we step off from the naive parton model and go to the QCD improved parton model, then Eq (3.1.22) can be regarded as a 0th order term in the series expansion of F_2 in orders of α_s . [15] If we want to include the term of order α_s , we have to take the following diagrams into account:

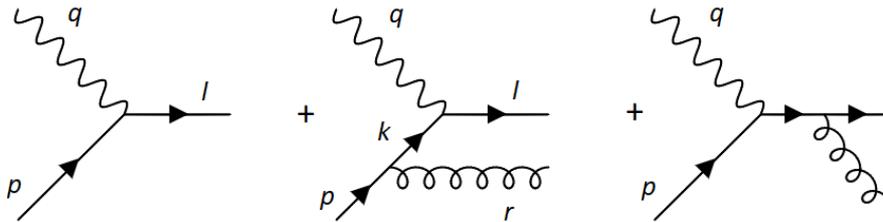


Figure 3.1: Contributing diagrams for deep inelastic scattering, up to first order in α_s .

The 0th order diagram is also included. In the two diagrams on the right, a collinear gluon is emitted. [15] It will be (nearly) collinear because of the high momentum of the quark that emits the gluon. We have strong ordering of transverse momenta: $p_{\perp} \ll k_{\perp} \ll \mu$. If we put the emphasis on $p_{\perp} \ll k_{\perp}$ then momentum p will be nearly collinear to the hadron. If we put the emphasis on $k_{\perp} \ll \mu$ then momentum k will be nearly collinear to momentum p .

If the third diagram in Figure 3.1 can be neglected, then we could consider the radiated gluon as part of the hadron structure. The third diagram cannot be neglected in order to ensure gauge invariance, but it will cancel the unphysical polarizations of the gluon. [15] We can choose a physical (axial) gauge for which only the second diagram contributes to the splitting function and the logarithm, while the third diagram only contributes finite terms. We can take even higher order contributions into account with multiple gluon radiation (ladder diagrams).

The structure function up to 1st order in α_s is given by [9] (p99-105):

$$F_2(x, Q) = x \sum_q e_q^2 \left[f_{q0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q0}(\xi) \left(P\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\mu_0^2}\right) + C\left(\frac{x}{\xi}\right) \right) + \dots \right] \quad (3.1.23)$$

Where f_{q0} is the bare quark distribution function, and P is a function we get from applying Feynman rules in the diagrams. P in particular describes the $q \rightarrow qg$ splitting. [15] Note that the transverse momentum k_{\perp} does not explicitly appear in the structure function. However, effects from the transverse momentum still appear as a logarithm $\ln(Q^2/\mu_0^2)$, which comes from an integration over transverse momentum of the form:

$$\int_{\mu_0^2}^{Q^2/x} \frac{dk_{\perp}^2}{k_{\perp}^2} \sim \ln\left(\frac{Q^2}{\mu_0^2 x}\right) = \ln\left(\frac{Q^2}{\mu_0^2}\right) - \ln(x) \quad (3.1.24)$$

if we would work with a momentum cut-off both at low and high transverse momentum. [9] (p98, p102) The cut-off μ_0^2 at low transverse momentum helps to treat the singularity. The integration can also be done by dimensional regularisation instead, which would give something of the form [9] (p104):

$$\int \frac{dk_{\perp}^2}{(k_{\perp}^2)^{1+\varepsilon}} \sim \left(-\frac{1}{\varepsilon}\right) \quad (3.1.25)$$

Where $\varepsilon < 0$.

f_{q0} can be seen as an unmeasurable, bare distribution. Collinear singularities (will be explained in section 3.2) are absorbed into this bare distribution (at factorization scale

μ) (Ref [9] p99-105). So we will define a renormalized distribution $f_q(x, \mu)$ as in [9] (p99-105):

$$f_q(x, \mu) = f_{q0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q0}(\xi) \left(P\left(\frac{x}{\xi}\right) \ln\left(\frac{\mu^2}{\mu_0^2}\right) + C\left(\frac{x}{\xi}\right) \right) + \dots \quad (3.1.26)$$

This means the structure function becomes [9] (p99-105):

$$F_2(x, Q) = x \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} f_q(\xi, \mu) \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\mu^2}\right) + \dots \right] \quad (3.1.27)$$

At any scale, we have as in [9] (p105):

$$F_2(x, Q) = x \sum_q e_q^2 f_q(x, Q) \quad (3.1.28)$$

Which is called the DIS scheme. The $f_q(x, \mu)$ can't be calculated from QCD because, even at first principles, since it is in the unperturbative regime. However we can determine it from structure function data via Eq (3.1.28) at any scale. We can also calculate its dependence on the scale μ . If we take a look at Eq (3.1.27), we notice that the left hand side is independent of μ . This means the right hand side must also be independent of μ . We will take the derivative of both sides, with respect to $\ln \mu^2$. This is similar to what we did in previous section. The resulting equation is [9] (p99-105):

$$\frac{df_q(x, \mu)}{d(\ln \mu^2)} = \frac{\alpha_s(\mu)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P\left(\frac{x}{\xi}\right) f_q(\xi, \mu) \quad (3.1.29)$$

Compare this with Eq (3.1.18), the DGLAP equation. They look similar. While this was not fully rigorous, a full treatment of the theory at first order in α_s would result in the equation [9] (p108):

$$\frac{df_q(x, \mu)}{d(\ln \mu^2)} = \frac{\alpha_s(\mu)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P\left(\frac{x}{\xi}, \alpha_s(\mu)\right) f_q(\xi, \mu) \quad (3.1.30)$$

This equation has the same form as the DGLAP equation Eq (3.1.18) with only a quark term:

$$\frac{df_q(x, \mu)}{d(\ln \mu^2)} = \int_x^1 \frac{d\xi}{\xi} f_q(\xi, \mu) P_{qq}(x/\xi, \alpha_s(\mu)) \quad (3.1.31)$$

Recall we had a series expansion for the splitting functions Eq (3.1.19), for which the lowest order term equals:

$$P_{ij}(z, \alpha_s) = \frac{\alpha_s}{2\pi} P_{ij}^{(0)}(z) + \mathcal{O}(\alpha_s^2) \quad (3.1.32)$$

Eq (3.1.30) is the DGLAP equation in 1st order of α_s . This means that combining Eq (3.1.30), (3.1.31) and (3.1.32), in 1st order of α_s , leads us to

$$P(x/\xi) = P_{qq}^{(0)}(x/\xi) \quad (3.1.33)$$

This means the splitting function P_{qq} up to lowest order is proportional to P , and can be represented by the $q \rightarrow qg$ splitting ($q \rightarrow q$ splitting with the emission of g). We can generalise this up to higher orders in α_s , where P_{qq} up to order α_s^n can be represented by $q \rightarrow qg$ splitting up to order α_s^n .

Even by including those, our description for first order in α_s would still not be complete. We would need to include other processes like $\gamma g \rightarrow q\bar{q}$. And analogous to $q \rightarrow qg$, it's possible to show that P_{qg} can be represented by $g \rightarrow q\bar{q}$ splitting. In general P_{ij} can be represented by $j \rightarrow i$ parton splitting with the emission of a third parton. [15]

If we go back to Figure 3.1 and the structure function Eq (3.1.27), We could go to higher orders by taking multiple gluon radiation (ladder diagrams) into account. In that case we will still have strong ordering of transverse momenta, which means that the transverse momentum of the parton after each new radiation of a gluon will be much larger than the previous. So we will get ordered nested integrals over transverse momenta. [9] (p138)

3.2 Splitting functions

In this section we will discuss the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [1] [2] [3] splitting functions more in detail. The start of this section be giving the important general definitions and formulas again that involve the splitting functions, what Feynman diagrams they're related to and what their physical interpretation is. This section will mostly focus on the leading order expressions of the splitting functions, but at the end we will also give the next-to-leading order expressions.

3.2.1 Main definitions and importance of the splitting functions

The DGLAP splitting functions first showed up in our derivation of the DGLAP equations. The DGLAP equations are the evolution equations for the parton distribution functions (PDFs).

$$\frac{df_i(x, \mu)}{d(\ln \mu^2)} = \sum_j \int_x^1 \frac{d\xi}{\xi} P_{ij}(x/\xi, \alpha_s(\mu)) f_j(\xi, \mu) \quad (3.2.1)$$

Where $P_{ij}(x/\xi, \alpha_s(\mu))$ are the splitting functions and $f_i(x, \mu)$ the PDFs. The splitting functions are calculable in QCD as a series in orders of α_s .

$$P_{ij}(z, \alpha_s) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n P_{ij}^{(n-1)}(z) \quad (3.2.2)$$

(Note that some sources define P_{ij} and the DGLAP differently, by letting index n start from 0 in the expansion, resulting in the appearance of an extra factor $\alpha_s/2\pi$ in the DGLAP or elsewhere.) In the way we defined the splitting function, the leading order term is

$$P_{ij}^{(LO)}(z, \alpha_s) = \frac{\alpha_s}{2\pi} P_{ij}^{(0)}(z) \quad (3.2.3)$$

The DGLAP equation and the perturbative nature of the splitting functions, allows us to calculate the PDF's μ scale dependence in perturbation theory, despite that the PDFs are otherwise uncalculable. PDFs always relate to the long timescale physics, and thus are in the unperturbative regime of QCD.

The splitting functions can also be defined via their relation to the anomalous dimension $\gamma_{N,ij}$ from the Renormalisation Group. They are simply related to each other via a Mellin transform:

$$\gamma_{N,ij}(\alpha_s(\mu)) = - \int_0^1 dz z^{N-1} P_{ij}(z, \alpha_s(\mu)) \quad (3.2.4)$$

Where we took a minus sign as convention so that the anomalous dimension would have a plus sign in the Callan-Symanzik equation. (Note that a lot of sources define it with the opposite sign.)

The splitting function also showed up in corrections for the structure function in orders of α_s :

$$F_2(x, Q) = x \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} f_q(\xi, \mu^2) \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P_{qq}^{(0)}\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\mu^2}\right) + \dots \right] \quad (3.2.5)$$

And also in the renormalized PDF:

$$f_q(x, \mu) = f_{q0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q0}(\xi) \left(P_{qq}^{(0)}\left(\frac{x}{\xi}\right) \ln\left(\frac{\mu^2}{\mu_0^2}\right) + C\left(\frac{x}{\xi}\right) \right) + \dots \quad (3.2.6)$$

The reason for this was due to corrections from gluon emission. This also made our first connection between splitting functions and a physical process. This idea that the splitting functions appear in correction terms in physical quantities due to parton emission, can be generalised beyond just quarks and beyond just deep inelastic scattering (DIS).

An important thing to remember is that the splitting functions in leading order are independent of the regularization or factorization scheme. The higher order terms on the other hand, are dependent on regularisation and factorisation scheme, and are Q dependent. [10], [9] (p99-105)

3.2.2 Physical interpretation of splitting functions

The splitting functions can be directly related to Feynman diagrams of parton emission, or what we call “splitting”. The splitting function $P_{ij}(z, \alpha_s(\mu))$ describes the probability for a parton j to go to a nearly collinear parton i (transverse momentum squared much smaller than μ^2) by emitting another parton, with z the ratio of collinear momenta for parton i versus parton j . Or in other words, the probability for a parton i to be found in a parton j (with z the ratio of collinear momentum, and transverse momentum being small). [9] (p109)

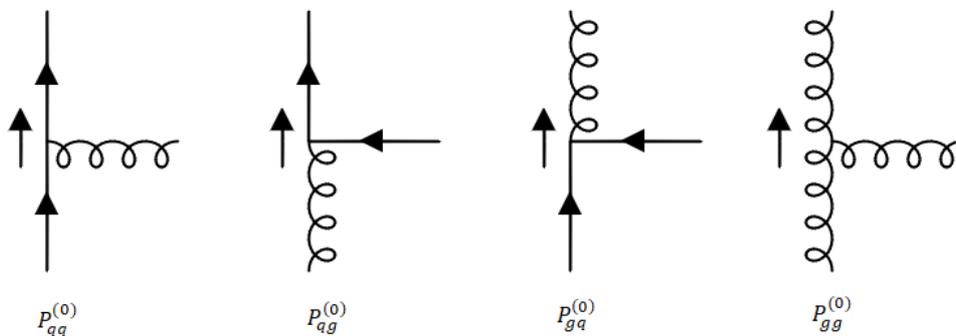


Figure 3.2: Leading order diagrams for the splittings functions. Time runs from bottom to top.

The probability is the norm squared of the S-matrix for a diagram.

The leading order (LO) splitting functions are directly related to observable scaling violations (which is why they are regularization and factorization independent). For example the measurable quantity $dF_2/d(\ln Q^2)$. If we had exact Bjorken scaling, then $dF_2/d(\ln Q^2)$ would be 0, but we do measure scaling violation in this quantity. In leading order (LO), $dF_2/d(\ln Q^2)$ will be proportional to an LO splitting function, $P_{qq}^{(0)}$ in case of Eq (3.2.5), because in LO that is the only part that the derivative picks out. (But for small x it will be proportional to $P_{qg}^{(0)}$ instead, because at small x the gluon distribution dominates over the quark distribution.) While in LO the scaling violation

that can be observed in $dF_2/d(\ln Q^2)$ is related to a LO splitting function, in next-to-leading order (NLO) on the other hand, the relation between splitting functions and observable scaling violations is not clear. In NLO this observable quantity requires an integrated product between PDFs and more splitting functions (both LO and NLO splitting functions). Because of this the observable quantity will not be proportional to the splitting functions anymore when we assign a factorisation scheme. Thus in NLO there is no longer a direct relation between splitting functions and observable scaling violations.

3.2.3 Splitting function expressions at leading order

In this section we give the expressions for the leading order splitting functions, and include some important relations and identities. For the calculations of the leading order splitting functions; refer to [16], [9] (p109), [5] (p590-591).

First we define:

$$\frac{1}{(1-x)_+} := \frac{1}{1-x} \quad \text{for } 0 \leq x < 1 \quad (3.2.7)$$

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} := \int_0^1 dx \frac{f(x) - f(1)}{1-x} \quad (3.2.8)$$

DGLAP splitting functions in leading order:

$$P_{qq}^{(0)}(z) = P_{\bar{q}\bar{q}}^{(0)}(z) = C_F \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right) \quad (3.2.9)$$

$$P_{qg}^{(0)}(z) = P_{\bar{q}g}^{(0)}(z) = T_R(z^2 + (1-z)^2) \quad (3.2.10)$$

$$P_{gq}^{(0)}(z) = P_{g\bar{q}}^{(0)}(z) = C_F \left(\frac{1+(1-z)^2}{z} \right) \quad (3.2.11)$$

$$P_{gg}^{(0)}(z) = 2C_A \left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right) + \left(\frac{11}{6} C_A + \frac{2}{3} T_R n_f \right) \delta(1-z) \quad (3.2.12)$$

$$P_{q\bar{q}}^{(0)}(z) = P_{\bar{q}q}^{(0)}(z) = 0 \quad (3.2.13)$$

And remember the $SU(N_c)$ factors:

$$C_A = N_c \quad , \quad C_F = \frac{N_c^2 - 1}{2N_c} \quad , \quad T_F = \frac{1}{2} \quad (3.2.14)$$

The splitting functions satisfy the sum rules

$$\int_0^1 dz P_{qq}^{(0)}(z) = 0 \quad (3.2.15)$$

$$\int_0^1 dz z \left(P_{qq}^{(0)}(z) + P_{gq}^{(0)}(z) \right) = 0 \quad (3.2.16)$$

$$\int_0^1 dz z \left(2n_f P_{qg}^{(0)}(z) + P_{gg}^{(0)}(z) \right) = 0 \quad (3.2.17)$$

3.2.4 Quark-to-quark splitting function in leading order

In section 3.1.2 we introduced the structure function for DIS up to 1st order in α_s (Eq (3.1.27))

$$F_2(x, Q) = x \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} f_q(\xi, \mu) \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\mu^2}\right) + \dots \right] \quad (3.2.18)$$

Where the 1st order effect came from the $q \rightarrow qg$ splitting. Now we will do the actual calculation and look at this effect in more detail. In this section we follow the calculation outlined in reference [9] (p100-105). We will also use the same momenta parametrisation (but with k_\perp instead of k_t):

$$k^\mu = \xi p^\mu + k_\perp^\mu + \frac{|k^2| + k_\perp^2}{2\xi} n^\mu \quad (3.2.19)$$

$$d^4k = \frac{d\xi}{2\xi} dk^2 d^2k_\perp \quad (3.2.20)$$

This means that we have as in [9] (p100-105):

$$(p - k)^2 = (1 - \xi) \frac{|k^2|}{\xi} - \frac{k_\perp^2}{\xi} \quad (3.2.21)$$

$$(k + q)^2 = 2\xi p \cdot q - Q^2 - |k^2| - 2q_\perp \cdot k_\perp \quad (3.2.22)$$

For the the Feynman amplitude we get [9] (p100-105):

$$\mathcal{M}^\mu = -ig e_q \bar{u}(l) \gamma^\mu \frac{1}{\not{k}} \varepsilon_\nu \gamma^\nu t^A u(p) \quad (3.2.23)$$

Where e_q is the quark electric charge, u and \bar{u} are the quark's spinors, and ε_ν the polarisation of the gluon.

We will use the optical theorem:

The diagram illustrates the optical theorem for quark-to-quark splitting. On the left, a vertical line represents a quark propagator with two external quark lines (upward arrows) and a gluon loop (wavy line) attached to it. This is enclosed in large vertical bars, and the entire expression is squared. On the right, the same tree-level diagram is shown inside large parentheses, with a vertical dashed line representing a cut in the propagator. The expression is equated to 2Im of this diagram.

Figure 3.3: Optical theorem for quark-to-quark splitting.

This gives us an averaged squared Feynman amplitude. We define the notation $\overline{|\mathcal{M}|_{\mu\nu}^2}$ as the average of the $|\mathcal{M}|_{\mu\nu}^2$ over colours and spins. Converting the Feynman diagram into math yields [9] (p100-105):

$$\overline{|\mathcal{M}|_{\mu\nu}^2} = \frac{1}{2} e_q^2 g^2 \sum_{pol} C_F \text{Tr}(\gamma^\nu (\not{k} + \not{q}) \gamma^\mu \not{k} \varepsilon_\alpha \gamma^\alpha \not{p} \varepsilon_\beta^* \gamma^\beta) \frac{1}{k^4} \quad (3.2.24)$$

Where we have the colour factor C_F

For the sum over the polarisation of the gluon, we use [9] (p100-105):

$$\sum_{pol} \varepsilon_\mu(r) \varepsilon_\nu^*(r) = -g_{\mu\nu} + \frac{n_\mu r_\nu + n_\nu r_\mu}{n \cdot r} \quad (3.2.25)$$

This is what we call a projector. When we apply this projector, only two physical polarisations will propagate, the transverse polarisations. This is the case cause we now have both $\varepsilon \cdot r = 0$ and $\varepsilon \cdot n = 0$. [9] (p100-105)

If we apply this projector in Eq (3.2.24), we can write the following expression for the Feynman amplitude [9] (p100-10):

$$\frac{1}{4\pi} n^\mu n^\nu \overline{|\mathcal{M}|_{\mu\nu}^2} = \frac{8e_q^2 \alpha_s}{|k^2|} \xi P(\xi) \quad (3.2.26)$$

Where the function P is defined as

$$P(\xi) = C_F \frac{1 + \xi^2}{1 - \xi} \quad (3.2.27)$$

The function $P(\xi)$ is $P_{qq}^{(0)}(\xi)$, the coefficient in the leading order term of our quark-to-quark splitting function $P_{qq}^{(LO)}(\xi, \alpha_s)$. (Ref [9] p100-105)) Using the convention we made in section 3.1, we have

$$P_{qq}^{(LO)}(\xi, \alpha_s) = \frac{\alpha_s}{2\pi} P_{qq}^{(0)}(\xi) = \frac{\alpha_s}{2\pi} P(\xi) \quad (3.2.28)$$

We only used the diagram in Figure 3.3, taking other diagrams into account as well would just result in an extra term in $P_{qq}^{(0)}(\xi)$, a term that is proportional to $\delta(1 - \xi)$, Ref [9] (p100-105). So we can make the replacement

$$P_{qq}^{(0)}(\xi) \rightarrow P_{qq}^{(0)}(\xi) + K\delta(1 - \xi) \quad (3.2.29)$$

And so we have

$$P_{qq}^{(0)}(\xi) = C_F \frac{1 + \xi^2}{1 - \xi} + K\delta(1 - \xi) \quad (3.2.30)$$

In general, this function does not allow for baryon number conservation (quark conservation). This can be seen by plugging the splitting function into Eq (3.1.26)

$$f_q(x, \mu) = f_{q0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q0}(\xi) \left(P\left(\frac{x}{\xi}\right) \ln\left(\frac{\mu^2}{\mu_0^2}\right) + C\left(\frac{x}{\xi}\right) \right) + \dots \quad (3.2.31)$$

In order to have conservation of baryon number, the solution is that the integral of the new splitting function in Eq (3.2.30) will be 0

$$\int_0^1 d\xi P_{qq}^{(0)}(\xi) = \int_0^1 d\xi \left(C_F \frac{1 + \xi^2}{1 - \xi} + K\delta(1 - \xi) \right) = 0 \quad (3.2.32)$$

Integration gives

$$K = C_F \left(\frac{3}{2} + \ln(1 - \xi)|_{\xi=1} \right) \quad (3.2.33)$$

There is a logarithmic singularity for $\xi = 1$. This issue has been solved by defining the function

$$\frac{1}{(1-x)_+} := \frac{1}{1-x} \quad \text{for } 0 \leq x < 1 \quad (3.2.7)$$

Which has the integration property

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} := \int_0^1 dx \frac{f(x) - f(1)}{1-x} \quad (3.2.8)$$

This trick will help us solve the integral by replacing $1/(1 - \xi)$ by $1/(1 - \xi)_+$ in the splitting function. In other words we define the splitting function as

$$P_{qq}^{(0)}(\xi) = C_F \frac{1 + \xi^2}{(1 - \xi)_+} + K\delta(1 - \xi) \quad (3.2.34)$$

We will now try again to performing the integral, and equating it to 0. This time we get

$$\int_0^1 d\xi P_{qq}^{(0)}(\xi) = K + C_F \left(\int_0^1 d\xi \frac{\xi^2 - 1}{1 - \xi} \right) = K - C_F \int_0^1 d\xi (\xi + 1) = K - \frac{3}{2} C_F = K - \frac{3}{2} C_F = 0 \quad (3.2.35)$$

Which means

$$K = \frac{3}{2} C_F \quad (3.2.36)$$

So our splitting function $P_{qq}^{(0)}(\xi)$ is equal to

$$P_{qq}^{(0)}(\xi) = C_F \left(\frac{1 + \xi^2}{(1 - \xi)_+} + \frac{3}{2} \delta(1 - \xi) \right) \quad (3.2.9)$$

3.2.5 Collinear divergences

Our splitting function $P_{qq}^{(0)}(\xi)$ exhibits a singularity for $\xi = 1$. Setting this value for ξ means that the emitted gluon is collinear to the parent quark, or in other words, the transverse momentum $k_{\perp} = 0$. This is what we call collinear divergence. And unlike divergences in the UV, for which we can for example introduce a maximum momentum (cut-off), we can't do the same in the limit to small momenta. Taking the limit $k_{\perp}^2 \rightarrow 0$ is equivalent to the long distance force in QCD, [9] (p100-105). Long distance effects in QCD are nonperturbative, so we cannot calculate this (atleast not in perturbation theory).

We can now take the original quark distribution function, and consider it a bare distribution, and then define a "renormalized" distribution function which absorbs the collinear singularity. Absorbing the collinear divergence will be done at a scale μ , which we refer to as the factorisation scale. We get for the "renormalized" quark distribution function (Ref [9] p100-105), which is a function we also introduced "ad hoc" (together with collinear singularities) in section 3.1.2, the following expression:

$$f_q(x, \mu) = f_{q0}(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_{q0}(\xi) \left(P\left(\frac{x}{\xi}\right) \ln\left(\frac{\mu^2}{\mu_0^2}\right) + C\left(\frac{x}{\xi}\right) \right) + \dots \quad (3.1.37)$$

And with this, the structure function (for hadron target) is equal to [9] (p100-105):

$$F_2(x, Q) = x \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} f_q(\xi, \mu) \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\mu^2}\right) + \dots \right] \quad (3.1.38)$$

This idea of “renormalisation” of the distribution function, is similar to renormalisation of the coupling strength, or renormalisation of other parameters in the Lagrangian.

The “renormalized” quark distribution function $f_q(x, \mu)$ cannot be calculated from (perturbative) QCD, not even from first principles, because the long distance effects from QCD contribute to it. In other words, $f_q(x, \mu)$ is nonperturbative. We determine the quark distribution function from experiments. The same is true for the other partons their distribution functions. All the PDFs will be nonperturbative, and need to be determined from experiments instead. However, there is one thing about the PDFs that we can calculate from perturbative QCD: the μ dependence. This is done by using the DGLAP equations (see section 3.1).

4 TMD Gluon-to-quark splitting function

In this section we will discuss and calculate the transverse momentum dependent gluon-to-quark splitting function. The calculation in this part is mostly based on 3 papers [17] [18] [19] [28]. Many of the concepts, conventions, notations and choices will be similar to those in the 3 papers. (An additional paper [20] was very briefly used to add a bit more to the understanding of some concepts.)

4.1 Collinear vs transverse momentum factorisation

Up till now we have only considered DGLAP equations and splitting functions in the collinear factorisation scheme, in which the incoming parton is considered to be collinear with the momentum of the hadron, $k = yp$. The transverse momentum k_\perp is set to 0. However, even in the collinear factorisation, effects from the transverse momentum still appear in higher order corrections. In case of the 1st order correction we discussed in chapter 3, Eq (3.1.27), the $\ln(Q^2/\mu^2)$ in the structure function $F_2(x, Q)$ comes from an integration over transverse momentum of the form Eq (3.1.24), which we can also write as

$$\int \frac{dk_\perp^2}{k_\perp^2} \sim \ln\left(\frac{Q^2}{\mu_0^2 x}\right) = \ln\left(\frac{Q^2}{\mu^2}\right) - \ln\left(\frac{\mu_0^2 x}{\mu^2}\right) \quad (4.1.1)$$

if we would work with a momentum cut-off both at low and high transverse momentum [9] (p99-105). The cut-off μ_0^2 at low transverse momentum helped to treat the collinear singularity. If we would do the integration by dimensional regularisation instead, it would give something of the form Eq (3.1.25):

$$\int \frac{dk_\perp^2}{(k_\perp^2)^{1+\varepsilon}} \sim \left(-\frac{1}{\varepsilon}\right) \quad (4.1.2)$$

Where $\varepsilon < 0$.

But what if we want to take the exact effects from transverse momentum into account already at leading order? Such a treatment will lead to what we call transverse-momentum-dependent (TMD) splitting functions and TMD PDFs (parton distribution functions), which are also called “unintegrated splitting functions” and “unintegrated PDFs”. TMD splitting functions and PDFs are naturally related to a hierarchy of scales in the phase space. [19]. For example, when we look at the situation for which we have the scale hierarchy $\hat{s} \gg M^2 \gg \Lambda_{\text{QCD}}^2$, with M^2 the hard scale and \hat{s} the COM (center-of-mass) energy squared, we will have large $\ln(\hat{s}/M^2)$ logarithms that can compensate the small α_s (which is small because we are in the perturbative region at high energy scale). It will be necessary to do a resummation of terms in $(\alpha_s/\ln(\hat{s}/M^2))^n$ to all orders [19]. This was done by Balitsky-Fadin-Kuraev-Lipatov

(BFKL) evolution equation [24-27]. The BFKL equation is based on a factorisation in the high energy limit $\hat{s} \gg M^2$, the “high energy factorisation”. In this factorisation scheme, structure functions and convolutions are automatically factorised into k_\perp dependent factors and the BFKL Green’s function. [19] The shortcoming of the high energy factorisation, is that it is limited to the low M^2/\hat{s} regime. A series of problems arise when trying to extrapolate it into the intermediate and higher M^2/\hat{s} regime. [19] For example elementary vertices that violate energy conservation, and quark contributions arising only in next-to-leading order (NLO). [19]

To do better, a resummation could be done which restores the DGLAP splitting functions. In essence, DGLAP preserves conservation of longitudinal momentum, while BFKL preserves the conservation of transverse momentum. [19] Attempts to unify DGLAP with BFKL already resulted in resummations which are stable in the low and intermediate M^2/\hat{s} regions up to $M^2/\hat{s} \sim 10^{-2}$. [19] Matching the collinear and high energy factorisations will be done by the k_\perp factorisation scheme. We will use the same approach as in the references [17] [18] [19] [28], which are based on the Curci-Furmanski-Petronzio (CFP) formalism [29] [17]

4.2 2PI kernels, Green functions and projector operators

We will now explain the approach to arrive at the Feynman diagram that we will use to calculate the TMD gluon-to-quark splitting function. It is based on the approaches of the references [17] [18] [19] [28]. Consider a single incoming hadron, we can define a dimensionless cross section

$$F(x, Q^2) \sim Q^2 \sigma(x, Q^2) \quad (4.2.1)$$

The dimensionless cross section is factorisable, just like the cross section and the structure functions:

$$F(x, Q^2) = \sum_i \int_x^1 dz C_i(x/z, \alpha_s(\mu_F), Q^2/\mu_F^2) f_i(z, \mu_F^2) + \mathcal{O}(\Lambda_{\text{QCD}}^2/Q^2) \quad (4.2.2)$$

With f_i the parton distribution functions and C_i process-dependent coefficient functions, analogous to those of the factorisation of the structure functions. μ_F is the factorisation scale, and is arbitrary as long as $\mu_F^2 \gg \Lambda_{\text{QCD}}^2$. The dimensionless cross section is independent of this factorisation scale. This means, just like with the structure functions, the μ_F dependence of C_i cancels the μ_F dependence of f_i . We can treat $F(x, Q^2)$ in the exact same way as we treated the structure functions $F_i(x, Q^2)$.

We will now briefly discuss the factorisation schemes that are used to reach the calculation of the gluon-to-quark splitting function.

4.2.1 Collinear factorisation

For collinear factorisation we will use the technique by Curci-Furmanski-Petronzio (CFP) [29] [17]. The dimensionless cross section can be factorised in the following way:

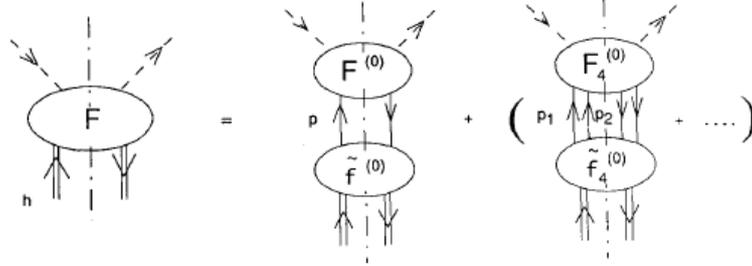


Figure 4.1: Factorisation of the dimensionless cross section F in terms of partonic cross sections and parton distributions. Time runs from bottom to top. Figure from [17] (p483)

$$F(x, Q^2) = \int F^{(0)}(\dots, p) \tilde{f}^{(0)}(p, \dots) + \left[\int F_4^{(0)}(\dots; p_1 p_2) \tilde{f}_4^{(0)}(p_1 p_2; \dots) + \dots \right] \quad (4.2.3)$$

Where $F^{(0)}, F_4^{(0)}, \dots$ are the dimensionless partonic cross sections (analogous to the parton structure functions, which are for a parton as target), and $\tilde{f}^{(0)}, \tilde{f}_4^{(0)}, \dots$ are the parton distribution functions. Both the $F^{(0)}$ and $\tilde{f}^{(0)}$ are considered bare quantities. [17] With the notation $\int AB$ we mean a convolution between A and B , which integrates over all internal momenta, and sums over all spin, colours and intermediate particle species.

Then we apply dimensional regularisation, to a $D = 4 + 2\varepsilon$ dimensional spacetime. This will introduce a regularisation scale μ . When we renormalise $F^{(0)}$, we automatically regularise collinear singularities into single poles in $1/\varepsilon$. [17] Factorisation allows us to subtract those poles from $F^{(0)}$, and it also allows us to factorise them to all orders of α_s into independent transition functions Γ . [17] This can be done according to:

$$F^{(0)} = \int C \Gamma \quad (4.2.4)$$

Where C is the coefficient function, which is finite for $\varepsilon \rightarrow 0$. We use the transition functions to define the “physical” parton density [17]:

$$\tilde{f} = \int \Gamma \tilde{f}^{(0)} \quad (4.2.5)$$

This recovers the factorisation formula at the start, Eq (4.2.2). [17]

We now choose a momentum parametrisation to simplify things. We will use Sudakov parametrisation, with incoming parton momentum p^μ , and any other momentum k^μ

$$p^\mu = P(1, \mathbf{0}, 1), \quad \bar{p}^\mu = P(1, \mathbf{0}, -1) \quad (4.2.6)$$

$$k^\mu = zp^\mu + k_\perp^\mu + \frac{k^2 + \mathbf{k}^2}{2zp \cdot \bar{p}} \bar{p}^\mu, \quad k_\perp^\mu = (0, \mathbf{k}, 0)$$

and for the gauge choice we will choose the axial gauge $\partial \cdot A$, which is a physical gauge. This gauge choice can also be expressed as $\bar{p} \cdot A = 0$. The summation over gluon polarisations is given by

$$d_{\mu\nu} = \sum_{pol} \varepsilon_\mu \varepsilon_\nu^* = -g_{\mu\nu} + \frac{k_\mu \bar{p}_\nu + \bar{p}_\nu k_\mu}{\bar{p} \cdot k} \quad (4.2.7)$$

We will now expand $F^{(0)}$ in two-particle irreducible (2PI) kernels $C^{(0)}$ and $K^{(0)}$ [17]:

$$F^{(0)} = \int C^{(0)} (1 + K^{(0)} + K^{(0)}K^{(0)} + \dots) = \int C^{(0)} G^{(0)} \quad (4.2.8)$$

$$G^{(0)} = \frac{1}{1 - K^{(0)}} \quad (4.2.9)$$

With $G^{(0)}$ the bare Green function. In the axial gauge we are working, there are no mass singularities in the two-particle irreducible (2PI) amplitudes, and thus all collinear singularities originate from integrations over the momenta coming out of the kernels $K^{(0)}$ and connecting them in the bare Green function $G^{(0)}$. [21], [17] We can introduce a projector operator \mathcal{P}_C (collinear projector operator) which will give us the factorisation formula Eq (4.2.4). This projector operator will decouple $C^{(0)}$ and $G^{(0)}$ in the spin indices. It will also decouple them in momentum space by extracting the $1/\varepsilon$ poles. [17] We can write for each kernel:

$$K^{(0)} = \int \left((1 - \mathcal{P}_C) K^{(0)} + \mathcal{P}_C K^{(0)} \right) \quad (4.2.10)$$

Where all the singularities are in $\mathcal{P}_C K^{(0)}$. If we now apply this in an iterative way, we get [17]:

$$G^{(0)} = \int G \Gamma \quad (4.2.11)$$

Where all $1/\varepsilon$ poles are subtracted from G , the “renormalized” Green function [17]:

$$G = \frac{1}{1 - \int (1 - \mathcal{P}_C) K^{(0)}} = 1 + \int (1 - \mathcal{P}_C) K^{(0)} + \int (1 - \mathcal{P}_C) K^{(0)} (1 - \mathcal{P}_C) K^{(0)} + \dots \quad (4.2.12)$$

And for the transition function we have [17]:

$$\Gamma = \frac{1}{1 - \int \mathcal{P}_C K} = 1 + \int \mathcal{P}_C K + \int \mathcal{P}_C K \mathcal{P}_C K + \dots \quad (4.2.13)$$

Where the kernel K is equal to

$$K = \int K^{(0)} G \quad (4.2.14)$$

Then the coefficient function C in the factorisation formula Eq (4.2.4) equals

$$C = \int C^{(0)} G \quad (4.2.15)$$

The projector operator can be expressed as $\mathcal{P}_C = \mathcal{P}_C^\varepsilon \otimes \mathcal{P}_C^s$ (Ref [17] p484), with \mathcal{P}_C^s and $\mathcal{P}_C^\varepsilon$ respectively the helicity and momentum space projectors.

Given two kernels A and B , connected by a parton with momentum k . The action of the helicity space projector \mathcal{P}_C^s on the helicity space, when the connecting parton is a quark, is equal to [17]:

$$\int A \mathcal{P}_C^s B = \int A_{\alpha'\beta'} \left(\frac{1}{2} \not{k} \right)_{\alpha'\beta'} \left(\frac{\bar{\not{p}}}{2\bar{p} \cdot k} \right)_{\alpha\beta} B_{\alpha\beta} \quad (4.2.16)$$

When the connecting parton is a gluon, the action of \mathcal{P}_C^s will be [17]:

$$\int A \mathcal{P}_C^s B = \int A_{\mu\nu\nu'} \frac{d^{\mu\nu\nu'}}{n-2} (-g^{\mu\nu}) B_{\mu\nu} \quad (4.2.17)$$

Which contains the sum over gluon polarisations, which is given by Eq (4.2.7). The momentum space operator $\mathcal{P}_C^\varepsilon$ will set $k^\mu = zp^\mu$ on A , and perform a k -momentum integration up to the factorisation scale μ_F on B . It also extracts the $1/\varepsilon$ poles. [17] Once $\mathcal{P}_C^\varepsilon$ is uniquely defined, it completely specifies the factorisation scheme.

A very common factorisation scheme is the $\overline{\text{MS}}$ scheme (Modified Minimal Substraction scheme). In this scheme, the transition functions have the expression [17]:

$$\Gamma_{ab,N} = \left[P_\alpha \exp \left(\frac{1}{\varepsilon} \int_0^h \frac{d\alpha}{\alpha} \gamma_N(\alpha) \right) \right]_{ab} \quad (4.2.18)$$

With γ_N the anomalous dimension, and with $h = \alpha_s (\mu_F^2 / \mu^2)^\varepsilon S_\varepsilon$, where S_ε is an ε -finite factor.

The effect of the projectors can be represented by the following Feynman diagrams:

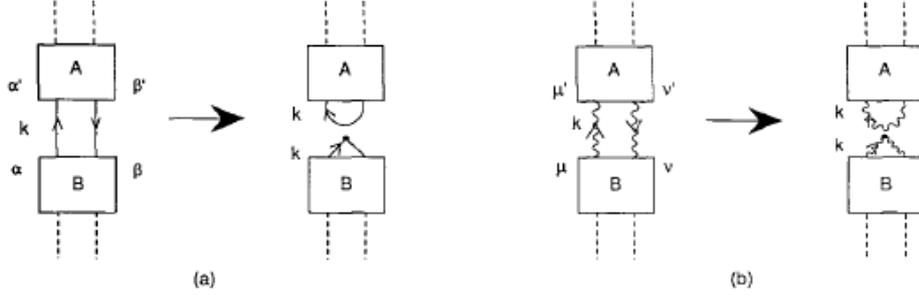


Figure 4.2: Effect of the projector operator \mathcal{P}_C on the spin indices of kernels A and B, in case of (a) a quark and (b) a gluon. Time runs from bottom to top. Figure from [17] (p485)

4.2.2 Factorisation at high energy

While in collinear factorisation we only had one scale, Q^2 , we will have two scales Q^2 and s in high energy factorisation. High energy means that $x = Q^2/s$ is small. We can do a Mellin transform between x -space and N -space (Eq (3.1.2) & Eq (3.1.3)), and small x will mean small N . Just like in the collinear case we have that α_s is small. Since both α_s and N are small, we will keep α_s/N fixed. Unlike in the collinear case, we cannot expand in powers of α_s now. Instead we need to expand in α_s/N .

For factorisation at high energy, we can also use Sudakov parametrisation, axial gauge, dimensional regularisation and 2PI kernels again. It is the k -integration from the gluon Green's functions $G_{qa}^{(0)}$ that generate the large perturbative $(\alpha_s/N)^k$. [17] This means we will specifically use kernels that are two-gluon irreducible (2GI). These kernels will be transverse momentum dependent (TMD). Multiple gluon exchange will generate logarithmic high energy (or small x) contributions [17]. Expansion in 2GI will allow us to perform a power counting at high energy.

We now want to decouple the 2GI kernels and the gluon Green's functions while keeping α_s/N fixed. [17] k_\perp factorisation can extract leading high energy behaviour, by the use of the k_\perp dependent projection operator \mathcal{P}_H [17]:

$$\int A_g G_{ga}^{(0)} = \int A_g \mathcal{P}_H G_{ga}^{(0)} + \dots \quad (4.2.19)$$

where A_g is the kernel. Similar as with the collinear projection operators, \mathcal{P}_H this projection operator can be expressed with a momentum space projector and a helicity (spin) projector, as $\mathcal{P}_H = \mathcal{P}_H^\varepsilon \otimes \mathcal{P}_H^s$. Diagrammatically, the action of the projection operator will be the same as Figure 4.2. However when it comes to the formulas, the action will be different than the collinear case. The action of \mathcal{P}_H^s will be [17]:

$$\int A_g \mathcal{P}_H^s G_{ga}^{(0)} = \int A_g^{\mu\nu\rho} \frac{k_{\perp\mu} k_{\perp\nu\rho}}{\mathbf{k}^2} (-g_{\mu\nu}) G_{ga}^{(0)\mu\nu} \quad (4.2.20)$$

This time we don't sum over all polarisations, but only over $k_{\perp\mu}, k_{\perp\nu\rho}/\mathbf{k}^2$.

$\mathcal{P}_H^\varepsilon$ will set $k^\mu = zp^\mu + k_\perp^\mu$ on A_g , and on $G_{ga}^{(0)}$ it will perform an integration over k^2 at fixed k_\perp . [17] The 2GI kernel has to be evaluated with an incoming gluon that is off-shell ($k^2 = -\mathbf{k}^2$).

Since \mathcal{P}_H is a true projection operator ($\mathcal{P}_H^2 = \mathcal{P}_H$), we are not neglecting any contributions of order $(\alpha_s/N)^k \varepsilon$. It also has the property $\mathcal{P}_H \supseteq \mathcal{P}_C$ (and $\mathcal{P}_H = \mathcal{P}_C$ if $k_\perp = 0$). [17] This means that consistency between collinear and high energy factorisation is guaranteed.

The high energy factorisation leads to a resummation of logarithms $\ln(s/Q^2)$. This resummation leads to the BFKL equation. In the BFKL equation we only use the unintegrated gluon distribution function because it is the only parton distribution function that picks α_s/N . This is considered the lowest log high energy factorisation. In next-to-leading log, we would have to include quark parton distribution functions as well.

If we compare the collinear and high energy factorisation schemes, collinear factorisation does only a resummation of $\ln(Q^2/\mu^2)$, while high energy factorisation does only a resummation of $\ln(s/Q^2)$. Trying to merge the two factorisations is nontrivial, and requires TMD splitting functions.

4.2.3 BFKL equation

The resummation of logarithms $\ln(s/Q^2)$ in high energy factorisation leads to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution equation. The BFKL equation in Mellin transformed space, and in $d = 4 + 2\varepsilon$ dimensional regularisation, can be expressed as [17]:

$$\begin{aligned} \mathcal{F}^{(0)}(N, \mathbf{k}, \alpha_s, \mu, \varepsilon) = & \delta^{(2+2\varepsilon)}(\mathbf{k}) + \frac{C_A \alpha_s}{\pi^2 N} \int \frac{d^{2+2\varepsilon} \mathbf{q}}{(2\pi\mu)^{2\varepsilon}} \frac{1}{\mathbf{q}^2} \left[\mathcal{F}^{(0)}(N, \mathbf{k} - \mathbf{q}, \alpha_s, \mu, \varepsilon) \right. \\ & \left. - \frac{\mathbf{k} \cdot (\mathbf{k} - \mathbf{q})}{(\mathbf{k} - \mathbf{q})^2} \mathcal{F}^{(0)}(N, \mathbf{k}, \alpha_s, \mu, \varepsilon) \right] \end{aligned} \quad (4.2.21)$$

With $\mathcal{F}^{(0)}(N, \mathbf{k}, \alpha_s, \mu, \varepsilon)$ the N -moment unintegrated gluon distribution function in Mellin transformed space, and where the boldface letters are defined as

$$k_\perp^\mu = (0, \mathbf{k}, 0), \quad q_\perp^\mu = (0, \mathbf{q}, 0) \quad (4.2.22)$$

The momentum q is internal in the kernel.

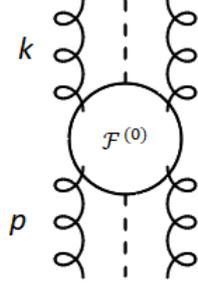


Figure 4.3: The corresponding diagram for the BFKL equation. Time runs from bottom to top.

The unintegrated gluon PDF can also be interpreted as an unintegrated gluon Green's function:

$$G_{gg}^{(0)} = \int d\mathbf{k} \mathcal{F}^{(0)} \quad (4.2.23)$$

With $G_{gg}^{(0)}$ the gluon Green's function. Resummation of terms in $(\alpha_s/\ln(\hat{s}/M^2))^n$ to all orders results in the following solution for the gluon-to-gluon anomalous dimension [17]:

$$\gamma_{N,gg} = \gamma_N(\alpha_s) + \mathcal{O}(\alpha_s(\alpha_s/N)^l) \quad (4.2.24)$$

With $\gamma_N(\alpha_s)$ the solution of the following implicit equation:

$$1 = \frac{C_A \alpha_s}{\pi N} \chi(\gamma_N(\alpha_s)) \quad (4.2.25)$$

Where the function χ can be expressed in terms of the Euler ψ -function:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) = \frac{1}{\gamma} \left(1 + \sum_{l=1}^{\infty} 2\zeta(2l+1)\gamma^{2l+1} \right) \quad (4.2.26)$$

Where ζ is the Riemann ζ -function.

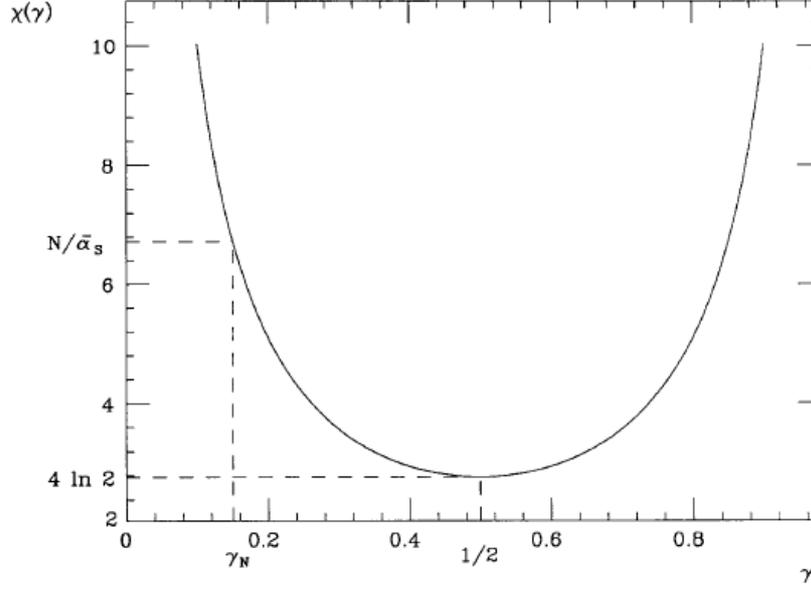


Figure 4.4: The characteristic function χ . Figure from [17]

Eq (4.2.26) results in the following power series for the anomalous dimension [17]:

$$\gamma_N(\alpha_s) = \sum_{n=1}^{\infty} g_n \left(\frac{C_A \alpha_s}{\pi N} \right)^n = \frac{C_A \alpha_s}{\pi N} + 2\zeta(3) \left(\frac{C_A \alpha_s}{\pi N} \right)^4 + 2\zeta(5) \left(\frac{C_A \alpha_s}{\pi N} \right)^6 + \mathcal{O} \left(\left(\frac{C_A \alpha_s}{\pi N} \right)^7 \right) \quad (4.2.27)$$

Note that powers of $1/N$ correspond to powers of $\ln(1/x)$ via Mellin transformation, and remind that $\ln(1/x)$ is equal to $\ln(s/Q^2)$. Remind that the anomalous dimension $\gamma_N(\alpha_s)$ is the Mellin transform of the splitting function, Eq (3.1.17). This means we get the following solution for the BFKL gluon-to-gluon splitting function:

$$P_{gg}(z, \alpha_s) = \frac{C_A \alpha_s}{\pi z} + 2\zeta(3) \left(\frac{C_A \alpha_s}{\pi} \right)^4 \frac{1}{z} (\ln(z))^3 + \dots \quad (4.2.28)$$

The first term is the leading order DGLAP term. The second term is the first BFKL correction, followed by the rest of the BFKL series.

The shortcoming of high energy factorisation and BFKL, is that it is limited to the low M^2/\hat{s} regime. Problems arise when trying to extrapolate it into the intermediate and higher M^2/\hat{s} regime. [19] For example elementary vertices that violate energy conservation, and quark contributions arising only in next-to-leading order (NLO). [19]

4.2.4 Kernel for gluon-to-quark splitting function

The quark Green function has the following expansion in 2GI kernels [17]:

$$G_{qa}^{(0)} = \int K_{qb}^{(0)} G_{ba}^{(0)} \quad (4.2.29)$$

By applying the high-energy factorisation formula Eq (4.2.19), it is possible to get the following k_{\perp} factorisation formula [17]:

$$G_{qa}^{(0)\alpha\beta}(q, p) = \int d^{2+2\varepsilon} \mathbf{k} \int_0^1 \frac{dy}{y} \left(\widehat{K}_{qa}^{(0)\alpha\beta}(q, p) \frac{k_{\perp}^{\mu} k_{\perp}^{\nu}}{\mathbf{k}^2} \right)_{k=y p+k_{\perp}} \cdot \mathcal{F}^{(0)}(y, \mathbf{k}, \alpha_s, \mu, \varepsilon) \quad (4.2.30)$$

With $\widehat{K}_{qa}^{(0)}$ the off-shell kernel and $\mathcal{F}^{(0)}$ the unintegrated parton distribution function.

When it comes to Feynman diagrams, the high-energy factorisation results in the following diagrams for the gluon-to-quark Green function:

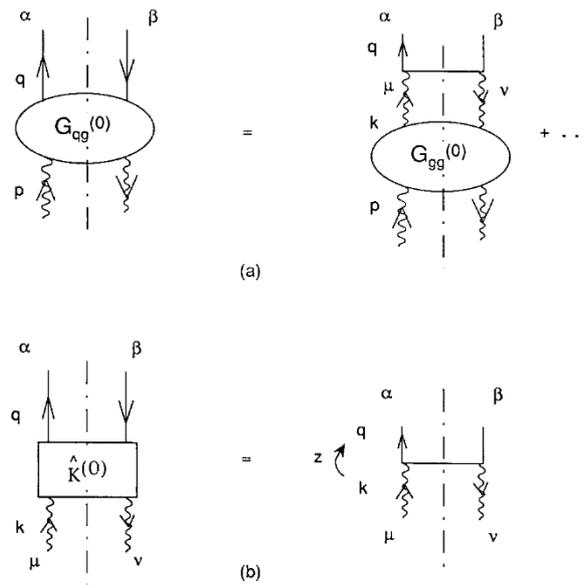


Figure 4.5: (a) High-energy factorisation of the gluon-to-quark Green function G_{qg}

(b) The corresponding off-shell kernel \widehat{K}_{qg}

Time runs from bottom to top.

Figure from [17] (p498)

Figure 4.5, which is from [17], and more specifically the lower right diagram (figure 4.5(b)), will be the starting point of our actual calculation for the gluon-to-quark splitting function. We now have the analogue of the k_{\perp} -factorisation formula, Ref [17] (p498, 4.2)

For the gluon-to-quark kernel we have

$$\hat{K}_{qg} \sim \int [dq] \hat{P}_{qg} \quad (4.2.31)$$

With \hat{P}_{qg} an unintegrated (TMD) splitting function. The notation above means that it integrates over unintegrated momenta or angles depending on what you want to average over. For example this could be the fully unintegrated TMD splitting function, or the angular averaged TMD splitting function, etc. Typically this will at least be an integral over the transverse momentum q_{\perp} , but possibly also over angle and other variables.

4.2.5 Identifying splitting functions in kernels

We can express a TMD kernel in the following way, while doing dimensional regularisation [19]:

$$\hat{K}_{ij} \left(z, \frac{\mathbf{k}^2}{\mu^2}, \varepsilon, \alpha_s \right) = z \int \frac{d(q^2) d^{2+2\varepsilon} \mathbf{q}}{2(2\pi)^{4+2\varepsilon}} \theta(\mu_F^2 + q^2) \mathcal{P}_{j,in} \otimes \hat{K}_{ij}^{(0)}(q, k) \otimes \mathcal{P}_{i,out} \quad (4.2.32)$$

With $\hat{K}_{ij}^{(0)}$ the squared matrix element for a parton of type j going to a parton of type i . μ_F^2 is the factorisation scale, and $\theta(\mu_F^2 + q^2)$ ensures that $\mu_F^2 + q^2$ remains larger than 0. Finding the splitting function comes down to identifying it in the integral expression for a kernel \hat{K}_{ij} . The equation that relates \hat{K}_{ij} to \hat{P}_{ij} is generally of the form

$$\hat{K}_{ij} \sim \int [dq] \hat{P}_{ij} \quad (4.2.33)$$

For Eq (4.2.32) we have [19]:

$$\hat{K}_{ij} \left(z, \frac{\mathbf{k}^2}{\mu^2}, \varepsilon, \alpha_s \right) = \frac{\alpha_s}{2\pi} z \frac{e^{-\varepsilon\gamma_E}}{\mu^{2\varepsilon}} \int \frac{d^{2+2\varepsilon} \tilde{q}}{\pi^{1+\varepsilon} \tilde{q}^2} \hat{P}_{ij}^{(0)} \theta(\mu_F^2 + q^2) \quad (4.2.34)$$

With $\hat{P}_{ij}^{(0)}$ the TMD splitting function (LO coefficient in the α_s expansion for the TMD splitting function)

In the next section we will calculate the $\hat{P}_{qg}^{(0)}$ splitting function.

4.3 From Feynman diagram to splitting function

We discussed the factorisation schemes and how they can be realised by projection operators, and that the projection operators we introduced are consistent with each other despite different factorisation schemes. This will be used to extend the use of projection operators to a situation where we take the transverse momentum into account from the start to the end, and which can work in the small x regime as well as

the collinear regime. This section will contain the detailed calculation of the TMD gluon-to-quark splitting function. During the calculation, we will do dimensional regularisation by working in a general dimensional spacetime.

4.3.1 Optical theorem and applying Feynman rules

The gluon-to-quark splitting function's kernel corresponds to the diagram:

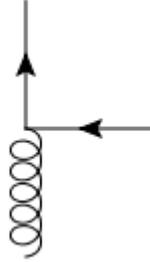


Figure 4.6: Gluon-to-quark splitting. (Time runs from bottom to top)

We will refer to this diagram as Y . The lower right diagram of figure 4.5 will be referred to as H . Due to the optical theorem [5] (p232-236) we have, conceptually represented, the following relation between the two diagrams:

$$\int d\Pi |Y|^2 = 2\text{Im}(H) \quad (4.3.1)$$

And expressed with the T-matrix this becomes

$$|T_{g \rightarrow q}|^2 = 2\text{Im}(T(H)) \quad (4.3.2)$$

With $T_{g \rightarrow q}$ the T-matrix for the Y diagram, and $T(H)$ the T-matrix for the H diagram. Let us call $T := T(Y) := |T_{g \rightarrow q}|^2$, $T' := T(H)$, and we can also write it as

$$T = 2\text{Im}(T') \quad (4.3.3)$$

We can express the S-matrix for the process as $S' = \mathbf{1} + T'$, where $\mathbf{1}$ is the identity matrix. We have

$$\langle \text{final} | iT' | \text{initial} \rangle = (2\pi)^4 \delta \left(\sum k_{\text{initial}} - \sum k_{\text{final}} \right) i\mathcal{M}' \quad (4.3.4)$$

We will now calculate the H diagram, which relates to the Y diagram via the optical theorem. The Y diagram corresponds to the gluon-to-quark splitting function.

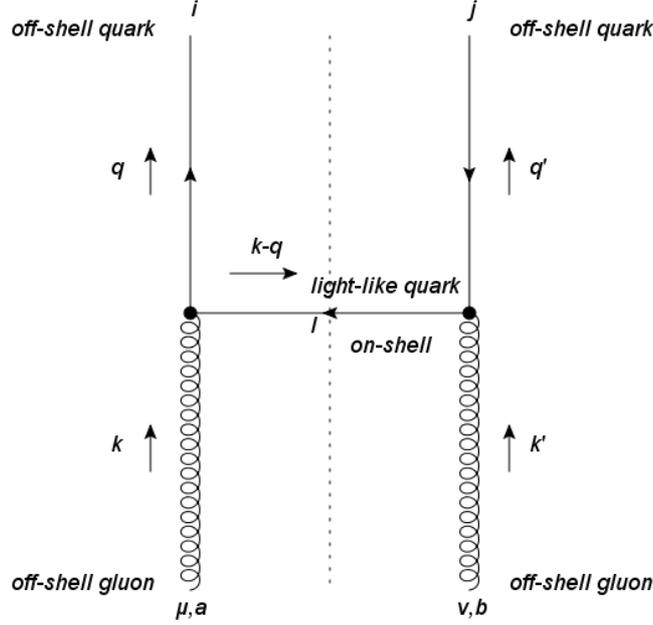


Figure 4.7: More detailed version of the H-diagram, corresponding to the off-shell kernel \hat{K}_{qg} . (Time runs from bottom to top.)

As illustrated on figure 4.7, the external particles will be off-shell. The only on-shell particle is the intermediate quark, which is on-shell due to the optical theorem, [5] (p232-236). The momentum conservation requires $k - q = q' - k'$, and we have for the matrix element:

$$\langle q, q' | iT' | k, k' \rangle = (2\pi)^4 \delta(k + k' - q - q') i\mathcal{M}' \quad (4.3.5)$$

We use Sudakov parametrisation:

$$k^\mu = yp^\mu + k_\perp^\mu, \quad q^\mu = xp^\mu + q_\perp^\mu + \frac{q^2 + \mathbf{q}^2}{2xp \cdot \bar{p}} \bar{p}^\mu, \quad z = x/y \quad (4.3.6)$$

$$k_\perp^\mu = (0, \mathbf{k}, 0), \quad p^\mu = P(1, \mathbf{0}, 1), \quad \bar{p}^\mu = P(1, \mathbf{0}, -1), \quad q_\perp^\mu = (0, \mathbf{q}, 0)$$

In this chapter, the symbols that are boldface are not three-momenta, even though they are similar to three-momenta. In fact, they are actually two-momenta, which is why we will also represent them as boldface symbols.

There are several properties and relations among the momenta:

$$k^2 = k_\perp^2 = -\mathbf{k}^2, \quad p^2 = 0, \quad \bar{p}^2 = 0 \quad (4.3.7)$$

$$k_\perp^\mu k_\mu = k_\perp^2, \quad k_\perp^\mu q_\mu = k_\perp^\mu q_{\perp\mu}, \quad k_\perp^\mu p_\mu = k_\perp^\mu \bar{p}_\mu = 0, \quad q_\perp^\mu p_\mu = q_\perp^\mu \bar{p}_\mu = 0$$

We also define the boost invariant transverse momentum

$$\tilde{\mathbf{q}} = \mathbf{q} - z\mathbf{k}, \quad \tilde{q}^\mu = q^\mu - zk^\mu \quad (4.3.8)$$

Due to the projector operators, we will have to take the external quarks their propagators into account as well.

For the off-shell quarks:

$$S(q) = \frac{i\delta^{ii'}}{\not{q} - m + i\epsilon} = \frac{i\delta^{ii'}(\not{q} + m)}{q^2 - m^2 + i\epsilon} \quad (4.3.9)$$

$$S(-q') = \frac{i\delta^{jj'}}{-\not{q}' - m + i\epsilon} = \frac{i\delta^{jj'}(-\not{q}' + m)}{q'^2 - m^2 + i\epsilon}$$

For the on-shell quark, we can apply Cutkosky cutting rules thanks to the cut provided by the optical theorem, [5] (p232-236):

$$S(q - k) = -2\pi i\delta^{ll'}(\not{q} - \not{k} + m - i\epsilon)\delta((q - k)^2 - m^2) \quad (4.3.10)$$

Now we write the expression for our gluon-to-quark splitting function kernel, and this for a general dimension (for dimensional regularisation). In a $D = 4 + 2\epsilon$ dimensional spacetime, we have the following integral expression for the kernel:

$$\widehat{K}_{qg} = \frac{1}{2} \int \frac{d(q^2)}{(2\pi)^{4+2\epsilon}} \int d^{2+2\epsilon} q_{\perp} \text{Tr}(\mathcal{P}_{g,in}^{\mu\nu}(k)\mathcal{P}_{q,out}(q)i\mathcal{M}') \quad (4.3.11)$$

With $\mathcal{P}_{g,in}^{\mu\nu}(k)$ and $\mathcal{P}_{q,out}(q)$ the projector operators that connect the two gluons at the bottom of the diagram, and connect the two quarks at the top of the diagram. (Fig 4.5) The Feynman amplitude \mathcal{M}' will be averaged over spin, polarisations and colours.

From the Feynman rules we find the following expression for \mathcal{M}' :

$$i\mathcal{M}' = S(q)ig\gamma_{\mu}T_{ii}^aS(q - k)ig\gamma_{\nu}T_{ij}^bS(-q') \quad (4.3.12)$$

4.3.2 Projection operators and trace

We will now define X to be the product of the propagators and the vertex factors, and with the projector operators acting on them:

$$\begin{aligned} X &= \mathcal{P}_{g,in}^{\mu\nu}(k)\mathcal{P}_{q,out}(q)i\mathcal{M}' \quad (4.3.13) \\ &= \mathcal{P}_{g,in}^{\mu\nu}(k)\mathcal{P}_{q,out}(q)S(q)ig\gamma_{\mu}T_{ii}^aS(q - k)ig\gamma_{\nu}T_{ij}^bS(-q') \\ &= -\mathcal{P}_{g,in}^{\mu\nu}(k)\mathcal{P}_{q,out}(q)\frac{i}{\not{q} - m + i\epsilon}ig\gamma_{\mu}T_{ii}^a2\pi i(\not{q} - \not{k} + m) \cdot \\ &\quad \cdot \delta((q - k)^2 - m^2)ig\gamma_{\nu}T_{ij}^b\frac{i}{-\not{q}' - m + i\epsilon} \end{aligned}$$

This means we can write the splitting function kernel as

$$\widehat{K}_{qg} = \frac{1}{2} \int \frac{d(q^2)}{(2\pi)^{4+2\varepsilon}} \int d^{2+2\varepsilon} q_{\perp} \text{Tr}(X) \quad (4.3.14)$$

For the projector operators, we will use the choice of Catani and Hautmann [19] (eq (43)) for $k^{\mu} = yp^{\mu} + k_{\perp}^{\mu}$

$$\mathcal{P}_{g,in}^{\mu\nu}(k) = -\frac{k_{\perp}^{\mu} k_{\perp}^{\nu}}{k_{\perp}^2} \frac{\delta^{ab}}{N^2 - 1} \quad (4.3.15)$$

$$\mathcal{P}_{q,out}(q) = \frac{\overline{\not{p}}}{2\bar{p}q} \delta_{ij} \delta_{q,-q} \quad (4.3.16)$$

The factor $\frac{\delta^{ab}}{N^2-1}$ arises because we will sum over the gluon's colour indices. If we would keep $a = b$ fixed from the start, then this factor wouldn't be needed.

Plugging this in X and carrying out some of the index contractions, gives us

$$X = i \frac{\pi g^2}{N^2 - 1} \frac{k_{\perp}^{\mu} k_{\perp}^{\nu}}{k_{\perp}^2} \frac{\overline{\not{p}}}{\bar{p}q} \frac{1}{\not{q} - m + i\varepsilon} \gamma_{\mu} T_{ii}^a (\not{q} - \not{k} + m) \cdot \delta((q - k)^2 - m^2) \gamma_{\nu} T_{ii}^a \frac{1}{\not{q} - m + i\varepsilon} \quad (4.3.17)$$

After rewriting this by changing the order of some factors, and using the rightmost forms of Eq (4.3.9), we get

$$X = i \frac{\pi g^2}{N^2 - 1} T_{ii}^a T_{ii}^a \frac{\overline{\not{p}}}{k_{\perp}^2 \bar{p}q} \frac{\delta((q - k)^2 - m^2)}{(q^2 - m^2 + i\varepsilon)^2} (\not{q} + m) \not{k}_{\perp} (\not{q} - \not{k} + m) \not{k}_{\perp} (\not{q} + m) \quad (4.3.18)$$

Remind that we cannot simply change the order of factors that contain gamma matrices (this also includes the factors that have a slash through). We also need to keep the order of the group generators, but in this case it doesn't matter because it happens to be symmetric. We use

$$T_{ii}^a T_{ii}^a = C_F N = T_F (N^2 - 1) \quad (4.3.19)$$

, and X is now equal to

$$X = i \frac{\pi g^2 T_F}{k_{\perp}^2 \bar{p}q} \frac{\delta((q - k)^2 - m^2)}{(q^2 - m^2 + i\varepsilon)^2} \overline{\not{p}} (\not{q} + m) \not{k}_{\perp} (\not{q} - \not{k} + m) \not{k}_{\perp} (\not{q} + m) \quad (4.3.20)$$

We are dealing with ultrarelativistic quarks. This means we can approximate $m = 0$, for both the on-shell and off-shell quarks. We could have done this from the start,

especially since we already knew we are working in the infinite momentum frame and all masses are negligible compared to the hard momenta involved. But for the sake of being more general, the mass was kept until this point. After setting the mass equal to 0, our expression will become

$$X = i \frac{\pi g^2 T_F}{k_{\perp}^2 \bar{p} q} \frac{\delta((q-k)^2)}{(q^2 + i\epsilon)^2} \bar{p} \not{\epsilon} \not{k}_{\perp} (\not{q} - \not{k}) \not{k}_{\perp} \not{q} \quad (4.3.21)$$

We will need to take the trace of this if we want to calculate the splitting function in Eq (4.3.15). We will define a function Z that only deals with the gamma matrices (contracted with momenta) and the trace:

$$Z := \text{Tr} \left(\bar{p} \not{\epsilon} \not{k}_{\perp} (\not{q} - \not{k}) \not{k}_{\perp} \not{q} \right) \quad (4.3.22)$$

This means we can now write $\text{Tr}(X)$ as

$$\text{Tr}(X) = i \frac{\pi g^2 T_F}{k_{\perp}^2 \bar{p} q} \frac{\delta((q-k)^2)}{(q^2 + i\epsilon)^2} Z \quad (4.3.23)$$

Calculating Z will require the use of trace identities for the gamma matrices. The ones that we need are [5]

$$\text{Tr}(\mathbf{A} \mathbf{B} \mathbf{C} \mathbf{D}) = 4((\mathbf{AB})(\mathbf{CD}) - (\mathbf{AC})(\mathbf{BD}) + (\mathbf{AD})(\mathbf{BC})) \quad (4.3.24)$$

$$\text{Tr}(\mathbf{A} \mathbf{B} \mathbf{C} \mathbf{D} \mathbf{E} \mathbf{F}) = 4(\mathbf{AB})((\mathbf{CD})(\mathbf{EF}) - (\mathbf{CE})(\mathbf{DF}) + (\mathbf{CF})(\mathbf{DE})) \quad (4.3.25)$$

$$-4(\mathbf{AC})((\mathbf{BD})(\mathbf{EF}) - (\mathbf{BE})(\mathbf{DF}) + (\mathbf{BF})(\mathbf{DE}))$$

$$+4(\mathbf{AD})((\mathbf{BC})(\mathbf{EF}) - (\mathbf{BE})(\mathbf{CF}) + (\mathbf{BF})(\mathbf{CE}))$$

$$-4(\mathbf{AE})((\mathbf{BC})(\mathbf{DF}) - (\mathbf{BD})(\mathbf{CF}) + (\mathbf{BF})(\mathbf{CD}))$$

$$+4(\mathbf{AF})((\mathbf{BC})(\mathbf{DE}) - (\mathbf{BD})(\mathbf{CE}) + (\mathbf{BE})(\mathbf{CD}))$$

These specific relations don't involve contraction between gamma matrices themselves, only contractions between momentum and gamma matrix. Because of this, these relations are independent of the dimension.

Using these trace identities in Z will lead to

$$Z = \text{Tr} \left(\bar{p} \not{\epsilon} \not{k}_{\perp} (\not{q} - \not{k}) \not{k}_{\perp} \not{q} \right) \quad (4.3.26)$$

$$= 4\{2(\bar{p}q)[2(qk_{\perp})(k_{\perp}(q-k)) - (q(q-k))k_{\perp}^2] - 2(\bar{p}k_{\perp})q^2(k_{\perp}(q-k))$$

$$+(\bar{p}(q-k))q^2k_{\perp}^2\}$$

$$= 4\{k_{\perp}^2(\bar{p}q)(q(2k-q)) - k_{\perp}^2q^2(\bar{p}k) + 4(\bar{p}q)(k_{\perp}q)(k_{\perp}(q-k)) - 2q^2(\bar{p}k_{\perp})(k_{\perp}(q-k))\}$$

We will use the properties of the parametrisation that we use, from Eq (4.3.7) we know

$$k_{\perp}k = k_{\perp}^2, \quad k_{\perp}q = k_{\perp}q_{\perp}, \quad k_{\perp}p = k_{\perp}\bar{p} = 0$$

This means that for our Z we now have

$$Z = 4\{k_{\perp}^2(\bar{p}q)(2kq - q^2 - 4k_{\perp}q_{\perp}) - k_{\perp}^2q^2(\bar{p}k) + 4(\bar{p}q)(k_{\perp}q_{\perp})^2\} \quad (4.3.27)$$

From our parametrisation we can also deduce the following relations:

$$\bar{p}k = y\bar{p}p + \bar{p}k_{\perp} = y\bar{p}p \quad (4.3.28)$$

$$\bar{p}q = x\bar{p}p + \bar{p}q_{\perp} + \frac{q^2 + \mathbf{q}^2}{2xp \cdot \bar{p}}\bar{p}^2 = x\bar{p}p \quad (4.3.29)$$

Because $\bar{p}k_{\perp} = \bar{p}q_{\perp} = \bar{p}^2 = 0$

Substituting this in Z yields

$$\begin{aligned} Z &= 4\bar{p}p\{k_{\perp}^2[x(2kq - q^2 - 4k_{\perp}q_{\perp}) - yq^2] + 4x(k_{\perp}q_{\perp})^2\} \\ &= 4x\bar{p}p\left(k_{\perp}^2\left(2kq - \left(1 + \frac{1}{z}\right)q^2 - 4k_{\perp}q_{\perp}\right) + 4(k_{\perp}q_{\perp})^2\right) \end{aligned} \quad (4.3.30)$$

Where in the last expression we used $z = x/y$

Another relation we have from our parametrisation, is

$$kq = k_{\perp}q_{\perp} + \frac{q^2 + \mathbf{q}^2}{2z} \quad (4.3.31)$$

So now we can write Z as

$$Z = 4\bar{p}q\left(k_{\perp}^2\left(\frac{\mathbf{q}^2}{z} - 2k_{\perp}q_{\perp} - q^2\right) + 4(k_{\perp}q_{\perp})^2\right) \quad (4.3.32)$$

Where we have changed the factor $x\bar{p}p$ back to $\bar{p}q$. Now that we have done the trace part, and simplified by using the parametrisation, we insert Z into Eq (4.3.23):

$$\begin{aligned} \text{Tr}(X) &= i \frac{\pi g^2 T_F}{k_{\perp}^2 \bar{p}q} \frac{\delta((q-k)^2)}{(q^2 + i\varepsilon)^2} Z \\ &= 4\pi i g^2 T_F \frac{\delta((q-k)^2)}{(q^2 + i\varepsilon)^2} \left(\frac{\mathbf{q}^2}{z} - 2k_{\perp}q_{\perp} - q^2 + 4\frac{(k_{\perp}q_{\perp})^2}{k_{\perp}^2}\right) \end{aligned} \quad (4.3.33)$$

We can choose to express $\text{Tr}(X)$ either in k_{\perp} and q_{\perp} , or in \mathbf{k} and \mathbf{q} , via the following relations that we have from the parametrisation:

$$k^2 = k_{\perp}^2 = -\mathbf{k}^2, \quad q^2 = -\mathbf{q}^2, \quad k_{\perp}q_{\perp} = -\mathbf{kq} \quad (4.3.34)$$

Remember that \mathbf{k} and \mathbf{q} each are “two-vectors”, so each have 2 components in the same plane. If we express $\text{Tr}(X)$ in \mathbf{k} and \mathbf{q} , we have

$$\text{Tr}(X) = 4\pi i g^2 T_F \frac{\delta((q-k)^2)}{(q^2 + i\varepsilon)^2} \left(\frac{\mathbf{q}^2}{z} + 2\mathbf{kq} - q^2 - 4 \frac{(\mathbf{kq})^2}{\mathbf{k}^2} \right) \quad (4.3.35)$$

If we work out the argument of the delta function, we get

$$\delta((q-k)^2) = \delta(q^2 - 2kq + k^2) = \delta\left(\frac{z-1}{z}q^2 - \mathbf{k}^2 - \frac{\mathbf{q}^2}{z} + 2\mathbf{kq}\right) \quad (4.3.36)$$

We will now use the following property of the delta function, which will allow us to rewrite $\text{Tr}(X)$ in a way that will be useful when we integrate it:

$$\delta(ax) = \frac{1}{|\alpha|} \delta(x) \quad (4.3.37)$$

This means we have

$$\delta\left(\frac{z-1}{z}q^2 - \mathbf{k}^2 - \frac{\mathbf{q}^2}{z} + 2\mathbf{kq}\right) = \left|\frac{z}{1-z}\right| \delta\left(q^2 + \frac{z}{1-z}(\mathbf{k}^2 - 2\mathbf{kq}) + \frac{\mathbf{q}^2}{1-z}\right) \quad (4.3.38)$$

Since $0 \leq z \leq 1$, we can leave out the absolute values. Using the expression above, we can write $\text{Tr}(X)$ as:

$$\text{Tr}(X) = 4\pi i g^2 T_F \frac{z}{1-z} \frac{\frac{\mathbf{q}^2}{z} + 2\mathbf{kq} - q^2 - 4 \frac{(\mathbf{kq})^2}{\mathbf{k}^2}}{(q^2 + i\varepsilon)^2} \delta\left(q^2 + \frac{z}{1-z}(\mathbf{k}^2 - 2\mathbf{kq}) + \frac{\mathbf{q}^2}{1-z}\right) \quad (4.3.39)$$

4.3.3 Integration over q^2

Now we go back to the expression for the splitting function kernel, Eq (4.3.14):

$$\hat{K}_{qg} = \frac{1}{2} \int \frac{d(q^2)}{(2\pi)^{4+2\varepsilon}} \int d^{2+2\varepsilon} q_{\perp} \text{Tr}(X) \quad (4.3.14)$$

We will plug $\text{Tr}(X)$ into Eq (4.3.14), and first perform the integral over $d(q^2)$. The kernel will now be expressed as

$$\hat{K}_{qg} = \frac{1}{2} \int \frac{d^{2+2\varepsilon} q_{\perp}}{(2\pi)^{4+2\varepsilon}} I \quad (4.3.40)$$

Where we introduce the integral I which we define as:

$$I = \int d(q^2) \text{Tr}(X) = 4\pi i g^2 T_F \frac{z}{1-z} \int d(q^2) \frac{\frac{\mathbf{q}^2}{z} + 2\mathbf{kq} - q^2 - 4 \frac{(\mathbf{kq})^2}{\mathbf{k}^2}}{(q^2 + i\varepsilon)^2} \cdot \delta\left(q^2 + \frac{z}{1-z}(\mathbf{k}^2 - 2\mathbf{kq}) + \frac{\mathbf{q}^2}{1-z}\right) \quad (4.3.41)$$

We will now solve this integral. Note that there are no other denominators aside from $1/(q^2 + i\varepsilon)^2$, and we can take a contour in a way that ε wouldn't matter. So we will have to solve the following integral:

$$I = 4\pi i g^2 T_F \frac{z}{1-z} \int d(q^2) \frac{\frac{\mathbf{q}^2}{z} + 2\mathbf{kq} - q^2 - 4 \frac{(\mathbf{kq})^2}{\mathbf{k}^2}}{q^4} \cdot \delta\left(q^2 + \frac{z}{1-z}(\mathbf{k}^2 - 2\mathbf{kq}) + \frac{\mathbf{q}^2}{1-z}\right) \quad (4.3.42)$$

If we integrate now, the delta function will set its argument to 0, which means

$$q^2 + \frac{z}{1-z}(\mathbf{k}^2 - 2\mathbf{kq}) + \frac{\mathbf{q}^2}{1-z} = 0 \quad (4.3.43)$$

This gives us the following expression for q^2

$$q^2 = \frac{z}{z-1}(\mathbf{k}^2 - 2\mathbf{kq}) + \frac{\mathbf{q}^2}{z-1} = \frac{1}{z-1}(z(\mathbf{k}^2 - 2\mathbf{kq}) + \mathbf{q}^2) \quad (4.3.44)$$

And for the integral we now have

$$I = 4\pi i g^2 T_F \frac{z}{1-z} \frac{\frac{z}{1-z}\mathbf{k}^2 + \frac{\mathbf{q}^2}{z(1-z)} + 2\frac{2z-1}{z-1}\mathbf{kq} - 4\frac{(\mathbf{kq})^2}{\mathbf{k}^2}}{\left(\frac{1}{z-1}(z(\mathbf{k}^2 - 2\mathbf{kq}) + \mathbf{q}^2)\right)^2} \\ = 4\pi i g^2 T_F \frac{z^2\mathbf{k}^2 + \mathbf{q}^2 - 2z(2z-1)\mathbf{kq} - 4z(1-z)\frac{(\mathbf{kq})^2}{\mathbf{k}^2}}{(z(\mathbf{k}^2 - 2\mathbf{kq}) + \mathbf{q}^2)^2} \quad (4.3.45)$$

4.3.4 Identification of the (unintegrated) splitting function

For the remaining integral for the splitting function kernel in Eq (4.3.40), we will use the boost invariant transverse momentum that we defined earlier (Eq 4.3.8), and change from q_\perp to $\tilde{\mathbf{q}}$

$$\tilde{\mathbf{q}} = \mathbf{q} - z\mathbf{k}, \quad \tilde{q}^\mu = q^\mu - zk^\mu$$

Doing this substitution in Eq (4.3.45) gives us the following expression for I

$$\begin{aligned}
I &= 4\pi i g^2 T_F \frac{z^2 \mathbf{k}^2 + \mathbf{q}^2 + 2z(1-2z)\mathbf{kq} + 4z(z-1) \frac{(\mathbf{kq})^2}{\mathbf{k}^2}}{(z(\mathbf{k}^2 - 2\mathbf{kq}) + \mathbf{q}^2)^2} \\
&= i \frac{4\pi g^2 T_F}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left(4z^2(1-z)^2 \mathbf{k}^2 + \tilde{\mathbf{q}}^2 + 4z(1-z)(1-2z)\mathbf{k}\tilde{\mathbf{q}} + 4z(z-1) \frac{(\mathbf{k}\tilde{\mathbf{q}})^2}{\mathbf{k}^2} \right)
\end{aligned} \tag{4.3.46}$$

The substitution $\tilde{\mathbf{q}} = \mathbf{q} - z\mathbf{k}$ will result in

$$d^{2+2\varepsilon} q_{\perp} \rightarrow d^{2+2\varepsilon} \tilde{\mathbf{q}} = |\tilde{\mathbf{q}}|^{1+2\varepsilon} d|\tilde{\mathbf{q}}| d\Omega_{2+2\varepsilon} \tag{4.3.47}$$

Going back to the expression for the kernel Eq (4.3.40), we now have:

$$\hat{K}_{qg} = \frac{1}{2} \int \frac{d^{2+2\varepsilon} q_{\perp}}{(2\pi)^{4+2\varepsilon}} I = \frac{1}{2} \int \frac{d^{2+2\varepsilon} \tilde{\mathbf{q}}}{(2\pi)^{4+2\varepsilon}} I = \frac{1}{2} \int \frac{d|\tilde{\mathbf{q}}| d\Omega_{2+2\varepsilon}}{(2\pi)^{4+2\varepsilon}} |\tilde{\mathbf{q}}|^{1+2\varepsilon} I \tag{4.3.48}$$

And plugging in the expression for I from Eq (4.3.46) into Eq (4.3.48) yields

$$\begin{aligned}
\hat{K}_{qg} &= i g^2 T_F \int \frac{d|\tilde{\mathbf{q}}| d\Omega_{2+2\varepsilon}}{(2\pi)^{3+2\varepsilon}} \frac{|\tilde{\mathbf{q}}|^{1+2\varepsilon}}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left[4z^2(1-z)^2 \mathbf{k}^2 + \tilde{\mathbf{q}}^2 + \right. \\
&\quad \left. + 4z(1-z)(1-2z)\mathbf{k}\tilde{\mathbf{q}} + 4z(z-1) \frac{(\mathbf{k}\tilde{\mathbf{q}})^2}{\mathbf{k}^2} \right]
\end{aligned} \tag{4.3.49}$$

When we do dimensional regularisation in the $\overline{\text{MS}}$ renormalisation scheme, we have the following expression for coupling g , [19]:

$$\frac{g^2}{2(2\pi)^{3+2\varepsilon}} = \frac{\alpha_s}{4\pi} \frac{e^{-\varepsilon\gamma_E}}{\pi^{1+\varepsilon} \mu^{2\varepsilon}} \tag{4.3.50}$$

With γ_E the Euler-Mascheroni constant.

If we plug this into Eq (4.3.49), we can rewrite the kernel as

$$\begin{aligned}
\hat{K}_{qg} &= iT_F \frac{\alpha_s}{2} \frac{e^{-\varepsilon\gamma_E}}{\pi^{2+\varepsilon} \mu^{2\varepsilon}} \int d|\tilde{\mathbf{q}}| d\Omega_{2+2\varepsilon} \frac{|\tilde{\mathbf{q}}|^{1+2\varepsilon}}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left[4z^2(1-z)^2 \mathbf{k}^2 + \tilde{\mathbf{q}}^2 + \right. \\
&\quad \left. + 4z(1-z)(1-2z)\mathbf{k}\tilde{\mathbf{q}} + 4z(z-1) \frac{(\mathbf{k}\tilde{\mathbf{q}})^2}{\mathbf{k}^2} \right]
\end{aligned} \tag{4.3.51}$$

We identify the unintegrated splitting function in the integrand of Eq (4.3.51):

$$\hat{K}_{qg} = i \frac{\alpha_s}{2} \frac{e^{-\varepsilon\gamma_E}}{\pi^{2+\varepsilon} \mu^{2\varepsilon}} \int \frac{d|\tilde{\mathbf{q}}|}{|\tilde{\mathbf{q}}|} |\tilde{\mathbf{q}}|^{2\varepsilon} d\Omega_{2+2\varepsilon} \hat{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) \tag{4.3.52}$$

The unintegrated TMD gluon-to-quark splitting function:

$$\begin{aligned}
\hat{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) &= T_F \frac{|\tilde{\mathbf{q}}|^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} [4z^2(1-z)^2\mathbf{k}^2 + \tilde{\mathbf{q}}^2 + 4z(1-z)(1-2z)\mathbf{k}\tilde{\mathbf{q}} + \\
&\quad + 4z(z-1)\frac{(\mathbf{k}\tilde{\mathbf{q}})^2}{\mathbf{k}^2}] \\
&= T_F \frac{|\tilde{\mathbf{q}}|^4}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left[1 + 4z^2(1-z)^2\frac{\mathbf{k}^2}{\tilde{\mathbf{q}}^2} + 4z(1-z)(1-2z)\frac{\mathbf{k}\tilde{\mathbf{q}}}{\tilde{\mathbf{q}}^2} + 4z(z-1)\frac{(\mathbf{k}\tilde{\mathbf{q}})^2}{\mathbf{k}^2\tilde{\mathbf{q}}^2} \right]
\end{aligned} \tag{4.3.53}$$

Note that when we turn α_s and the factors due to dimensional regularisation, back into g , the only factors in front of the integral are ig^2T_F . The i just originates from the factors in front of the propagators, and g^2 originates from the vertices. The decision we make here is to only absorb the constant factor T_F into the splitting function.

This result for the unintegrated TMD gluon-to-quark splitting function is exactly the same as in [19].

In the collinear limit, $\mathbf{k}^2 \rightarrow 0$, the unintegrated TMD splitting function becomes simply T_F .

4.3.5 Calculation of the angular averaged splitting function

We will now calculate the angular averaged TMD gluon-to-quark splitting function:

$$\bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) = \frac{\int d\Omega_{2+2\varepsilon} \hat{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}})}{\int d\Omega_{2+2\varepsilon}} \tag{4.3.54}$$

Where the denominator is the ‘‘area’’ of the d -dimensional unit sphere, since we are taking the average and not just the sum. Or in other words, it is the complete solid angle in d dimensions, [5] (p249):

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{4.3.55}$$

For example for $d = 2$ we would get $\int d\Omega_2 = 2\pi$, which is the perimeter of the unit circle. Using Eq (4.3.55) in Eq (4.3.54) gives us

$$\bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) = \frac{\Gamma(1+\varepsilon)}{2\pi^{1+\varepsilon}} \int d\Omega_{2+2\varepsilon} \hat{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) \tag{4.3.56}$$

Based on Eq (4.3.53) we define:

$$\hat{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) = \hat{A} + \hat{B} + \hat{C} + \hat{D} \tag{4.3.57}$$

With the terms \hat{A} , \hat{B} , \hat{C} and \hat{D} defined as:

$$\hat{A} = T_F \frac{\tilde{\mathbf{q}}^4}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \quad (4.3.58)$$

$$\hat{B} = T_F \frac{4z^2(1-z)^2\mathbf{k}^2\tilde{\mathbf{q}}^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \quad (4.3.59)$$

$$\hat{C} = T_F \frac{4z(1-z)(1-2z)\tilde{\mathbf{q}}^2(\mathbf{k}\tilde{\mathbf{q}})}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \quad (4.3.60)$$

$$\hat{D} = T_F \frac{1}{\mathbf{k}^2} \frac{4z(z-1)\tilde{\mathbf{q}}^2(\mathbf{k}\tilde{\mathbf{q}})^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \quad (4.3.61)$$

For \hat{A} and \hat{B} we have no angle dependence, this means:

$$\int d\Omega_{2+2\varepsilon}(\hat{A} + \hat{B}) = (\hat{A} + \hat{B}) \int d\Omega_{2+2\varepsilon} = \frac{2\pi^{1+\varepsilon}}{\Gamma(1+\varepsilon)}(\hat{A} + \hat{B}) \quad (4.3.62)$$

For the angular average of \hat{C} we have to deal with the scalar product.

$$\int d\Omega_{2+2\varepsilon}\hat{C} = T_F \frac{4z(1-z)(1-2z)\tilde{\mathbf{q}}^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \int d\Omega_{2+2\varepsilon}\mathbf{k}\tilde{\mathbf{q}} \quad (4.3.63)$$

To solve this, we define an angle in the plane of the two “vector”:

$$\mathbf{k}\tilde{\mathbf{q}} = \mathbf{k} \cdot \tilde{\mathbf{q}} := |\mathbf{k}||\tilde{\mathbf{q}}| \cos \varphi \quad (4.3.64)$$

In 2 dimensions, we would have to do the following integral, which vanishes:

$$\int d\Omega_2 \cos \varphi = \int_0^{2\pi} d\varphi \cos \varphi = 0 \quad (4.3.65)$$

Similarly, the integral in $2 + 2\varepsilon$ dimensions also vanishes

$$\int d\Omega_{2+2\varepsilon} \cos \varphi = 0 \quad (4.3.66)$$

This means that the integral in Eq (4.3.63) vanishes

$$\int d\Omega_{2+2\varepsilon}\hat{C} = 0 \quad (4.3.67)$$

For the angular average of \hat{D} we should calculate:

$$\int d\Omega_{2+2\varepsilon}\hat{D} = T_F \frac{1}{\mathbf{k}^2} \frac{4z(z-1)\tilde{\mathbf{q}}^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \int d\Omega_{2+2\varepsilon}(\mathbf{k}\tilde{\mathbf{q}})^2 \quad (4.3.68)$$

Again we use the same angle φ from Eq (4.3.64):

$$\int d\Omega_{2+2\varepsilon}(\mathbf{k}\tilde{\mathbf{q}})^2 = \mathbf{k}^2\tilde{\mathbf{q}}^2 \int d\Omega_{2+2\varepsilon}\cos^2(\varphi) \quad (4.3.69)$$

In 2 dimensions we would have:

$$\int d\Omega_2\cos^2(\varphi) = \int_0^{2\pi} d\varphi \cos^2(\varphi) = \pi \quad (4.3.70)$$

In $2 + 2\varepsilon$ dimensions however, we have:

$$\int d\Omega_{2+2\varepsilon}\cos^2(\varphi) = \frac{2\pi^{1+\varepsilon}}{\Gamma(1+\varepsilon)} \frac{1}{2+2\varepsilon} \quad (4.3.71)$$

Using Eq (4.3.69) and Eq (4.3.71) in Eq (4.3.68), we get the following result for the integral of \hat{D} :

$$\int d\Omega_{2+2\varepsilon}\hat{D} = T_F \frac{4\pi^{1+\varepsilon}}{(1+\varepsilon)\Gamma(1+\varepsilon)} \frac{z(z-1)\tilde{\mathbf{q}}^4}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \quad (4.3.72)$$

Now we go back to Eq (4.3.56) and Eq (4.3.57), and use the solutions for the integrals of \hat{A} , \hat{B} , \hat{C} and \hat{D} to get a result for $\bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}})$

$$\begin{aligned} \bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) &= \frac{\Gamma(1+\varepsilon)}{2\pi^{1+\varepsilon}} \int d\Omega_{2+2\varepsilon}(\hat{A} + \hat{B} + \hat{C} + \hat{D}) \quad (4.3.73) \\ &= \frac{\Gamma(1+\varepsilon)}{2\pi^{1+\varepsilon}} \left(\frac{2\pi^{1+\varepsilon}}{\Gamma(1+\varepsilon)} (\hat{A} + \hat{B}) + \int d\Omega_{2+2\varepsilon}\hat{D} \right) \\ &= T_F \frac{\tilde{\mathbf{q}}^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left(\tilde{\mathbf{q}}^2 + 4z^2(1-z)^2\mathbf{k}^2 + \frac{2z(z-1)}{1+\varepsilon}\tilde{\mathbf{q}}^2 \right) \\ &= T_F \frac{\tilde{\mathbf{q}}^2}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left(\frac{2z(z-1) + 1 + \varepsilon}{1+\varepsilon}\tilde{\mathbf{q}}^2 + 4z^2(1-z)^2\mathbf{k}^2 \right) \end{aligned}$$

We can rewrite $2z(z-1) + 1 = z^2 + (1-z)^2$, and we get the following expression for the angular averaged TMD gluon-to-quark splitting function:

$$\bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) = T_F \frac{\tilde{\mathbf{q}}^4}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left(\frac{z^2 + (1-z)^2 + \varepsilon}{1+\varepsilon} + 4z^2(1-z)^2 \frac{\mathbf{k}^2}{\tilde{\mathbf{q}}^2} \right) \quad (4.3.74)$$

This is exactly the same as in [17]

When we take the limit $\varepsilon \rightarrow 0$, the angular averaged splitting function becomes

$$\bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) = T_F \frac{\tilde{\mathbf{q}}^4}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left(z^2 + (1-z)^2 + 4z^2(1-z)^2 \frac{\mathbf{k}^2}{\tilde{\mathbf{q}}^2} \right) \quad (4.3.75)$$

In the collinear limit we have $\mathbf{k}^2 \rightarrow 0$, for the splitting function this means

$$\bar{P}_{qg}^{(0)}(z, \mathbf{k}^2 \rightarrow 0, \tilde{\mathbf{q}}) = T_F(z^2 + (z-1)^2) \quad (4.3.76)$$

This is the same as the collinear gluon-to-quark splitting function in Eq (3.2.10).

In the collinear limit we get the splitting function from DGLAP, and in high energy we can use our TMD splitting function together with the gluon PDF from BFKL. For BFKL we would need to calculate the resummed anomalous dimension by taking the Mellin transform of the convolution of the angular averaged TMD splitting function and the TMD gluon PDF. By equating this convolution to the collinearly factorised form of the Green's function, we would obtain a result that has the BFKL resummation of the DGLAP anomalous dimension [24-27] [30]. So our factorisation overlaps with both DGLAP and BFKL.

4.3.6 Calculation of the splitting kernel

We will now calculate the TMD gluon-to-quark splitting kernel \hat{K}_{qg} by doing the last remaining integration over $|\tilde{\mathbf{q}}|$. We will first do this for the whole $|\tilde{\mathbf{q}}|$ space, not for the restricted $\theta(\mu_F^2 + q^2)$.

When we look at Eq (4.3.52) and Eq (4.3.56), we can now write the splitting kernel as an integral over the angular averaged splitting function from Eq (4.3.74):

$$\hat{K}_{qg} = i \frac{\alpha_s e^{-\varepsilon\gamma_E}}{\pi\mu^{2\varepsilon}\Gamma(1+\varepsilon)} \int \frac{d|\tilde{\mathbf{q}}|}{|\tilde{\mathbf{q}}|} |\tilde{\mathbf{q}}|^{2\varepsilon} \bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}}) \quad (4.3.77)$$

When we plug in Eq (4.3.74) for $\bar{P}_{qg}^{(0)}(z, \mathbf{k}, \tilde{\mathbf{q}})$, and when we allow $|\tilde{\mathbf{q}}|$ to take any value larger or equal to 0, the splitting kernel will be the following integration:

$$\hat{K}_{qg} = i \frac{T_F \alpha_s e^{-\varepsilon\gamma_E}}{\pi\mu^{2\varepsilon}\Gamma(1+\varepsilon)} \int_0^\infty d|\tilde{\mathbf{q}}| \frac{|\tilde{\mathbf{q}}|^{3+2\varepsilon}}{(z(1-z)\mathbf{k}^2 + \tilde{\mathbf{q}}^2)^2} \left(\frac{z^2 + (1-z)^2 + \varepsilon}{1+\varepsilon} + 4z^2(1-z)^2 \frac{\mathbf{k}^2}{\tilde{\mathbf{q}}^2} \right) \quad (4.3.78)$$

We do the following substitution:

$$u := \tilde{\mathbf{q}}^2 \quad (4.3.79)$$

$$du = 2|\tilde{\mathbf{q}}|d|\tilde{\mathbf{q}}| \quad (4.3.80)$$

With this, the expression for the splitting kernel is now

$$\hat{K}_{qg} = i \frac{T_F \alpha_s e^{-\varepsilon\gamma_E}}{2\pi\mu^{2\varepsilon}\Gamma(1+\varepsilon)} \int_0^\infty du \frac{u^{1+\varepsilon}}{(z(1-z)\mathbf{k}^2 + u)^2} \left(\frac{z^2 + (1-z)^2 + \varepsilon}{1+\varepsilon} + 4z^2(1-z)^2 \frac{\mathbf{k}^2}{u} \right) \quad (4.3.81)$$

Now we want to change this integral such that the denominator $(z(1-z)\mathbf{k}^2 + u)^2$ is replaced by a denominator of the form $(Au + B(1-u))^2$, and such that the integration limits are changed from $[0, \infty]$ to $[0, 1]$. A denominator of the form $(Au + B(1-u))^2$ is something you see in Feynman parametrisation, a technique that can solve many integrals in QFT. There is a very easy substitution that does this, and that also keeps the function under the integral a rational function:

$$v := \frac{u}{u+1} \quad (4.3.82)$$

$$dv = \frac{du}{(u+1)^2} \quad (4.3.83)$$

With this substitution, the splitting kernel becomes

$$\begin{aligned} \widehat{K}_{qg} = i \frac{T_F \alpha_s e^{-\varepsilon \gamma_E}}{2\pi \mu^{2\varepsilon} \Gamma(1+\varepsilon)} \int_0^1 dv \left(\frac{v}{1-v}\right)^{1+\varepsilon} \frac{1}{(z(1-z)\mathbf{k}^2(1-v) + v)^2} \\ \cdot \left(\frac{z^2 + (1-z)^2 + \varepsilon}{1+\varepsilon} + 4z^2(1-z)^2 \mathbf{k}^2 \frac{1-v}{v} \right) \end{aligned} \quad (4.3.84)$$

We split the kernel up in two terms:

$$\widehat{K}_{qg} = i \frac{T_F \alpha_s e^{-\varepsilon \gamma_E}}{2\pi \mu^{2\varepsilon} \Gamma(1+\varepsilon)} (I_1 + I_2) \quad (4.3.85)$$

With I_1 and I_2 the integrals:

$$I_1 = 4z^2(1-z)^2 \mathbf{k}^2 \int_0^1 dv \left(\frac{v}{1-v}\right)^\varepsilon \frac{1}{(z(1-z)\mathbf{k}^2(1-v) + v)^2} \quad (4.3.86)$$

$$I_2 = \frac{z^2 + (1-z)^2 + \varepsilon}{1+\varepsilon} \int_0^1 dv \left(\frac{v}{1-v}\right)^{1+\varepsilon} \frac{1}{(z(1-z)\mathbf{k}^2(1-v) + v)^2} \quad (4.3.87)$$

We can try to solve this via Feynman parametrisation, or some similar techniques. Notice that if we already go back to 4D spacetime now by taking $\varepsilon = 0$, then Eq (4.3.86) becomes simply the basic integral for Feynman parametrisation:

$$\frac{1}{AB} = \int_0^1 \frac{dv}{(vA + (1-v)B)^2} \quad (4.3.88)$$

So for $\varepsilon = 0$, the term I_1 will be

$$\begin{aligned}
I_1 &= 4z^2(1-z)^2 \mathbf{k}^2 \int_0^1 \frac{dv}{(z(1-z)\mathbf{k}^2(1-v) + v)^2} = \frac{4z^2(1-z)^2 \mathbf{k}^2}{z(1-z)\mathbf{k}^2} \\
&= 4z(1-z)
\end{aligned} \tag{4.3.89}$$

The term I_2 for $\varepsilon = 0$ will be

$$I_2 = (z^2 + (1-z)^2) \int_0^1 dv \frac{v}{1-v} \frac{1}{(z(1-z)\mathbf{k}^2(1-v) + v)^2} \tag{4.3.90}$$

This will be solved by partial fraction decomposition, by rewriting I_2 in the following form:

$$I_2 = (z^2 + (1-z)^2) \int_0^1 dv \frac{v}{(1-v)(av + b)^2} \tag{4.3.91}$$

Identifying a and b yields:

$$a = 1 - z(1-z)\mathbf{k}^2 \quad , \quad b = z(1-z)\mathbf{k}^2 \tag{4.3.92}$$

Thus we have $a = 1 - b$, and we can write I_2 as

$$I_2 = (z^2 + (1-z)^2) \int_0^1 dv \frac{v}{(1-v)((1-b)v + b)^2} \tag{4.3.93}$$

Now we will do the partial fraction decomposition:

$$\frac{v}{(1-v)((1-b)v + b)^2} = \frac{A}{1-v} + \frac{B}{(1-b)v + b} + \frac{C}{((1-b)v + b)^2} \tag{4.3.94}$$

Putting everything on the same denominator gives us the system of equations:

$$\begin{cases}
A(1-b)^2 - B(1-b) = 0 \\
2Ab(1-b) + B(1-b) - Bb - C = 1 \\
Ab^2 + Bb + C = 0
\end{cases} \tag{4.3.95}$$

It's solution is given by

$$\begin{cases}
A = 1 \\
B = 1 - b \\
C = -b
\end{cases} \tag{4.3.96}$$

Using this solution in Eq (4.3.94) and plugging the result in Eq (4.3.93) gives us

$$I_2 = (z^2 + (1-z)^2) \int_0^1 dv \left(\frac{1}{1-v} + \frac{1-b}{(1-b)v + b} - \frac{b}{((1-b)v + b)^2} \right) \tag{4.3.97}$$

The first term in I_2 gives integrates to

$$\int_0^1 \frac{dv}{1-v} = (-\ln|v-1|)|_{v=0}^{v=1} = -\ln(0) = +\infty \quad (4.3.98)$$

This term gives us a pole.

The second term in I_2 integrates to

$$\begin{aligned} \int_0^1 \frac{dv}{v + b/(1-b)} &= \ln|1/(1-b)| - \ln|b/(1-b)| = -\ln|b| \\ &= \ln(z(1-z)\mathbf{k}^2) \end{aligned} \quad (4.3.99)$$

The third term in I_2 integrates to

$$\begin{aligned} -\int_0^1 \frac{b dv}{((1-b)v + b)^2} &= \frac{b}{(1-b)^2} \frac{1}{v + b/(1-b)} \Big|_0^1 \\ &= \frac{b}{(1-b)^2} \left(\frac{1}{1 + b/(1-b)} - \frac{1}{b/(1-b)} \right) \\ &= -1 \end{aligned} \quad (4.3.100)$$

Combining these three results in Eq (4.3.97) yields

$$I_2 = (z^2 + (1-z)^2)(\ln(z(1-z)\mathbf{k}^2) - 1 - \ln(0)) \quad (4.3.101)$$

Using Eq (4.3.89) and Eq (4.3.101) in Eq (4.3.85) for the TMD splitting kernel, where we set $\varepsilon = 0$ before integration, gives us

$$\hat{K}_{qg} = iT_F \frac{\alpha_s}{2\pi} [4z(1-z) + (z^2 + (1-z)^2)(\ln(z(1-z)\mathbf{k}^2) - 1 - \ln(0))] \quad (4.3.102)$$

This splitting kernel has a logarithmic pole, coming from I_2 .

We go back to Eq (4.3.86) and Eq (4.3.87) and will now keep the ε during integration.

We notice that if we write the denominator $(z(1-z)\mathbf{k}^2(1-v) + v)^2$ as $(av + b)^2$, it might be possible that I_1 and I_2 are hypergeometric functions.

$$I_1 = 4z^2(1-z)^2\mathbf{k}^2 \int_0^1 dv \left(\frac{v}{1-v} \right)^\varepsilon \frac{1}{(av + b)^2} \quad (4.3.103)$$

$$I_2 = \frac{z^2 + (1-z)^2 + \varepsilon}{1 + \varepsilon} \int_0^1 dv \left(\frac{v}{1-v} \right)^{1+\varepsilon} \frac{1}{(av + b)^2} \quad (4.3.104)$$

Where a and b are given by Eq (4.3.92).

The hypergeometric functions (or in this case specifically Gauss's hypergeometric functions) are a type of special functions first studied by Euler, but the full treatment was done first by Gauss. The hypergeometric functions ${}_2F_1(\alpha, \beta; \gamma; w)$ are the solutions to the hypergeometric differential equation [37]

$$w(1-w)\frac{d^2f(w)}{dw^2} + (\gamma - (\alpha + \beta + 1)w)\frac{df(w)}{dw} - \alpha\beta f(w) = 0 \quad (4.3.105)$$

The hypergeometric functions can be expressed by the hypergeometric series [37]:

$${}_2F_1(\alpha, \beta; \gamma; w) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{w^n}{n!} \quad (4.3.106)$$

Where the notation $(x)_n$ is a Pochhammer symbol, defined as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1) \dots (x+n-1) \quad (4.3.107)$$

With Γ the Gamma function.

There is also the following integral for the hypergeometric functions [37]:

$${}_2F_1(\alpha, \beta; \gamma; w) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-wx)^{-\alpha} dx \quad (4.3.108)$$

With B the Euler Beta function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (4.3.109)$$

Eq (4.3.108) is only defined for $\text{Re}(w) < 1$, and for $\text{Re}(\gamma) > \text{Re}(\beta) > 0$.

The integral Eq (4.3.108) looks a lot like the integrals in Eq (4.3.103) and (4.3.104), with $x = v$. We can rewrite Eq (4.3.103) and (4.3.104) as

$$I_1 = \frac{4z^2(1-z)^2\mathbf{k}^2}{b^2} \int_0^1 dv \left(\frac{v}{1-v}\right)^\varepsilon \frac{1}{(1+av/b)^2} \quad (4.3.110)$$

$$I_2 = \frac{z^2 + (1-z)^2 + \varepsilon}{(1+\varepsilon)b^2} \int_0^1 dv \left(\frac{v}{1-v}\right)^{1+\varepsilon} \frac{1}{(1+av/b)^2} \quad (4.3.111)$$

Where the factor in front of the integral in I_1 simplifies to $4/\mathbf{k}^2$ because from Eq (4.3.92) we have $b = z(1-z)\mathbf{k}^2$. We look at Eq (4.3.108) and (4.3.110), and we identify the arguments of the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; w)$ for I_1 :

$$\alpha = 2, \quad \beta = 1 + \varepsilon, \quad \gamma = 2, \quad w = -a/b \quad (4.3.112)$$

The condition $\text{Re}(\gamma) > \text{Re}(\beta) > 0$ is satisfied as long as $-1 < \varepsilon < 1$. Is the condition $\text{Re}(w) < 1$ satisfied?

$$w = -a/b = 1 - \frac{1}{b} = 1 - \frac{1}{z(1-z)\mathbf{k}^2} < 1 \quad (4.3.113)$$

This means we need to have

$$\frac{1}{z(1-z)\mathbf{k}^2} > 0 \quad (4.3.114)$$

In order to use Eq (4.3.108). The condition Eq (4.3.114) is always valid because $0 < z < 1$, and because \mathbf{k}^2 practically never reaches infinity. So we can solve I_1 in Eq (4.3.110) by using Eq (4.3.108):

$$\begin{aligned} I_1 &= \frac{4}{\mathbf{k}^2} B(\beta, \gamma - \beta) {}_2F_1(\alpha, \beta; \gamma; w) \\ &= \frac{4}{\mathbf{k}^2} \Gamma(1 + \varepsilon) \Gamma(1 - \varepsilon) {}_2F_1\left(2, 1 + \varepsilon; 2; 1 - \frac{1}{z(1-z)\mathbf{k}^2}\right) \end{aligned} \quad (4.3.115)$$

We will now look at Eq (4.3.108) and (4.3.111), and we identify the arguments of the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; w)$ for I_2 :

$$\alpha = 2, \quad \beta = 2 + \varepsilon, \quad \gamma = 2, \quad w = -a/b \quad (4.3.116)$$

The condition $\text{Re}(\gamma) > \text{Re}(\beta) > 0$ is satisfied as long as $-2 < \varepsilon < 0$. And the condition $\text{Re}(w) < 1$ is again satisfied because it is the same as with I_1 . So we can solve I_2 in Eq (4.3.111) by using Eq (4.3.108):

$$\begin{aligned} I_2 &= \frac{z^2 + (1-z)^2 + \varepsilon}{(1+\varepsilon)z^2(1-z)^2\mathbf{k}^4} B(\beta, \gamma - \beta) {}_2F_1(\alpha, \beta; \gamma; w) \\ &= \frac{z^2 + (1-z)^2 + \varepsilon}{z^2(1-z)^2\mathbf{k}^4} \Gamma(1 + \varepsilon) \Gamma(-\varepsilon) {}_2F_1\left(2, 2 + \varepsilon; 2; 1 - \frac{1}{z(1-z)\mathbf{k}^2}\right) \end{aligned} \quad (4.3.117)$$

Using Eq (4.3.115) and Eq (4.3.117) in Eq (4.3.85) we get the TMD gluon-to-quark splitting kernel, in dimensional regularisation:

$$\begin{aligned} \hat{K}_{qg} &= i \frac{T_F \alpha_s e^{-\varepsilon \gamma_E}}{2\pi \mu^{2\varepsilon} \mathbf{k}^2} \left(4\Gamma(1 - \varepsilon) {}_2F_1\left(2, 1 + \varepsilon; 2; 1 - \frac{1}{z(1-z)\mathbf{k}^2}\right) + \right. \\ &\quad \left. + \frac{z^2 + (1-z)^2 + \varepsilon}{z^2(1-z)^2\mathbf{k}^2} \Gamma(-\varepsilon) {}_2F_1\left(2, 2 + \varepsilon; 2; 1 - \frac{1}{z(1-z)\mathbf{k}^2}\right) \right) \end{aligned} \quad (4.3.118)$$

In order to combine I_1 and I_2 into \hat{K}_{qg} with only one ε , we are required to combine the conditions that ε had to obey in both integrals. This results in the condition

$$-1 < \varepsilon < 0 \quad (4.3.119)$$

The splitting kernel has a pole, coming from I_2 . When we take the limit $\varepsilon \rightarrow 0$ in Eq (4.3.118), $\Gamma(-\varepsilon)$ will become infinite. For small ε we can expand the Gamma function near its pole:

$$\Gamma(-\varepsilon) = -\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon) \quad (4.3.120)$$

With γ_E the Euler-Mascheroni constant. When we take the limit $\varepsilon \rightarrow 0$, the splitting kernel becomes:

$$\begin{aligned} \hat{K}_{qg} = i \frac{T_F \alpha_s}{2\pi \mathbf{k}^2} & \left(4 {}_2F_1 \left(2, 1; 2; 1 - \frac{1}{z(1-z)\mathbf{k}^2} \right) + \right. \\ & \left. + \frac{z^2 + (1-z)^2}{z^2(1-z)^2 \mathbf{k}^2} \left(-\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon) \right) {}_2F_1 \left(2, 2; 2; 1 - \frac{1}{z(1-z)\mathbf{k}^2} \right) \right) \end{aligned} \quad (4.3.121)$$

In the collinear limit we have $\mathbf{k}^2 \rightarrow 0$, which makes the splitting kernel collinear divergent.

4.4 Importance of TMD splitting functions

TMD splitting functions take into account the transverse momentum, already at leading order. The collinear splitting functions do not do this, and effects from transverse momentum enters the collinear splitting functions only in higher orders. The TMD splitting functions can be used in the small x region (BFKL region), and they resum terms in $(\alpha_s/\ln(\hat{s}/M^2))^n$ to all orders. If we want to calculate the resummed anomalous dimension, we need to take the Mellin transform of the convolution of the angular averaged TMD splitting function and the TMD gluon PDF. By equating this convolution to the collinearly factorised form of the Green's function, we would obtain a result that has the BFKL resummation of the DGLAP anomalous dimension. In the collinear limit, the TMD splitting functions reduce to the collinear splitting functions, which we explicitly checked for the gluon-to-quark TMD splitting function in the previous section, which are the ones used for the DGLAP equation. This means that the TMD splitting functions overlap with both BFKL and DGLAP.

5 Monte Carlo method in parton branching

In this chapter I perform numerical calculations for the evolution of parton distribution functions, both collinear and TMD, by making use of the Parton Branching (PB) method [22] and of Monte Carlo simulations based on the computer code uPDFevolv [23]. The numerical calculations in this chapter implement the evolution equations and splitting functions discussed at analytical level in the previous chapters. I start in section 5.1 with a brief description of the PB approach; then in section 5.2 I illustrate the Monte Carlo method of solution of the evolutions; in section 5.3 I present numerical results, based on this method, for collinear and TMD parton distributions.

5.1 Parton branching

In the parton branching approach [22] to the evolution equations, a soft-gluon resolution scale is introduced in the DGLAP equations, which results in resolvable and non-resolvable parton emissions. This allows us to write the evolution equations in a way that can be solved by iteration. Parton Branching is not restricted to DGLAP only. Instead of collinear parton distribution functions (PDFs) we can also apply the method for transverse momentum dependent (TMD) PDFs, by using a different type of ordering. In this section we discuss both cases.

5.1.1 Parton Branching for DGLAP

We start from DGLAP Eq (3.1.18), which we can express as

$$\frac{df_i(x, \mu)}{d(\ln \mu^2)} = \sum_j \int_x^1 \frac{dz}{z} P_{ij}(z, \alpha_s(\mu)) f_j(x/z, \mu) \quad (5.1.1)$$

where $z = x/\xi$, $f_i(x, \mu)$ are the parton distribution functions (PDFs), and $P_{ij}(z, \alpha_s(\mu))$ are the collinear splitting functions. The indices i and j will run over all partons, which in general are the gluon and all quark and antiquark flavours (so $2N_f + 1$ different index values). The collinear splitting functions can be expressed as a power series Eq (3.1.19). In parton branching, instead of working with the actual PDFs, we will usually work with the PDFs weighted by x , which we call the momentum-weighted PDFs:

$$\tilde{f}_i(x, \mu) := x f_i(x, \mu) \quad (5.1.2)$$

Using the momentum-weighted PDFs the DGLAP equation becomes

$$\frac{d\tilde{f}_i(x, \mu)}{d(\ln \mu^2)} = \sum_j \int_x^1 dz P_{ij}(z, \alpha_s(\mu)) \tilde{f}_j(x/z, \mu) \quad (5.1.3)$$

If the transverse distance between emitted partons has a finite resolution scale, then due to energy-momentum conservation we cannot resolve partons that are radiated with z closer to 1 than a cut-off value. On the other hand, if we would remove non-resolvable radiative contributions, we would get a violation of unitarity. [22] In the parton branching method we will restore unitarity by recasting the DGLAP equation in terms of no-branching probabilities and real-emission branching probabilities. The splitting functions can be decomposed in terms that classify the singular behaviour for $z \rightarrow 1$ as follows [22]:

$$P_{ij}(z, \alpha_s) = D_{ij}(\alpha_s) \delta(1-z) + \frac{K_{ij}(\alpha_s)}{(1-z)_+} + R_{ij}(z, \alpha_s) \quad (5.1.4)$$

Where $1/(1-z)_+$ is defined according to Eq (3.2.7) and Eq (3.2.8)., and $R_{ij}(z, \alpha_s)$ contains logarithmic terms which are analytic for $z \rightarrow 1$. The functions $D_{ij}(\alpha_s)$, $K_{ij}(\alpha_s)$ and $R_{ij}(z, \alpha_s)$ can each be decomposed in a power series analogous to the power series decomposition of $P_{ij}(z, \alpha_s)$ Eq (3.1.19). $D_{ij}(\alpha_s)$ and $K_{ij}(\alpha_s)$ are diagonal in parton flavour [22].

We introduce the soft-gluon resolution parameter z_M , where $1 - z_M \sim \mathcal{O}(\Lambda_{\text{QCD}}/\mu)$. In the DGLAP equation in Eq (5.1.3) we split the integral up in the resolvable region ($z < z_M$) and the non-resolvable region ($z > z_M$). By making use of the decomposition Eq (5.1.4) and the momentum sum rule

$$\sum_i \int_0^1 z P_{ij}(z, \alpha_s) dz = 0 \quad (5.1.5)$$

we can write the evolution equations as

$$\frac{d\tilde{f}_i(x, \mu)}{d(\ln \mu^2)} = \sum_j \left[\int_x^{z_M} dz P_{ij}^{(R)}(z, \alpha_s(\mu)) \tilde{f}_j(x/z, \mu) - \int_0^{z_M} dz z P_{ji}^{(R)}(z, \alpha_s(\mu)) \tilde{f}_i(x, \mu) \right] \quad (5.1.6)$$

where $P_{ij}^{(R)}(z, \alpha_s)$ is defined as the real-emission branching probabilities (real-emission splitting functions):

$$P_{ij}^{(R)}(z, \alpha_s) = \frac{K_{ij}(\alpha_s)}{1-z} + R_{ij}(z, \alpha_s) \quad (5.1.7)$$

The derivation of Eq (5.1.6) can be found in [22]. The first term in Eq (5.1.6) are the contributions from real parton emission, while the second term are the contributions from virtual corrections.

We introduce the Sudakov form factor:

$$\Delta_i(z_M, \mu, \mu_0) = \exp\left(-\sum_j \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_0^{z_M} dz z P_{ji}^{(R)}(z, \alpha_s(\mu'))\right) \quad (5.1.8)$$

The Sudakov form factor $\Delta_i(z_M, \mu, \mu_0)$ can be interpreted as the probability that a parton i undergoes no branching between evolution scales μ_0 and μ , and where the branchings are classified according to the resolution parameter z_M . The evolution equation Eq (5.1.6) can be written in terms of $P_{ij}^{(R)}$ and Sudakov form factors [22]:

$$\frac{d}{d(\ln \mu^2)} \left(\frac{\tilde{f}_i(x, \mu)}{\Delta_i(\mu)} \right) = \sum_j \int_x^{z_M} dz P_{ij}^{(R)}(z, \alpha_s(\mu)) \frac{\tilde{f}_j(x/z, \mu)}{\Delta_i(\mu)} \quad (5.1.9)$$

where we don't sum over i , and where we defined $\Delta_i(\mu) := \Delta_i(z_M, \mu, \mu_0)$ to make notation shorter. If we integrate this evolution equation, we get

$$\tilde{f}_i(x, \mu) = \Delta_i(\mu) \tilde{f}_i(x, \mu_0) + \sum_j \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \frac{\Delta_i(\mu)}{\Delta_i(\mu')} \int_x^{z_M} dz P_{ij}^{(R)}(z, \alpha_s(\mu')) \tilde{f}_j(x/z, \mu') \quad (5.1.10)$$

Note that this equation uses collinear splitting functions and collinear PDFs. This is the equation that we want to solve in the Parton Branching method. It is a Fredholm type equation, which can be solved by iteration as a series.

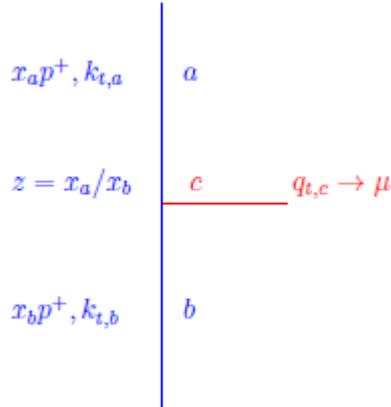


Figure 5.1: A single branching process $b \rightarrow a + c$. The p^+ is the “plus lightcone momentum”, which is defined in the same way as p in Eq (4.2.6) and is equal to $P(1, \mathbf{0}, 1)$. The k_t and q_t are the transverse momenta. Figure from [22]

In the Parton Branching method with the collinear PDFs, Eq (5.1.10), we have strong ordering of transverse momenta. This means that the transverse momentum of the evolving parton after an emission will be much larger than before the emission.

(Transverse momentum ordering was also discussed in section 3.1.2) In case of Figure 5.1 this means $k_{t,a} \gg k_{t,b}$.

5.1.2 Parton Branching for TMD PDFs

Instead of restricting Parton Branching to collinear PDFs, we can also do Parton Branching with transverse momentum dependent (TMD) PDFs. However, Parton Branching is ill defined when using transverse momentum ordering for TMD PDFs (for variation of z_{\max}). Instead, angular ordering will be used. In Figure 5.1 the evolution scale μ can be related to the angle θ of the momentum of particle c with respect to the beam direction. The resulting angular ordering means that the angle is much larger for each subsequent emitted parton: $\theta_{i+1} \gg \theta_i$. Parton Branching is well defined for TMD PDFs if we use angular ordering.

In the transverse momentum ordered situation, z_M will be fixed, while in the angular ordered situation z_M will be dynamic. Another difference between the orderings is how α_s is treated. In transverse momentum ordering we have μ dependant $\alpha_s(\mu)$, and μ equals q_\perp . In angular ordering we still have q_\perp dependant $\alpha_s(q_\perp)$, but this time q_\perp depends both on μ and z .

For the collinear PDFs both orderings can be used. If $\mathcal{F}_i(x, \mathbf{k}, \mu)$ is the TMD PDF (also called unintegrated PDF) for parton type i , with \mathbf{k} the transverse 2-momentum (like the \mathbf{k} in Eq (4.2.6), with $k_\perp^\mu = (0, \mathbf{k}, 0)$), then the momentum-weighted collinear PDF equals [22]

$$\tilde{f}_i(x, \mu) = \int \frac{d^2 \mathbf{k}}{\pi} x \mathcal{F}_i(x, \mathbf{k}, \mu) \quad (5.1.11)$$

We can also define the momentum-weighted TMD PDF:

$$\tilde{\mathcal{F}}_i(x, \mathbf{k}, \mu) := x \mathcal{F}_i(x, \mathbf{k}, \mu) \quad (5.1.12)$$

By using Eq (5.1.10) and angular ordering we can write the following evolution equation for the TMD PDFs [22]:

$$\begin{aligned} \tilde{\mathcal{F}}_i(x, \mathbf{k}, \mu) = & \Delta_i(\mu) \tilde{\mathcal{F}}_i(x, \mathbf{k}, \mu_0) + \sum_j \int \frac{d^2 \mathbf{q}'}{\pi \mathbf{q}'} \frac{\Delta_i(\mu)}{\Delta_i(\mathbf{q}')} \Theta(\mu^2 - \mathbf{q}'^2) \Theta(\mathbf{q}'^2 - \mu_0^2) \cdot \\ & \cdot \int_x^{z_M} dz P_{ij}^{(R)}(z, \alpha_s(\mathbf{q}')) \tilde{\mathcal{F}}_j(x/z, \mathbf{k} + (1-z)\mathbf{q}', \mathbf{q}') \end{aligned} \quad (5.1.13)$$

This can be solved by iteration as [22]

$$\tilde{\mathcal{F}}_a(x, \mathbf{k}, \mu) = \sum_{i=0}^{\infty} \tilde{\mathcal{F}}_a^{(i)}(x, \mathbf{k}, \mu) \quad (5.1.14)$$

Note that in this section the splitting functions are collinear splitting functions. We can do Parton Branching up to a certain order. Leading order (LO) Parton Branching uses the LO splitting functions and LO α_s , NLO Parton Branching uses the NLO splitting functions and NLO α_s , etc.

5.2 Monte Carlo method

To solve the evolution equation numerically we will apply a Monte Carlo method (also called a Monte Carlo simulation), by reducing the problem to generating z and μ . The Monte Carlo method is a class of algorithms that uses repeated random sampling to estimate probabilistic results. It generates variables according to a probability distribution to calculate expectation values or integrals. Monte Carlo can be used to solve any problem that has a probabilistic nature, or any problem that can't (or is too difficult to) be solved analytically. The main classes of problems that are most commonly solved by Monte Carlo are integration, optimisation and inverse problems. In our case, Monte Carlo will be used to solve an integration problem.

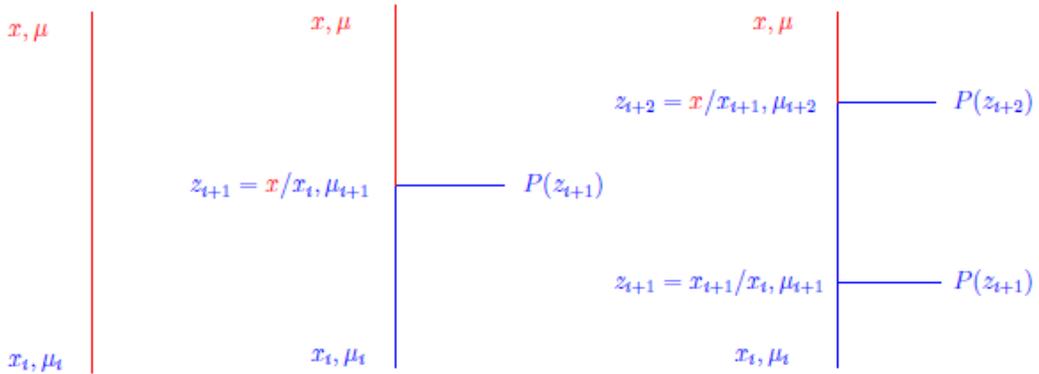


Figure 5.2: Parton evolution process by iteration, where a parton evolves from scale μ_i to μ . (Time runs from bottom to top.) We have evolution without any branching (left), with only one branching (middle), with two branchings (right), etc. Figure from [22]

In the parton evolution we start from parton a and evolve the scale from μ_i to μ either without branching, or with one branching at scale μ_{i+1} , or with a second branching at scale μ_{i+2} , etc. The probability to evolve from scale μ_i to μ_{i+1} without any resolvable branching is given by the Sudakov form factor $\Delta_a(z_M, \mu_{i+1}, \mu_i)$. [22] We introduce a random number $R_0 \in [0,1]$, and we generate the value for μ_{i+1} by solving Eq (5.1.8) for μ_{i+1} at a given μ_i (Ref [22]),

$$R_0 \int_{\mu_i^2}^{\infty} d\Delta_a(z_M, \mu, \mu_0) = \int_{\mu_i^2}^{\mu_{i+1}^2} d\Delta_a(z_M, \mu, \mu_0) \quad (5.2.1)$$

Which leads to the following equation after integration:

$$R_0 = 1 - \frac{\Delta_a(\mu_{i+1})}{\Delta_a(\mu_i)} \quad (5.2.2)$$

We introduce a random number $R_1 \in [0,1]$, and we generate the splitting variable z by

$$\int_{z_{\min}}^{z_{i+1}} dz' P_{ab}^{(R)}(z', \alpha_s(\mu_{i+1})) = R_1 \int_{z_{\min}}^{z_M} dz' P_{ab}^{(R)}(z', \alpha_s(\mu_{i+1})) \quad (5.2.3)$$

where z_{\min} is the lowest kinematically allowed value. In case of TMD PDFs, we introduce an extra random number to generate the angle θ_i uniformly. When we generate a pair of z_i, μ_i (and θ_i) values many times, we obtain a true and unbiased estimate of the integrals and we obtain a solution for the evolution equations. [22] In order to do this method we require a starting distribution, which could come from experiment or from a model.

This Monte Carlo method for parton branching was implemented in a numerical program. The code is the same whether we want collinear PDFs or TMD PDFs, since the former is just obtained via an integration over the latter. As starting distributions we can use default sets from evolution packages.

5.3 Numerical solutions of parton branching

In this section we use the numerical (Monte Carlo) Parton Branching program “updfevolv” to simulate parton distribution functions (PDF), given a starting distribution at an initial scale μ_0 as input, by evolving the PDFs from scale μ_0 to scale μ via the method described in the previous sections. We can simulate both collinear PDFs (as a function of x) and transverse momentum dependent (TMD) PDFs (as a function of x and k_{\perp}). x is the Bjorken scaling variable Eq (2.2.26), and k_{\perp} (also called k_t) is the transverse momentum of the parton. To improve convergence, momentum-weighted PDFs are used in the Parton Branching program. The definition of momentum-weighted PDFs are given by Eq (5.1.2) and Eq (5.1.12). The program uses TMD PDFs during the iteration, and collinear PDFs are obtained via integration of the TMD PDFs according to Eq (5.1.11).

The program used in this thesis, updfevolv can simulate both collinear and TMD PDFs, and it can use both collinear and TMD splitting functions. We have the choice

to use either LO (leading order) or NLO (next-to-leading order) splitting functions and α_s . There are several other choices that we can make in the program, including but not limited to $\alpha_s(M_Z)$ (strong coupling strength at the scale of the Z boson mass), the masses of the heavy quark flavours (c, b, t), the soft-gluon resolution parameter z_M , the ordering, the starting distribution. As starting distributions use default sets from evolution packages. In this thesis we will take the starting distributions from the packages QCDNUM, HERAPDF2.0LO and HERAPDF2.0NLO. First we will also demonstrate the flavour composition with PB-NLO-HERA I+II 2018 set2, a standard PDF set.

QCDNUM: Is based on polynomial spline interpolation and are basically “toy distributions”. The starting distributions are parametrised at the scale $\mu_0^2 = 2 \text{ GeV}^2$, with only 3 light quark flavours (u, d, s). However, the heavier quarks can be produced during the evolution to scales higher than their respective masses. [31]

HERAPDF2.0LO: Is based on fits to experimental data from the HERA collider, a lepton-proton collider at DESY in Germany. HERAPDF is a fitting group for collinear PDFs, only fitted to deep inelastic scattering (DIS). This particular one, 2.0 LO, uses leading order (LO) α_s and splitting functions. Again it uses only 3 light quark flavours (u, d, s) for the starting distributions, parametrised at the scale $\mu_0^2 = 2 \text{ GeV}^2$. [32] [33] [34]

HERAPDF2.0NLO: Is the same as HERAPDF2.0LO, but with next-to-leading order (NLO) α_s and splitting functions instead. [32] [33] [35]

PB-NLO-HERA I+II 2018 set2: Is a standard PDF set. It is based on a fit that produces TMD PDFs, is next-to-leading order, and was evolved to other scales via Parton Branching (PB) method. It is fitted to DIS and uses the same experimental data as HERAPDF. In set1, the collinear PDFs have the same parametrisation as HERAPDF. For set2 however, it is not the same as HERAPDF, because set2 uses angular ordering and different α_s . [36]

5.3.1 Flavour composition of the proton

We will first discuss the composition of the proton by showing the PDFs for each flavour for a standard set. The quantum numbers of a hadron are provided by what we call “valence quarks”, which are the quarks that are typically listed as the hadron content. But as we have seen in the previous chapters, there are also gluons and sea quarks inside a hadron that carry a part of the hadron’s momentum but don’t contribute to the quantum numbers, where the sea consists of quark-antiquark pairs. In case of the proton, the valence quarks are 2 up quarks and a down quark. Naively we would expect the valence up PDF to be twice the valence down PDF.

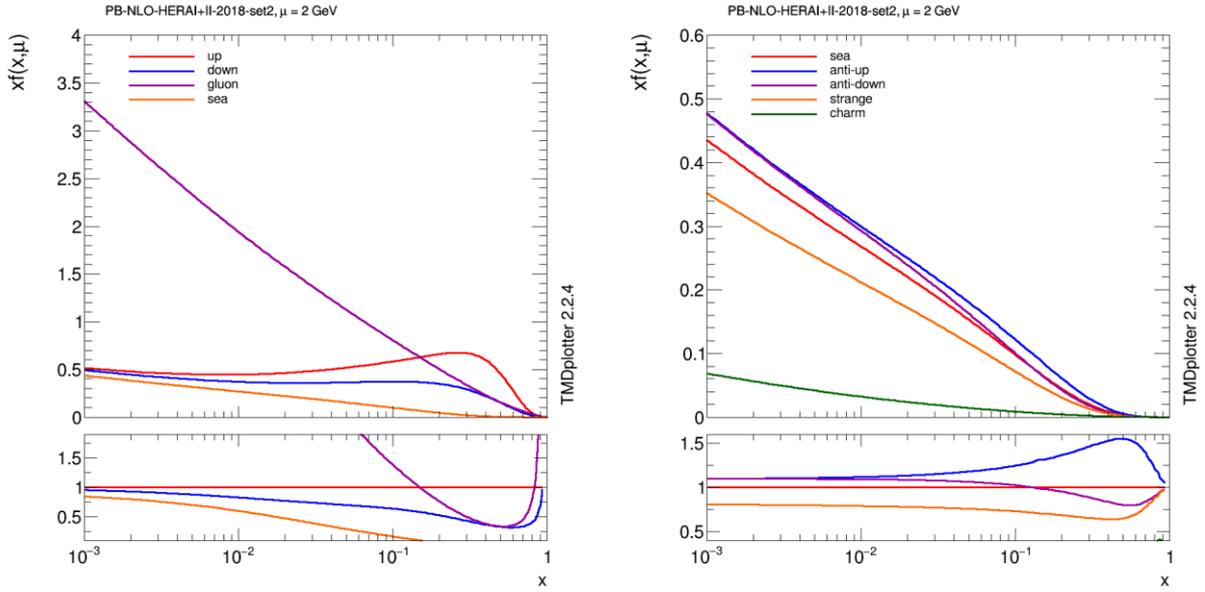


Figure 5.3: Momentum-weighted collinear PDFs for the different flavours, as a function of Bjorken x , from the standard set PB-NLO-HERA set 2, at scale $\mu = 2 \text{ GeV}$. The smaller graphs at the bottom are the ratios. Note that for sea quarks, quark and antiquark have the exact same PDF. (Thus “sea up” equals “anti-up”, “sea down” equals “anti-down”, “strange” equals “anti-strange”, “charm” equals “anti-charm”). Important to note is that on these graphs the “sea” curve is not the average of the whole sea, but the average of only the sea up, sea down and strange.

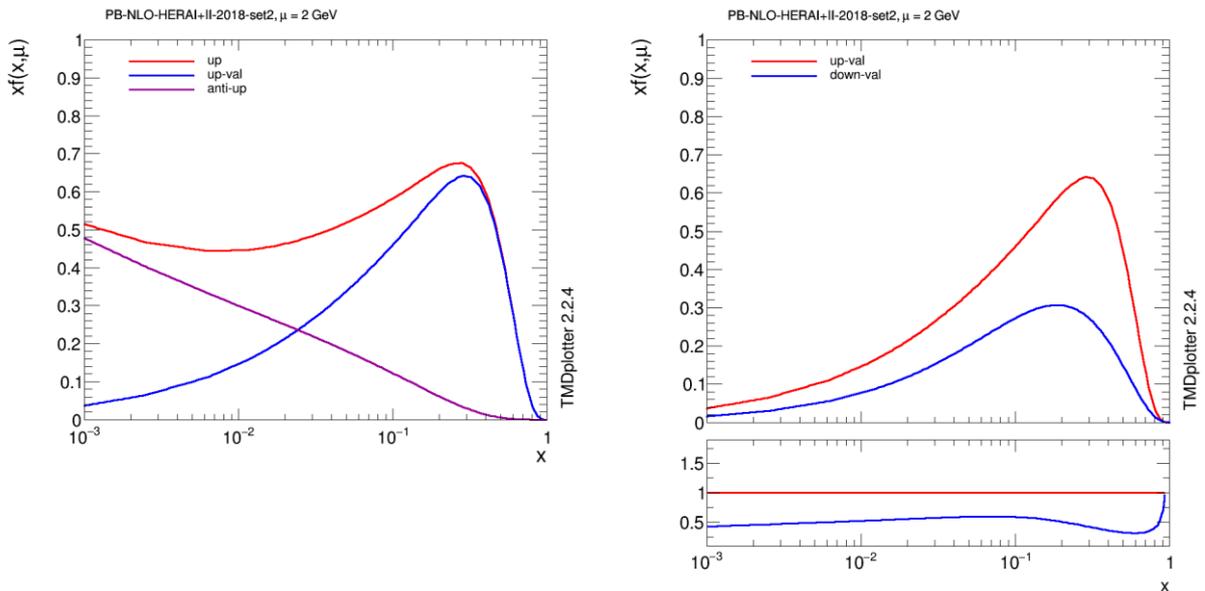


Figure 5.4: Comparison and decomposition of total, valence and sea up and down quark PDFs, at the scale $\mu = 2 \text{ GeV}$. The left graph shows how the up quark PDF is decomposed in the valence up quarks and the sea up quarks (which equals the anti-up). The decomposition of down quark PDF in valence down and sea down is analogous to this, but not shown. The right graphs show both valence quark PDFs and the ratio between them.

We took a NLO standard PDF set based on experimental data from HERA to show the flavour composition of the proton at a certain scale μ . We show this in Figure 5.3

for $\mu = 2 \text{ GeV}$, which is a scale that is not too high, but high enough to allow a small charm and anticharm component. We can see in the figure that at high x most of the proton's momentum is carried by up quark, down quark and gluons. At small x most of the proton's momentum will be carried by gluons. We also notice the amount of sea quarks and antiquarks increases for decreasing x , and gets close the up and down PDFs at very small x . In Figure 5.4 we can see that the up and down quarks at high x will almost solely be valence quarks, while at small x they will mostly be sea up and sea down quarks. We also notice in Figure 5.4 that the ratio of valence up to valence down is approximately 2 as expected, except for high x . If we look at the sea composition in Figure 5.3, we notice that the ratios of the sea up, down and strange remain constant for small x . Due to the small masses of the up and down quark, we would naively expect them to have the same PDF in the sea, and this seems to be true for very small x . For x above 10^{-2} however, we see that they deviate from each other. At the scale of $\mu = 2 \text{ GeV}$ we would also naively expect the strange quark to have the same PDF as the sea up and sea down, which is shown to be not true. Figure 5.3 also shows that there is a small charm component in the sea, as expected since the energy scale is higher than the charm quark mass. (While the charm is part of the sea, in the figures the charm is not part of the curve for the "sea", due to the definition of "sea" in the plotting tool being the average of the three lightest flavours only.)

In Figure 5.5 we show the flavour composition at a higher scale, $\mu = 10 \text{ GeV}$. This scale also allows a small bottom and antibottom component. The figures show that at small x the ratio between gluon PDF and quark PDFs has increased compared to the case of $\mu = 2 \text{ GeV}$. Another observation we can make is that the ratio between the sea quarks has gotten closer to 1. These two effects are generally true for increasing μ scale.

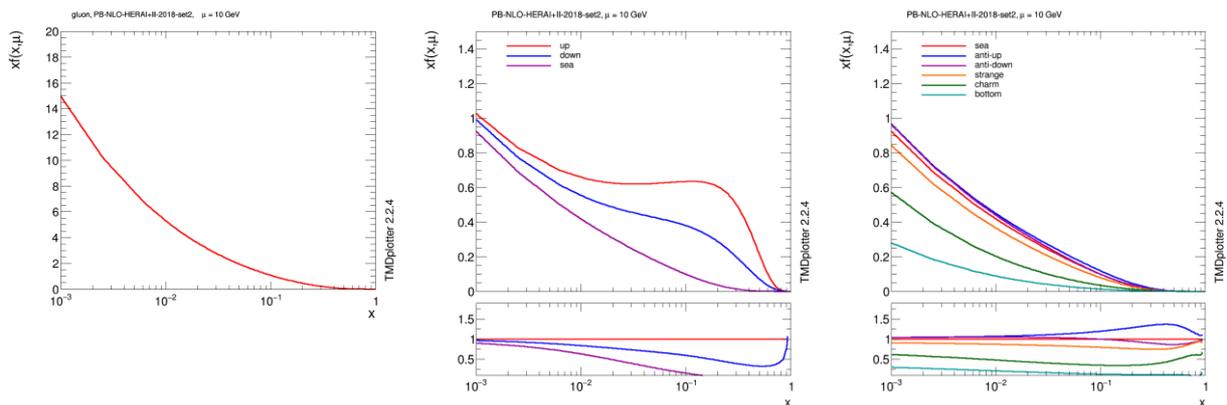


Figure 5.5: Momentum-weighted collinear PDFs for the different flavours, as a function of Bjorken x , from the standard set PB-NLO-HERA set 2, at scale $\mu = 10 \text{ GeV}$. The smaller graphs at the bottom are the ratios. Note that for sea quarks, quark and antiquark have the exact same PDF. ("sea up" equals "anti-up", "sea down" equals "anti-down", "strange" equals "anti-strange", etc.) Important to note is that on these graphs the "sea" curve is not the average of the whole sea, but the average of only the sea up, sea down and strange.

5.3.2 Initial distributions, LO vs NLO, and ordering

In this section we will discuss some of the different choices we can make in the settings of the Parton Branching program. In this thesis we used QCDNUM, HERAPDF2.0LO and HERAPDF2.0NLO as initial distributions. The explanation and references for these initial distributions is given in the intro of section 5.3. Figure 5.6 shows Parton Branching with QCDNUM as initial distribution, and Figure 5.7 shows this for HERAPDF2.0NLO as initial distribution. For QCDNUM we can also choose between leading order (LO) and next-to-leading order (NLO) α_s and splitting functions. In case of QCDNUM both LO and NLO will have the same initial distribution, while HERAPDF2.0 LO and NLO have different initial distributions. Figure 5.6 shows the differences between LO and NLO for QCDNUM.

For the ordering we restricted ourselves to two choices in this thesis: we either use transverse momentum ordering (DGLAP) or angular ordering (AO) of the parton branchings during the simulation. Collinear PDFs are well defined for both types of ordering, while TMD PDFs are well defined for angular ordering but not for transverse momentum ordering. So if we want to show TMD PDFs we should use angular ordering, if we want to show collinear PDFs we could use either ordering. In the transverse momentum ordered situation, z_M will be fixed, while in the angular ordered situation z_M will be dynamical. z_M is the parameter that separates soft gluons (non-resolvable) from hard gluons (resolvable). Radiated gluons outside of the angular ordered region will interfere destructively with each other. An abelian example of this is bremsstrahlung in QED (quantum electrodynamics), where the photons are radiated inside cones centered around the incoming and outgoing electron. In our situation (QCD) we have a similar effect, but non-abelian. Another difference between the orderings is how α_s is treated. In transverse momentum ordering we have μ dependant $\alpha_s(\mu)$, and μ equals q_\perp . In angular ordering we still have q_\perp dependant $\alpha_s(q_\perp)$, but this time q_\perp depends both on μ and z . Figures 5.6 and 5.7 show the differences between transverse momentum ordering (DGLAP) and angular ordering (AO), for the collinear PDFs.

In this section we only restrict ourselves to up quark and gluon PDFs. The effects on the down quark PDF will be analogous to the up quark, and the effect on the sea is expected to be similar as well.

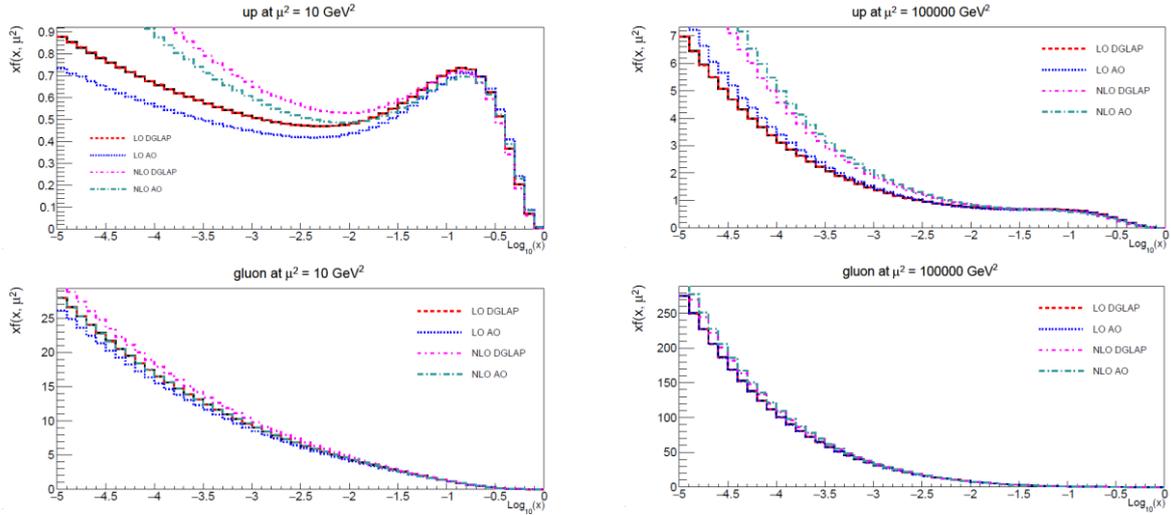


Figure 5.6: Simulated momentum-weighted collinear PDFs via Parton Branching, for the up quark and gluon at two different μ scales, taking QCDNUM as initial PDF. The four simulated PDFs on each graph were made with different settings: LO vs NLO, DGLAP vs AO.

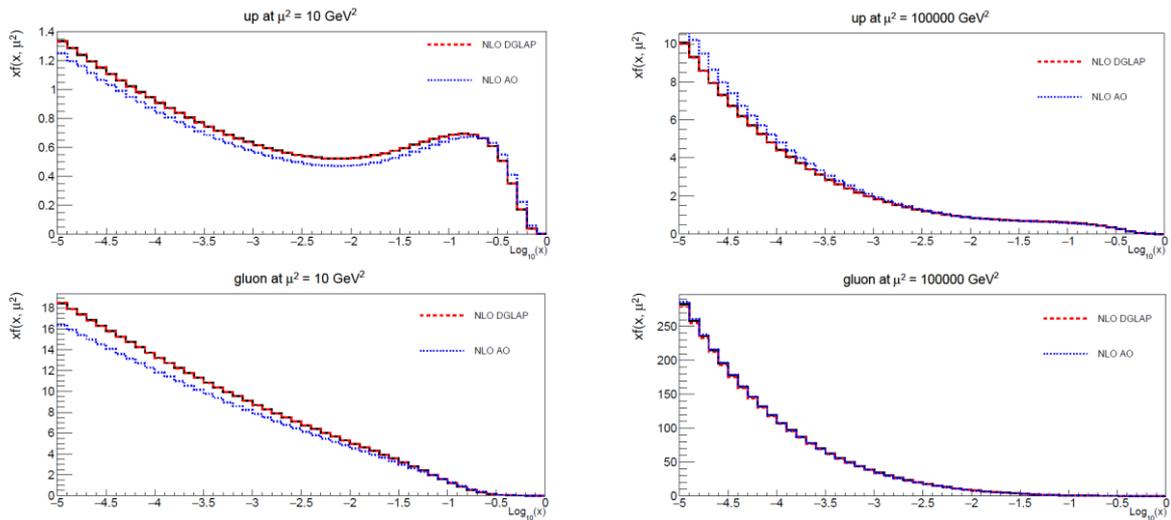


Figure 5.7: Simulated momentum-weighted collinear PDFs via Parton Branching, for the up quark and gluon at two different μ scales, taking HERAPDF2.0NLO as initial PDF. The two simulated PDFs on each graph are NLO but have different ordering settings: DGLAP vs AO.

In Figure 5.6, we show the simulated PDFs starting from initial QCDNUM PDF at scale $\mu_0^2 = 2 \text{ GeV}^2$, evolved to higher scales μ . In Figure 5.6, the largest change in PDF is caused by NLO evolution for the up quark. In Figure 5.7 we show the simulated PDFs starting from initial HERAPDF2.0NLO PDF, again evolved from scale $\mu_0^2 = 2 \text{ GeV}^2$ to higher scales. We can see that the LO AO evolution in Figure 5.6 and the NLO AO evolution in Figure 5.7 have for the most part a very similar effect on the PDF. A general behaviour we can notice for all simulated PDFs, except for the ones that remain almost the exact same as the initial PDF, is that they all significantly deviate from the initial PDF in the ratio at the high end of $\log(x)$.

5.3.3 Effect on PDF for collinear vs TMD splitting functions.

We will now look at the effects of TMD (transverse momentum dependent) splitting functions on the PDFs simulated by Parton Branching. First we look at the effects on the collinear PDFs in Figure 5.8, where we compare different LO PDFs, including one simulated via TMD splitting functions, starting from HERAPDF2.0LO as initial distribution. The PDF that uses TMD splitting functions, uses these everywhere in the Parton Branching method, both in the real emissions and in the Sudakov form factor. As an extra comparison we also include a NLO AO PDF that uses HERAPDF2.0NLO as initial distribution. After the collinear PDFs, we will look at the effects on the TMD PDFs. For those we will only use PDFs simulated via angular ordering (AO). Each time we will look what the effect is if you use LO AO with TMD splitting functions instead of collinear splitting functions, and what the effect is if you use NLO AO instead of LO AO (but both collinear splitting functions). Unfortunately we have no PDFs simulated via NLO AO with TMD splitting functions, because currently no one knows how to get NLO TMD splitting functions. This is because a TMD splitting function already includes some higher order corrections due to resummation.

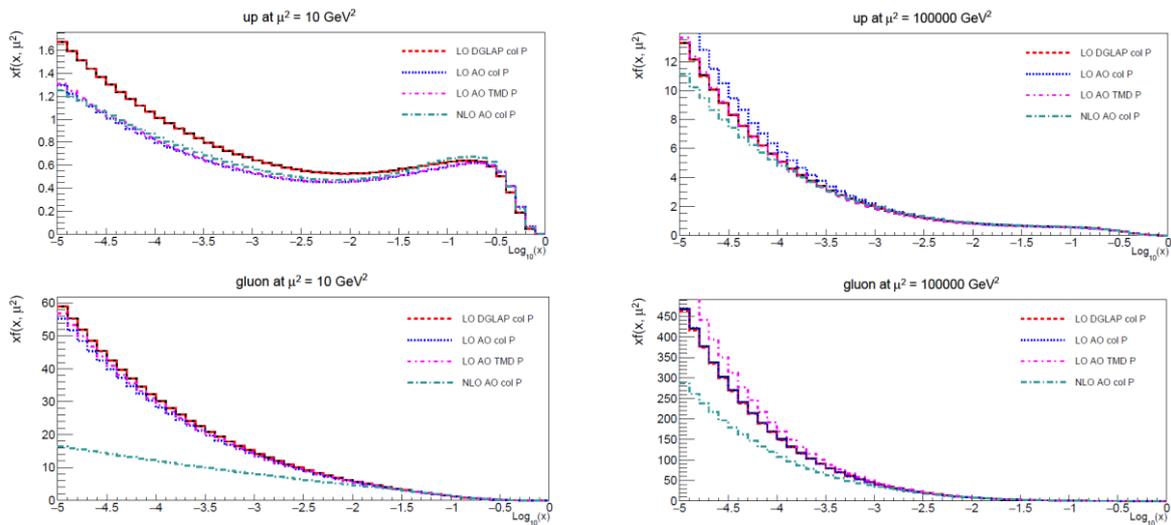


Figure 5.8: Simulated momentum-weighted collinear PDFs via Parton Branching, for the up quark and gluon at two different μ scales. “Col P” means they were simulated via collinear splitting functions, while “TMD P” means simulated via TMD splitting functions. All of the LO PDFs take HERAPDF2.0LO as initial PDF. As an extra comparison we also include the NLO AO PDF, which has HERAPDF2.0NLO as initial PDF instead. All PDFs with collinear splitting function were made with the typical updfevolv program. The “LO AO TMD P” was simulated via a modified program that uses TMD splitting functions instead.

Looking at the collinear PDFs in Figure 5.8, When we compare the LO AO PDF that uses collinear splitting functions with the one that uses TMD splitting functions, we notice that at $\mu^2 = 10 \text{ GeV}^2$ they are almost equal to each other. Apparently using TMD splitting functions at this scale and below, barely makes a difference. At $\mu^2 = 100000 \text{ GeV}^2$, a much higher scale, we do see a clear difference between collinear and TMD splitting functions, for x below $10^{-1.5}$ a 10^{-1} .

For the simulation using NLO AO, the resulting gluon PDF is very different than the others shown in Figure 5.8. It is much smaller than the others for small x . For the up quark at $\mu^2 = 10 \text{ GeV}^2$ on the other hand, the NLO AO PDF is near the two LO AO PDFs. For the up quark at $\mu^2 = 100000 \text{ GeV}^2$ the NLO AO PDF is clearly lower than the others for very small x .

Next we will show and investigate TMD PDFs, for simulations with angular ordering (AO), as a function of x with fixed k_{\perp} , or as a function of k_{\perp} with fixed x .

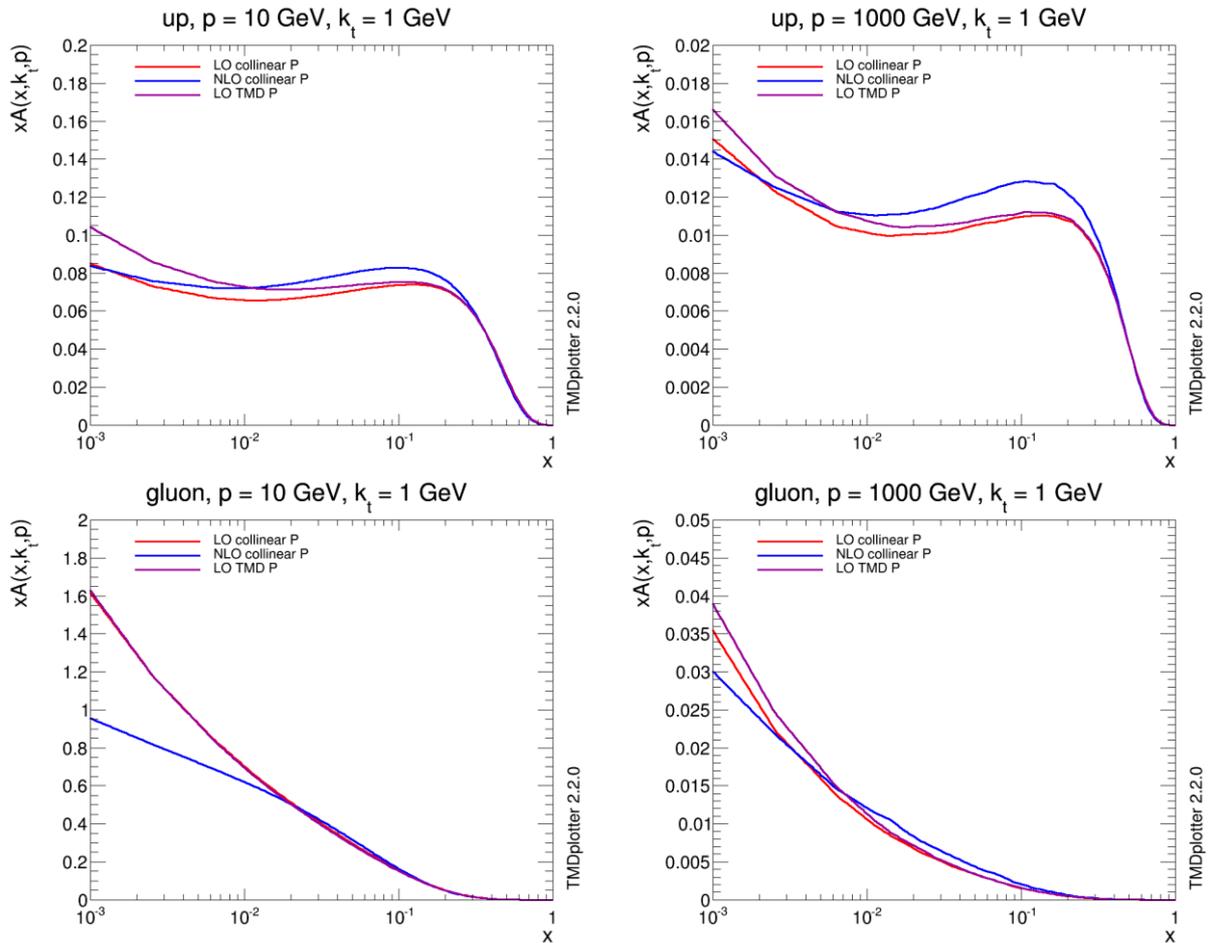


Figure 5.9: Simulated momentum-weighted TMD PDFs via Parton Branching, as a function of x , at transverse momentum $k_{\perp} = 1 \text{ GeV}$, for the up quark and gluon at two different μ scales (μ is called p in these figures). The red and blue PDFs used collinear splitting functions, while the purple PDF used TMD splitting functions. The two LO PDFs (red and purple) have HERAPDF2.0LO as initial PDF, while the NLO PDF (blue) has HERAPDF2.0NLO as initial PDF. All PDFs used angular ordering (AO). In the bottom left graph, the red and purple PDFs are almost exactly the same.

For the x dependence of the TMD PDFs, shown in Figure 5.9, we chose a fixed k_{\perp} that is much smaller than the two scales μ that we show. The TMD PDFs seem to very roughly have a similar shape as the collinear PDFs. All three considered TMD PDFs look almost equal to each other at the higher end of $\log(x)$. In case of the gluon PDF at scale $\mu = 10 \text{ GeV}$ we can even say that it is almost exactly the same for LO evolution with either collinear splitting functions or TMD splitting functions, at least

for the whole region of x that is shown. For the gluon at a much higher scale $\mu = 1$ TeV, and for the up quark at both scales, the effect of simulating the LO PDF via TMD splitting functions, compared to collinear splitting functions, is just a small increase in the PDF at small x . (For the up quark this is visible in Figure 5.9 this is visible for x on the order of 10^{-1} and below, while for the gluon it is visible for x on the order of 10^{-2} and below.)

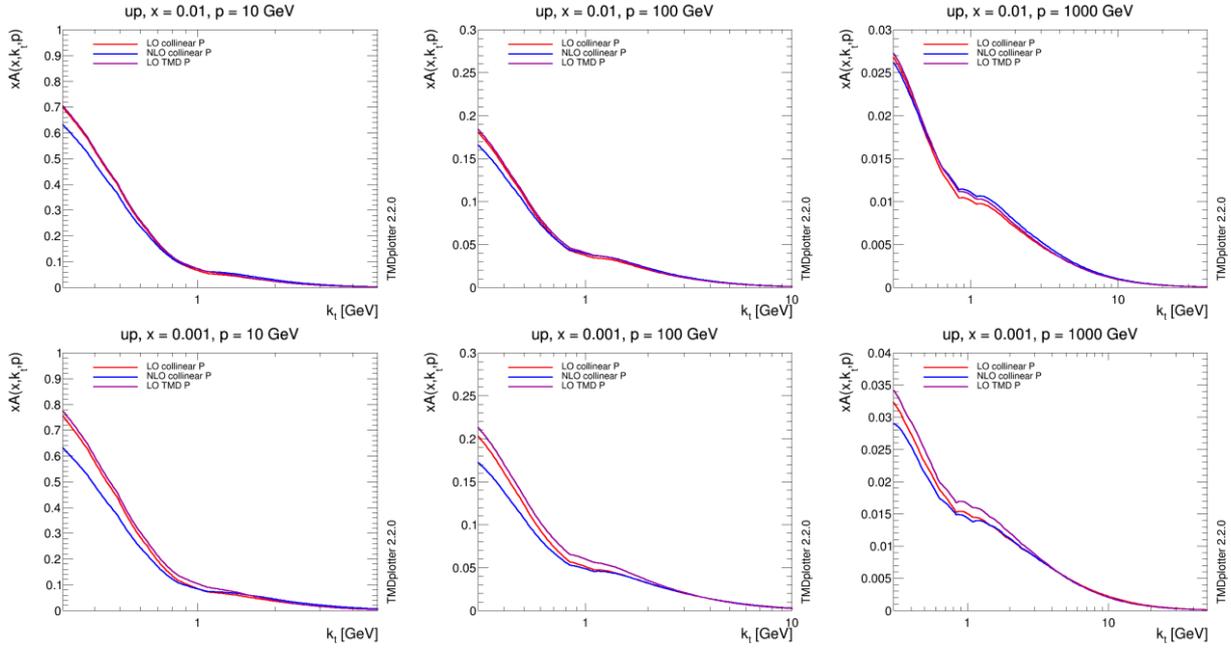
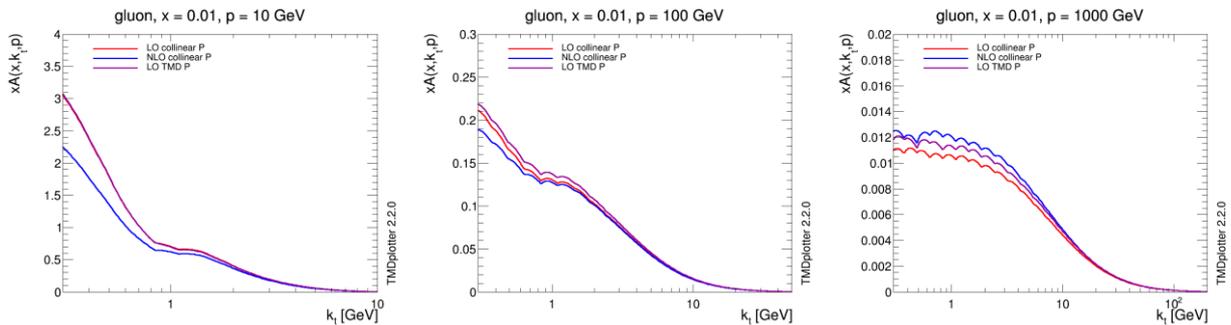


Figure 5.10: Simulated momentum-weighted up quark TMD PDFs via Parton Branching, as a function of k_{\perp} , at $x = 0.01$ and $x = 0.001$, and at three different μ scales (μ is called p in these figures). The red and blue PDFs used collinear splitting functions, while the purple PDF used TMD splitting functions. The two LO PDFs (red and purple) have HERAPDF2.0LO as initial PDF, while the NLO PDF (blue) has HERAPDF2.0NLO as initial PDF. All PDFs used angular ordering (AO). In the graphs for $x = 0.01$ with $\mu = 10$ GeV and 100 GeV, the red and purple PDFs are almost the same.



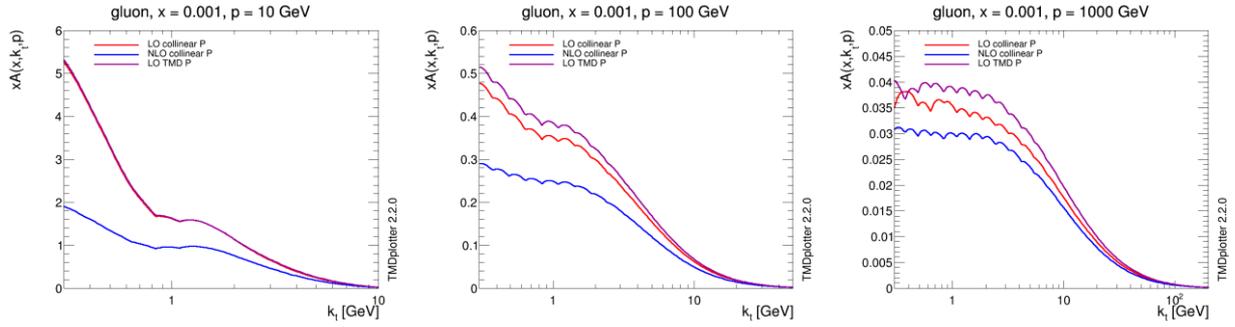


Figure 5.11: Simulated momentum-weighted gluon TMD PDFs via Parton Branching, as a function of k_{\perp} , at $x = 0.01$ and $x = 0.001$, and at three different μ scales (μ is called p in these figures). The red and blue PDFs used collinear splitting functions, while the purple PDF used TMD splitting functions. The two LO PDFs (red and purple) have HERAPDF2.0LO as initial PDF, while the NLO PDF (blue) has HERAPDF2.0NLO as initial PDF. All PDFs used angular ordering (AO). In the graphs $\mu = 10$ GeV, the red and purple PDFs are almost the same.

We show the k_{\perp} dependence of the simulated TMD PDFs for two fixed x values, and at three different μ scales. For both the up quark and gluons, shown in Figures 5.10 and 5.11 respectively, the general behaviour seems to be a decreasing PDF for increasing k_{\perp} , and the PDF practically goes to 0 at some k_{\perp} . We might expect this value for k_{\perp} to be the scale μ , because in transverse momentum ordering we would have $k_{\perp} \ll \mu$. In Figures 5.10 and 5.11 we can see that this is not true. The PDFs under consideration are angular ordered, not transverse momentum ordered. We can also see that on each graph the three PDFs look almost equal at their high k_{\perp} tail. We also see that in the x and μ regime that we consider in Figures 5.10 and 5.11, it seems that the TMD PDF is increased if you go from LO PDF with collinear splitting functions to LO PDF with TMD splitting functions. However, this effect is small at $\mu = 10$ GeV, but the effect increases with increasing μ . The effect also increases with decreasing x . In all the situations that are shown in Figures 5.10 and 5.11, the NLO PDF is clearly different from the others, except for the up quark at $\mu = 1$ TeV for $x = 0.01$ in Figure 5.10.

6 Conclusion

In this thesis we tried to calculate the transverse momentum dependent (TMD) gluon-to-quark splitting function and study its physical effects numerically. In order to do so, we studied and discussed the framework of quantum field theory (QFT), mostly in the context of gauge field theory, and the theory of quantum chromodynamics (QCD) in particular. We studied how to quantise gauge field theories, and how to renormalize them. We studied the effects of renormalisation at high energy, which in the case of QCD were asymptotic freedom, the QCD scale and confinement. Then we studied QCD modified parton model and the factorisation method in order to treat deep inelastic scattering (DIS). Deep inelastic scattering was the main example and we kept using it throughout most of the high energy part of the thesis. It allowed us to study the inner structure of hadrons. We studied radiation processes in high energy QCD, and the DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) evolution equations for parton densities (parton distribution functions). This is where we introduced the DGLAP splitting functions. We studied the splitting functions in detail near the end of the thesis, and explained their physical meaning. First we did this for the case of collinear momentum, and noticed the collinear divergences that arise. We studied factorisation schemes in the context of the splitting functions and we studied an extension to include the transverse momentum as well, already in leading order. Then we calculated the TMD gluon-to-quark splitting function in detail, and gave physical meaning to the results. We also fully integrated the TMD splitting function (integration over momentum q) and showed that it can be expressed by hypergeometric functions. We presented the results of Monte Carlo calculations for QCD parton densities, and studied the numerical effects of the splitting functions.

7 References

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