

EXTENDING COMPACT HAMILTONIAN \mathbb{S}^1 -SPACES TO INTEGRABLE SYSTEMS WITH MILD DEGENERACIES IN DIMENSION FOUR

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ABSTRACT. We show that all compact four-dimensional Hamiltonian \mathbb{S}^1 -spaces can be extended to a completely integrable system on the same manifold such that all singularities are non-degenerate, except possibly for a finite number of degenerate orbits of parabolic (also called cuspidal) type – we call such systems hypersemitoric.

More precisely, given any compact four dimensional Hamiltonian \mathbb{S}^1 -space (M, ω, J) we show that there exists a smooth $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is a completely integrable system of hypersemitoric type. Hypersemitoric systems generalize semitoric systems. In addition to elliptic-elliptic, elliptic-regular, and focus-focus singular points which can occur in semitoric systems, hypersemitoric systems may also have hyperbolic-regular and hyperbolic-elliptic singular points (hyperbolic-hyperbolic points cannot appear due to the presence of the global \mathbb{S}^1 -action) and moreover degenerate singular points of a relatively tame type called parabolic.

Admitting the existence of degenerate points is necessary since there exist compact four-dimensional Hamiltonian \mathbb{S}^1 -spaces whose extensions *must* include degenerate singular points of *some* kind as we show in the present paper. Parabolic points are among the most common and natural degenerate points, and we show that it is sufficient to only admit these degenerate points in order to extend all Hamiltonian \mathbb{S}^1 -spaces. In this sense, hypersemitoric systems are thus the “nicest and smallest” class of systems to which all Hamiltonian \mathbb{S}^1 -spaces can be extended. Moreover, we prove several foundational results about these systems, such as the non-existence of loops of hyperbolic-regular points and properties about their fibers.

1. INTRODUCTION

1.1. **Background.** For many years interactions between the classical field of integrable systems and the relatively modern field of compact Hamiltonian group actions on symplectic manifolds have yielded interesting results. Some of the earliest, and best known, examples of results in this direction are those of Atiyah [Ati82], Guillemin & Sternberg [GS82], and Delzant [Del88]. Taken together these results produce a classification of effective Hamiltonian \mathbb{T}^n -actions on compact symplectic n -manifolds, which can equivalently be thought of as a classification of compact integrable systems for which all integrals generate periodic flows of the same period. Karshon & Lerman [KL15] generalized this to the non-compact case. Another breakthrough in this area is the work of Karshon [Kar99], which classifies effective Hamiltonian \mathbb{S}^1 -actions on compact symplectic 4-manifolds. Given a fixed compact symplectic four manifold, the circle and 2-torus actions on that manifold have also been classified by Holm & Kessler [HK19] and Karshon & Kessler & Pinsonnault [KKP15]. Another recent important classification result is the classification of so-called *semitoric integrable systems*, due to Pelayo & Vũ Ngọc [PVuN09, PVuN11] with a generalization by Palmer & Pelayo & Tang

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[PPT19]. Note that the semitoric classification includes both compact and non-compact systems.

For an integrable system to be semitoric it must satisfy several properties, but in particular only one of the two integrals is required to have periodic flow, so semitoric systems can be thought of as a bridge connecting the toric classification to more general situations in integrable systems.

Also, the periodic flow of the integral means that each semitoric system naturally comes with the structure of a Hamiltonian \mathbb{S}^1 -action. The relationship between the semitoric classification and Karshon’s classification of Hamiltonian \mathbb{S}^1 -actions on compact 4-manifolds was studied by Hohloch & Sabatini & Sepe [HSS15]. Karshon [Kar99] answered the question of which \mathbb{S}^1 -spaces can be obtained from toric integrable systems, and Hohloch & Sabatini & Sepe [HSS15] show that there are some \mathbb{S}^1 -spaces that can be obtained from semitoric systems but not from toric ones, though it is important to note that not all Hamiltonian \mathbb{S}^1 -actions on compact symplectic 4-manifolds can be obtained from a semitoric integrable system in this way.

In this paper we introduce a class of integrable systems which further generalize semitoric integrable systems, which we call *hypersemitoric systems*, while still requiring that one of the integrals generate a periodic flow. This new class of systems represents a further step towards general integrable systems, and moreover enjoys the property that *all* Hamiltonian \mathbb{S}^1 -actions on symplectic four-manifolds can be obtained from an integrable system of hypersemitoric type.

Hypersemitoric systems get as far away from the rigid situation of toric systems as the continued existence of an underlying effective Hamiltonian \mathbb{S}^1 -action allows while avoiding “really bad” degeneracies. Nevertheless, as we will see, the presence of the \mathbb{S}^1 -action enables some of the techniques related to toric and semitoric geometry to be made to work in this situation. Pelayo & Vũ Ngọc [PVuN12, Section 2.3] discuss the expected difficulty in classifying systems with hyperbolic singular points. One of the main extra difficulties that make these systems more challenging, but also more interesting, is that the fibers of the momentum map are often disconnected. The existence of disconnected fibers prevents the application of many standard techniques, and therefore analyzing these systems is a non-trivial endeavor. These extra difficulties are unavoidable though, since hyperbolic singularities, and also disconnected fibers, are a common feature in natural systems. For instance, the Lagrange top and the two body problem are two fundamental physical systems which include an \mathbb{S}^1 -symmetry and also have hyperbolic points. Examples with hyperbolic singular points also often include a type of degenerate singular point called a *parabolic singularity* [BGK18, EG12], and in certain situations hyperbolic singular points force the presence of parabolic degenerate points (c.f. Corollary 4.6). Intuitively, one can think of parabolic points as generic among degenerate points. Degenerate points of parabolic type, like non-degenerate points, are stable under perturbations [Gia07], and they are therefore common in nature as well. For instance, they appear in the Kovalevskaya top [BRF00] and many other systems from rigid body dynamics, see the references in [BGK18]. Therefore, we define hypersemitoric systems to be those systems where one of the integrals generates an \mathbb{S}^1 -action and the singular points are all of non-degenerate or parabolic type. This is a natural definition because:

- as discussed above, non-degenerate singularities and parabolic singularities are the most common ones in nature;

- all \mathbb{S}^1 -spaces can be extended to a hypersemitoric system (c.f. Theorem 1.6) and the class of hypersemitoric systems is in some sense the “easiest and smallest” class with this property.

1.2. \mathbb{S}^1 -spaces and integrable systems. *Throughout this paper we will assume that all manifolds M are connected.* If (M, ω) is a symplectic manifold then any smooth $J: M \rightarrow \mathbb{R}$ determines a vector field \mathcal{X}^J (called *Hamiltonian vector field of J*) on M via the equation $\omega(\mathcal{X}^J, \cdot) = -dJ$.

Definition 1.1. If (M, ω) is a compact four dimensional symplectic manifold and the flow of \mathcal{X}^J is periodic of minimal period 2π then we call (M, ω, J) a *Hamiltonian \mathbb{S}^1 -space*, which we will often shorten to simply an *\mathbb{S}^1 -space*, and we call J the *Hamiltonian*. In other words, the Hamiltonian flow of such a J generates an effective action of $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ on M .

Such \mathbb{S}^1 -spaces are classified up to isomorphism by the work of Karshon [Kar99] in terms of a labeled graph encoding information about the fixed points and isotropy groups (\mathbb{Z}_k -spheres) of the \mathbb{S}^1 -action (see Section 2.1). That is, she constructed a bijection between isomorphism classes of such systems and their associated graphs.

On the other hand, a triple $(M, \omega, F = (f_1, \dots, f_n))$ is an *$2n$ -dimensional integrable system* if (M, ω) is a $2n$ -dimensional symplectic manifold and $F: M \rightarrow \mathbb{R}^n$, known as the *momentum map*, satisfies:

- (1) $\omega(\mathcal{X}^{f_i}, \mathcal{X}^{f_j}) = 0$ for all $i, j \in \{1, \dots, n\}$;
- (2) $(\mathcal{X}^{f_1})_p, \dots, (\mathcal{X}^{f_n})_p \in T_p M$ are linearly independent for almost all $p \in M$.

We say that an integrable system (M, ω, F) is *compact* if M is compact. The points in M at which Condition (2) fails are called *singular points*, and the other points of M are called *regular points*. If the vector fields $\mathcal{X}^{f_1}, \dots, \mathcal{X}^{f_n}$ are complete, which is automatic if M is compact, then by Condition (1) their flows commute and thus generate an action of \mathbb{R}^n on M . An integrable system is called *toric* if M is compact and each \mathcal{X}^{f_i} is periodic of minimal period 2π , which implies that their flows generate an effective action of the n -torus \mathbb{T}^n on M . Atiyah [Ati82] and Guillemin & Sternberg [GS82] showed that if (M, ω, F) is a toric integrable system then the image $F(M)$ is a convex n -dimensional polytope, and moreover Delzant [Del88] showed that toric integrable systems are classified up to isomorphism by this convex polytope. If the isomorphism is only required to intertwine the torus actions then the toric systems are classified by Delzant polytopes up to the action of the affine group $\mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

The following class of systems generalize toric systems in dimension four:

Definition 1.2. A four dimensional integrable system $(M, \omega, F = (J, H))$ is a *semitoric integrable system*, or briefly a *semitoric system*, if:

- (1) J is proper and generates an effective \mathbb{S}^1 -action;
- (2) all singular points of $F = (J, H)$ are non-degenerate and do not include hyperbolic blocks (i.e. there are no singular points of hyperbolic-regular, hyperbolic-elliptic, or hyperbolic-hyperbolic type, as described in Section 2.5).

Semitoric systems were classified in terms of five invariants, generalizing the toric classification, by Pelayo & Vũ Ngọc [PVuN09, PVuN11]. The original classification has the extra assumption that the systems must be *simple* (see Section 2.7), but this assumption has been removed recently by Palmer & Pelayo & Tang [PPT19].

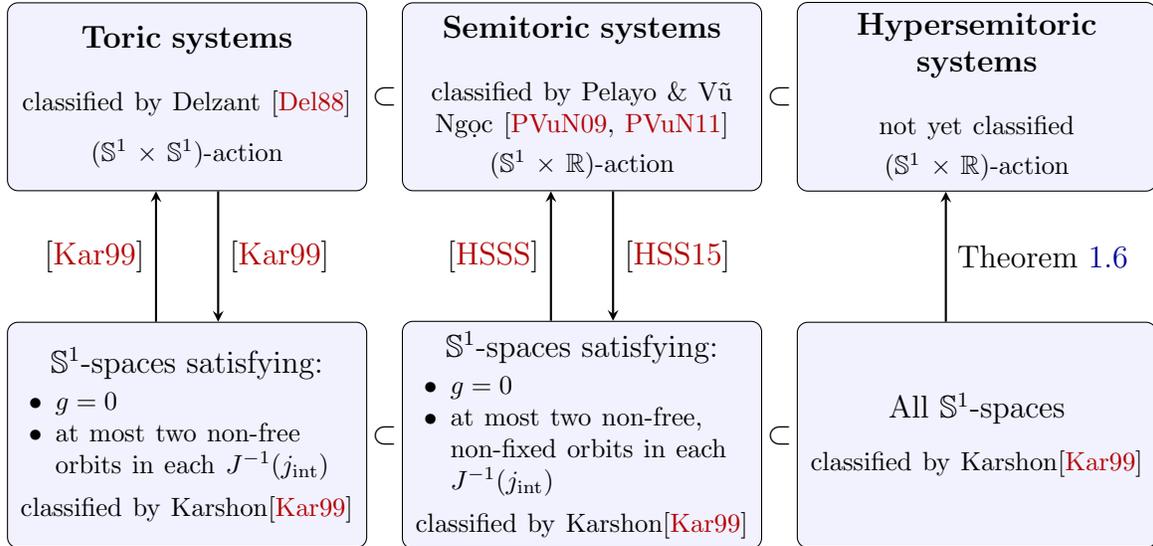


FIGURE 1. Relationships between integrable systems and Hamiltonian \mathbb{S}^1 -spaces. Above, g refers to the genus of any fixed surfaces, if they exist, and j_{int} refers to an element of the interior of the image of J . This diagram is commutative in the sense that the inclusions and projections downwards are compatible. The upwards arrows represent extending, and the downwards arrows represent using the classification of the integrable system to recover the Karshon graph. Since hypersemitoric systems are not yet symplectically classified, there is no downwards arrow below them in the diagram.

Semitoric systems are much more general than toric systems, and their behavior is much more complicated due to the presence of focus-focus singularities which cannot occur in toric systems.

Definition 1.3. If (M, ω, J) is an \mathbb{S}^1 -space and $H: M \rightarrow \mathbb{R}$ is such that $(M, \omega, (J, H))$ is an integrable system, then we say that $(M, \omega, (J, H))$ is an *extension* of (M, ω, J) and that (M, ω, J) is the *underlying* \mathbb{S}^1 -space of $(M, \omega, (J, H))$.

1.3. Motivation. Given a compact integrable system $(M, \omega, (J, H))$ such that J generates an \mathbb{S}^1 -action, it is possible to obtain an \mathbb{S}^1 -space by simply forgetting the function H . But obtaining an integrable system from a given \mathbb{S}^1 -space can be more complicated.

Question 1.4. *Given a compact four dimensional Hamiltonian \mathbb{S}^1 -space (M, ω, J) , when can we find an $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is an integrable system? What are the “nicest possible” extensions?*

Karshon already was considering this question in her paper [Kar99] containing the original classification of \mathbb{S}^1 -spaces, where she proved exactly which \mathbb{S}^1 -spaces can be extended to a toric integrable system, and more recently¹ Hohloch & Sabatini & Sepe & Symington [HSSS] have described which \mathbb{S}^1 -spaces exactly can be extended to a semitoric integrable system.

The motivation of this paper is two-fold:

¹first announced at Poisson 2014 in a talk by Daniele Sepe.

- **Motivation 1:** Condition (2) in the definition of semitoric systems explicitly prohibits hyperbolic singular points, which are abundant and important in physical systems and other applications, and are furthermore the next natural singularities to study after focus-focus singular points;
- **Motivation 2:** There are \mathbb{S}^1 -spaces which cannot be extended to semitoric systems, so to extend all \mathbb{S}^1 -spaces to a certain class of integrable systems we must generalize the notion of semitoric systems.

1.4. **Results.** We now define *hypersemitoric systems*, which represent a substantial generalization of semitoric systems and, in particular, they include singularities with hyperbolic components and certain degenerate singular points.

Definition 1.5. An integrable system $(M, \omega, (J, H))$ is *hypersemitoric* if:

- (1) J is proper and generates an effective \mathbb{S}^1 -action;
- (2) all degenerate singular points of F (if any) are of parabolic type (as in Definition 2.26).

Hypersemitoric systems form a significantly more general class than semitoric systems (cf. Remark 1.8), which are in turn significantly more abundant than toric ones. Dullin & Pelayo [DP16] constructed in fact a hypersemitoric system by starting with a semitoric system and changing the momentum map near a focus-focus point to induce a Hamiltonian-Hopf bifurcation and produce a family of singular points (which include hyperbolic-regular and degenerate singular points) — for more details see Section 3.1.

In the first part of this paper we give several examples and prove a few first properties of such systems. For instance, we show that hypersemitoric systems do not admit “loops of hyperbolic-regular values” (see Corollary 4.3 for a more precise statement).

One way in which hypersemitoric systems are more complicated than semitoric systems is that the momentum map of hypersemitoric systems often has disconnected fibers, so the base of the Lagrangian fibration induced by the integrable system cannot be naturally identified with the momentum map image. Additionally, hyperbolic-regular singular fibers occur in one-parameter families, and in many cases the two endpoints of the family correspond to fibers that include degenerate singular points, which is why it is somewhat necessary to allow degenerate singularities in this definition. In fact, there exist systems which cannot be extended to an integrable system with no degenerate singular points (Corollary 4.6), so to have any hope to be able to extend all \mathbb{S}^1 -spaces to a class of integrable systems, we must include some type of degenerate points. Degenerate singularities of parabolic type are one of the simplest classes of degenerate singularities among the typical degenerate singularities discussed in Bolsinov & Fomenko [BF04], and they are stable under perturbation [Gia07]. That is, parabolic points cannot be removed from a system by completely integrable perturbations. Moreover, these are the degenerate singularities which occur in the examples which motivated this paper (see Section 3) — there they occur when a family of hyperbolic-regular singularities meets with a family of elliptic-regular singularities. For these reasons we opted to investigate if working only with parabolic degenerate points would be sufficient in order to extend all Hamiltonian \mathbb{S}^1 -spaces — and indeed it did turn out to be sufficient as Theorem 1.6 will show. Apart from that, it is worthwhile mentioning that local and semilocal invariants of parabolic points were recently described by Bolsinov & Guglielmi & Kudryavtseva [BGK18].

In the second part of this paper we show that *any* compact four dimensional Hamiltonian \mathbb{S}^1 -space can be extended to a hypersemitoric integrable system. That is, we prove:

Theorem 1.6. *Let (M, ω, J) be a 4-dimensional Hamiltonian \mathbb{S}^1 -space where (M, ω) is a compact symplectic manifold. Then there exists a smooth function $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is an integrable system of hypersemitoric type.*

The proof of Theorem 1.6 actually gives a slightly more refined result about the properties of the resulting hypersemitoric system, which we state as Corollary 5.10. Combining Theorem 1.6 with the results of Karshon [Kar99] and those of Hohloch & Sabatini & Sepe & Symington [HSSS], we obtain the following theorem, illustrated in Figure 1.

Theorem 1.7. *Let (M, ω, J) be a 4-dimensional Hamiltonian \mathbb{S}^1 -space where (M, ω) is a compact symplectic manifold. Then:*

- (1) *there exists $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is a toric integrable system if and only if each fixed surface (if any exists) has genus zero and each non-extremal level set of J contains at most two non-free orbits of the \mathbb{S}^1 -action (see Karshon [Kar99, Proposition 5.21]);*
- (2) *there exists $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is a semitoric system if and only if each fixed surface has genus zero (if any exists) and each non-extremal level set of J contains at most two non-free orbits of the \mathbb{S}^1 -action which are not fixed points (see Hohloch & Sabatini & Sepe & Symington [HSSS]);*
- (3) *in all cases, there exists $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is hypersemitoric (Theorem 1.6).*

The idea of the proof: The proof of Theorem 1.6 makes use of the minimal model results in Karshon's classification [Kar99]. Karshon proved that all \mathbb{S}^1 -spaces can be obtained from a list of certain *minimal models* by a finite sequence of \mathbb{S}^1 -equivariant blowups. We first show that all of the minimal models can be extended to hypersemitoric systems (Proposition 5.1). To complete the proof we then show that these hypersemitoric systems can be used to construct a hypersemitoric system on any \mathbb{S}^1 -space obtained from a minimal model via a sequence of \mathbb{S}^1 -equivariant blowups, i.e. all \mathbb{S}^1 -spaces. We show this by arguing that any \mathbb{S}^1 -equivariant blowup on the underlying \mathbb{S}^1 -space can be obtained by performing a certain operation on the extended system, which preserves the fact that it is hypersemitoric. One of the main new ideas is related to performing an \mathbb{S}^1 -equivariant blowup at a focus-focus singular point. This proceeds by incorporating the technique described by Dullin & Pelayo [DP16] to use a supercritical Hamiltonian-Hopf bifurcation to replace a neighborhood of the focus-focus singular value in $F(M)$ with a triangle of singular values known as a *flap* while preserving the structure of the integrable system and the \mathbb{S}^1 -action. The flap includes two families of elliptic-regular points, one family of hyperbolic-regular points, one elliptic-elliptic point, and two degenerate fibers, and then a usual toric-type blowup can be performed on the elliptic-elliptic point. This process is shown in Figure 2. Using our techniques, we also obtain a short proof of item (2) above, recovering the results of [HSSS], see Corollary 5.11.

Remark 1.8. Hypersemitoric systems represent a very general class of systems with \mathbb{S}^1 -symmetries. To analyze this more deeply is beyond the scope of the present paper (we will consider this in a future project), but, given an \mathbb{S}^1 -space (M, ω, J) , it seems reasonable to

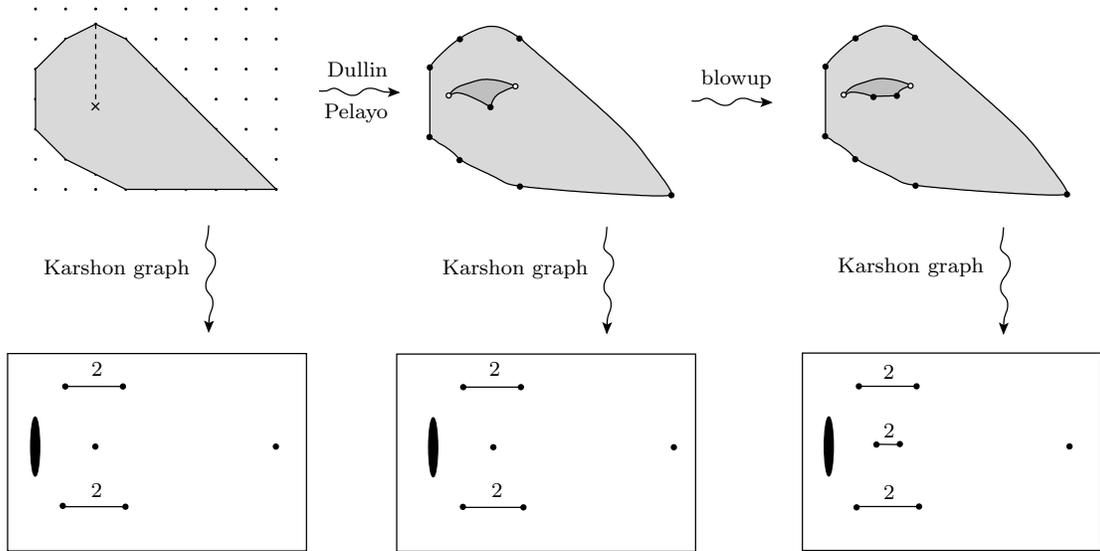


FIGURE 2. In order to perform an \mathbb{S}^1 -equivariant blowup at a fixed point in the \mathbb{S}^1 -space which corresponds to a focus-focus singular point in the extended system, we can first change the focus-focus point into an elliptic-elliptic point by a Hamiltonian-Hopf bifurcation using the technique of Dullin & Pelayo [DP16].

conjecture that the set

$$\{H: M \rightarrow \mathbb{R} \mid (M, \omega, (J, H)) \text{ is hypersemitoric}\}$$

is an open and dense subset of

$$\{H: M \rightarrow \mathbb{R} \mid (M, \omega, (J, H)) \text{ is integrable}\}.$$

This seems plausible by the following sketched argument: such an integral H can (roughly) be thought of as a one-parameter family of functions on the reduced space. By the work of Cerf [Cer70], one parameter families of functions on smooth manifolds can be perturbed to be Morse at all but finitely many times, at which times the function takes a form which lifts to be a parabolic point. Lifting such a perturbed function should yield a hypersemitoric system. One of the technical problems that would need to be overcome, is that in this situation the reduced space is in general not a smooth manifold but has a finite number of singular points.

1.5. Outline of paper: In Section 2 we recall various results we will need throughout the paper. In Section 3 we give some motivating examples. In Section 4 we prove some results about properties of integrable systems for which one of the integrals generates an \mathbb{S}^1 -action. In Section 5 we prove Theorem 1.6.

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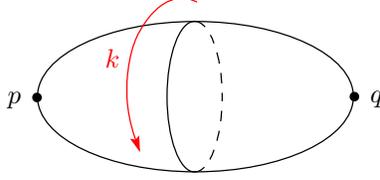


FIGURE 3. A \mathbb{Z}_k -sphere connecting the fixed points p and q .

2. PRELIMINARIES

In this section we briefly recall the results we will need and give ample references for the details. In particular, this section mainly summarizes the work of Karshon [Kar99], Delzant [Del88], Pelayo & Vũ Ngọc [PVuN09, PVuN11], Hohloch & Sabatini & Sepe [HSS15], Efstathiou & Giacobbe [EG12], and Bolsinov & Guglielmi & Kudryavtseva [BGK18].

2.1. \mathbb{S}^1 -spaces and their Karshon graphs. Let (M, ω, J) be an \mathbb{S}^1 -space, which recall always means a compact four-dimensional Hamiltonian \mathbb{S}^1 -space. Following [Kar99], we will construct a labeled graph associated to this space. Let $M^{\mathbb{S}^1}$ be the fixed point set for the \mathbb{S}^1 -action.

Lemma 2.1 ([Kar99, Lemma 2.1]). *Let (M, ω, J) be a four dimensional compact Hamiltonian \mathbb{S}^1 -space. Then $M^{\mathbb{S}^1}$ has finitely many components, each of which is either an isolated point or a symplectic surface, and any such surface, if it exists, is exactly the preimage (under J) of the maximum or minimum value of J . Moreover, the preimages under J of its maximum and minimum values are each connected.*

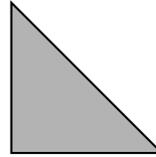
For $k \in \mathbb{Z}_{>0}$ let $\mathbb{Z}_k = \{\lambda \in \mathbb{S}^1 \mid k\lambda \in 2\pi\mathbb{Z}\}$. Connected components of the set of points with isotropy subgroup \mathbb{Z}_k , $k > 1$, are homeomorphic to cylinders, and the closure of each such component is an embedded sphere in M which is rotated by the \mathbb{S}^1 -action and whose poles are fixed points. These are known as \mathbb{Z}_k -spheres, see Figure 3. The \mathbb{Z}_k -spheres connect distinct components of $M^{\mathbb{S}^1}$.

Now we will construct the graph. The set of nodes is the set of connected components of $M^{\mathbb{S}^1}$ each labeled by the value of J on that component. The fixed surfaces Σ are represented by “fat nodes,” which we draw as large nodes, that are labeled by the value of J and additionally the normalized symplectic area of the surface, $A = \frac{1}{2\pi} \int_{\Sigma} \omega$, and its genus g . If the genus label is 0 we often omit it in figures. Two nodes are connected by an edge if and only if there exists a \mathbb{Z}_k -sphere, $k > 1$, connecting the two associated fixed points in M , in which case the edge is labeled by k . We use the horizontal position of the nodes to indicate the J -value, and we will often omit the volume label as well. Notice this means our graphs are rotated by $\pi/2$ compared to [Kar99] in which she uses the vertical position of the nodes to indicate the J -value. We use the horizontal position to more easily compare them with semitoric polygons.

Given any fixed point $p \in M^{\mathbb{S}^1}$, there exist integers $m, n \in \mathbb{Z}$ and complex coordinates w, z around p such that $t \cdot (w, z) = (e^{imt}w, e^{int}z)$ and the symplectic form is locally given by $\frac{i}{2}(dw \wedge d\bar{w} + dz \wedge d\bar{z})$. These integers are called the *weights* of the \mathbb{S}^1 -action at p , and they are also easy to see in the graph: for $k > 1$ a fixed point has $-k$ as one of its weights if and only if it is at the north pole of a \mathbb{Z}_k -sphere and k as one of its weights if and only if it is at the south pole of a \mathbb{Z}_k -sphere. The point p has zero as one of its weights if and only if it lies



(a) The Karshon graph for $\mathbb{C}\mathbb{P}^2$.



(b) The Delzant polygon of $\mathbb{C}\mathbb{P}^2$.

FIGURE 4. The Karshon graph and Delzant polygon for the standard actions of \mathbb{S}^1 and \mathbb{T}^2 on $\mathbb{C}\mathbb{P}^2$. Notice that they are related as described in Section 2.3.

in a fixed surface. All weights not determined by these rules are ± 1 . Furthermore, if p is in the preimage of the maximum value of J then p has two non-positive weights, if p is in the preimage of the minimum value of J then it has two non-negative weights, and otherwise p has one positive and one negative weight.

Example 2.2. Consider the usual action of \mathbb{S}^1 on $\mathbb{C}\mathbb{P}^2$ given by $t \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it} z_1 : z_2]$ for $t \in \mathbb{S}^1$ with Hamiltonian $J([z_0 : z_1 : z_2]) = |z_1|^2 / (|z_0|^2 + |z_1|^2 + |z_2|^2)$. Then $J^{-1}(0) = \{z_1 = 0\}$ is a sphere which is fixed by the \mathbb{S}^1 -action and $J^{-1}(1) = \{z_0 = z_2 = 0\}$ is a point fixed by the \mathbb{S}^1 -action. There are no other fixed points and the action is free away from these sets. The fixed sphere is represented by a fat node at $J = 0$ with normalized area $A = 1$, the fixed point is represented by a regular node at $J = 1$, and there are no edges. The graph is shown in Figure 4a.

An *isomorphism* between two \mathbb{S}^1 -spaces (M_1, ω_1, J_1) and (M_2, ω_2, J_2) is a symplectomorphism $\Psi: M_1 \rightarrow M_2$ such that $\Psi^* J_2 = J_1$, in which case Ψ is also automatically equivariant with respect to the \mathbb{S}^1 -actions. One of the main results of [Kar99] is that the graphs contain all of the information of the isomorphism class of the associated \mathbb{S}^1 -space.

Theorem 2.3 ([Kar99, Theorem 4.1]). *Two four-dimensional compact Hamiltonian \mathbb{S}^1 -spaces are associated to the same Karshon graph if and only if they are isomorphic as \mathbb{S}^1 -spaces.*

To complete the classification, Karshon also describes exactly which graphs occur. We discuss this in Section 2.4.

Remark 2.4. *Complexity-one spaces* are the higher dimensional analogue of \mathbb{S}^1 -spaces, and they consist of a $2n$ -dimensional symplectic manifold with a Hamiltonian action of the torus \mathbb{T}^{n-1} . A complexity-one space is called *tall* if all reduced spaces are two-dimensional. Tall complexity-one spaces are classified by the work of Karshon & Tolman [KT01, KT03, KT14], which extends the classification of \mathbb{S}^1 -spaces presented in the above section by including additional invariants. An extension of the present paper would be to study extending such Hamiltonian torus actions to integrable systems. For instance, Wacheux [Wac13] studied six dimensional integrable systems where two components of the momentum map are both periodic, and thus generate a Hamiltonian \mathbb{T}^2 -action.

2.2. Classification of toric integrable systems. Let $(M, \omega, F = (f_1, \dots, f_n))$ be a $2n$ -dimensional toric integrable system, and recall that in toric systems M is compact. The flows of f_1, \dots, f_n are each periodic of minimal period 2π , and thus induce an effective action of

$\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. Such a system can also be thought of as a Hamiltonian action of \mathbb{T}^n on M . Atiyah [Ati82] and Guillemin & Sternberg [GS82] showed that the image $F(M)$ is a convex n -dimensional polytope and, moreover, it is the convex hull of the images of the fixed points of the torus action on M . Furthermore, Delzant [Del88] showed that the polytope $\Delta := F(M)$ always satisfies three conditions:

- (1) *simplicity*: exactly n edges meet at each vertex of Δ (note that this is automatic if $n = 2$, the case we will consider in this paper);
- (2) *rationality*: each face of Δ admits an integral normal vector (i.e. a normal vector in \mathbb{Z}^n);
- (3) *smoothness*: given any vertex, the set of integral inwards pointing normal vectors (from item (2)) of the faces adjacent to that vertex can be chosen such that they span \mathbb{Z}^n .

Delzant also showed that any n -dimensional polytope satisfying these conditions arises as the image of the momentum map for some toric integrable system, and that two toric integrable systems (M_1, ω_1, F_1) and (M_2, ω_2, F_2) have the same momentum map image if and only if there exists a symplectomorphism $\Phi: M_1 \rightarrow M_2$ such that $\Phi^*F_2 = F_1$, called an *isomorphism of toric integrable systems*. Thus, Delzant completed the classification of toric integrable systems up to isomorphism in terms of a convex polytope, the image of the momentum map. We will call this polytope the *Delzant polytope* of the system, or the *Delzant polygon* if $n = 2$.

Example 2.5. Consider the toric integrable system $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}}, F = (J, H))$ where ω_{FS} is the usual Fubini-Study symplectic form, J is as in Example 2.2, and $H([z_0, z_1, z_2]) = |z_2|^2 / (|z_0|^2 + |z_1|^2 + |z_2|^2)$. Then the associated Delzant polygon is the triangle with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$ as in Figure 4b.

2.3. \mathbb{S}^1 -spaces and toric systems. Let $(M, \omega, F = (J, H))$ be a compact toric integrable system with Delzant polygon $\Delta = F(M)$. Then (M, ω, J) is an \mathbb{S}^1 -space with \mathbb{S}^1 -action coming from the subgroup $\mathbb{S}^1 \times \{0\} \subset \mathbb{T}^2$ of the torus acting on M . The fixed surfaces of this action are the preimages of the (closed) vertical edges of Δ , if any, which have normalized symplectic area equal to the length of the edge and are always genus zero. The isolated fixed points of the action are the vertices of Δ which are not on vertical edges. The \mathbb{Z}_k -spheres are the preimages of the edges of Δ which have slope b/k where $k, b \in \mathbb{Z}$ are relatively prime and $k > 1$. Thus, it is straightforward to construct the Karshon graph from the Delzant polygon, compare the Delzant polygon to the Karshon graph for the standard action on $\mathbb{C}\mathbb{P}^2$ in Figure 4. Note that not all \mathbb{S}^1 -spaces come from toric manifolds in such a way:

Lemma 2.6 ([Kar99, Proposition 5.21]). *An \mathbb{S}^1 -space can be extended to a toric integrable system if and only if each fixed surface of the \mathbb{S}^1 -space, if any, has genus zero and each non-extremal level set of J contains at most two non-free orbits of the \mathbb{S}^1 -action.*

Here by *non-extremal level set* we mean the preimage of any point in the image of J except for its maximum or minimum values. The following result will also be useful for us.

Lemma 2.7 ([Kar99, Corollary 5.19]). *Let (M, ω, J) be an \mathbb{S}^1 -space. If all fixed points of the \mathbb{S}^1 -action are isolated then (M, ω, J) extends to a toric integrable system.*

2.4. Minimal \mathbb{S}^1 -spaces and \mathbb{S}^1 -equivariant blowups. Not every possible labeled graph can actually be obtained as the Karshon graph associated to an \mathbb{S}^1 -space. Again following

Karshon [Kar99], we will describe the set of labeled graphs which do correspond to an \mathbb{S}^1 -space in terms of minimal models. We will describe the effect of \mathbb{S}^1 -equivariant blowups and blowdowns on the labeled graph and then describe the *minimal \mathbb{S}^1 -spaces*, which are the \mathbb{S}^1 -spaces which do not admit blowdowns, and their Karshon graphs. Thus, the set of all graphs that can be obtained from \mathbb{S}^1 -spaces is equal to the set of graphs which can be produced from one of these minimal graphs via a finite sequence of blowups.

2.4.1. *Equivariant blowups.* Let (M, ω) be a symplectic four-manifold. A (symplectic) blowup of (M, ω) essentially amounts to removing an embedded 4-ball and collapsing the boundary via the Hopf fibration, and a blowdown is the inverse operation. Specifically, let $p \in M$ and let $U \subset M$ be a neighborhood of p and $\phi: U \rightarrow V \subset \mathbb{C}^2$ be a symplectomorphism with $\phi(p) = (0, 0)$. Then given any $r > 0$ such that the standard ball of radius r in \mathbb{C}^2 is contained in V we can define the *blowup at p of size $\lambda := \frac{r^2}{2}$* by removing the preimage of the ball and collapsing the boundary via the usual Hopf fibration given by $\mathbb{S}^3 \rightarrow \mathbb{C}\mathbb{P}^1$, $(z_0, z_1) \mapsto [z_0 : z_1]$ where the coordinates (z_0, z_1) are from the inclusion $\mathbb{S}^3 \subset \mathbb{C}^2$.

These operations are described in detail, including how to equip the resulting manifold with a symplectic form, in [MS17, Section 7.1]. We will use $\text{Bl}^k(M)$ to denote the manifold obtained by blowing up M at k points (the diffeomorphism type is independent of the choice of points and size of the blowups).

Now suppose that (M, ω) is equipped with the extra structure of an \mathbb{S}^1 -space or toric integrable system. In the first case, it comes with a Hamiltonian \mathbb{S}^1 -action so that we can require that the map ϕ be equivariant with respect to this action, meaning with respect to rotation of the first coordinate in \mathbb{C}^2 . Notice that such a ϕ only exists if p is a fixed point of the action. Taking the blowup with respect to these equivariant coordinates and restricting the momentum map to the resulting space yields the *\mathbb{S}^1 -equivariant blowup at p of size λ* .

In the second case, given a Hamiltonian \mathbb{T}^2 -action and assuming ϕ to be \mathbb{T}^2 -equivariant, i.e., invariant under rotation of both coordinates of \mathbb{C}^2 , we may take the blowup with respect to these equivariant coordinates. Restricting the momentum map to the resulting space yields the *\mathbb{T}^2 -equivariant blowup at p of size λ* .

The resulting Karshon graph (for an \mathbb{S}^1 -space) or Delzant polygon (for a toric integrable system) is independent of all choices, and since these objects classify \mathbb{S}^1 -spaces respectively toric integrable systems, we conclude that the isomorphism class of the result of the \mathbb{S}^1 -equivariant and \mathbb{T}^2 -equivariant blowups described above is independent of all choices (this is the same argument from [Kar99, Proposition 6.1]).

2.4.2. *Blowups on Delzant polygons.* A vector $v \in \mathbb{Z}^2$ is *primitive* if it is not a multiple of a shorter integral vector (that is, that $v = ku$ for $u \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$ implies $k = \pm 1$). Given primitive vectors $u_1, u_2 \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$ let

$$\text{Simp}_x^\lambda(u_1, u_2) = \{x + t_1 u_1 + t_2 u_2 \mid t_1, t_2 > 0, t_1 + t_2 < \lambda\}.$$

So taking x to be the origin and u_1, u_2 to be the standard basis vectors of \mathbb{R}^2 yields a right triangle with two sides of length λ , and other choices of x and primitive $u_1, u_2 \in \mathbb{Z}^2$ gives all translations of images under $\text{GL}(2, \mathbb{Z})$ of that triangle. Let (M, ω, F) be a toric integrable system with associated Delzant polygon $\Delta = F(M)$. Performing a \mathbb{T}^2 -equivariant blowup of size $\lambda > 0$ on a fixed point $p \in M$ changes the Delzant polygon by an operation known as a *corner chop*, see Figure 5. If (M', ω', F') is the resulting toric integrable system then the

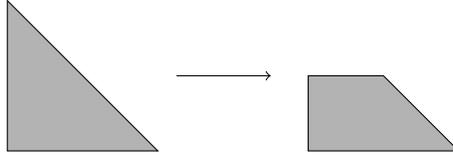


FIGURE 5. Performing a blowup at one of the fixed points of $\mathbb{C}\mathbb{P}^2$ corresponds to performing a corner chop on the associated Delzant polygon.

new polygon $\Delta' = F'(M')$ is given by

$$\Delta' = \Delta \setminus \text{Simp}_{F(p)}^\lambda(v_1, v_2)$$

where $v_1, v_2 \in \mathbb{Z}^2$ are the primitive vectors directing the edges adjacent to the corner $F(p)$. Furthermore, the system (M, ω, F) associated to Δ admits a \mathbb{T}^2 -equivariant blowup of size $\lambda > 0$ at a fixed point $p \in M$ if and only if $F(p)$ is the unique vertex of Δ contained in the simplex $\text{Simp}_{F(p)}^\lambda(v_1, v_2)$, where v_1, v_2 are primitive vectors directing the edges adjacent to $F(p)$.

2.4.3. *Blowups on Karshon graphs.* Let (M, ω, J) be an \mathbb{S}^1 -space and we will see the effect of an \mathbb{S}^1 -equivariant blowup of size $\lambda > 0$ at a point $p \in M$ on the associated Karshon graph. Recall that such a blowup can only be performed at p if it is fixed by the \mathbb{S}^1 -action, which means it is either an isolated fixed point or it lies in a fixed surface. Let $j_{\min}, j_{\max} \in \mathbb{R}$ denote the minimum, respectively maximum, values achieved by J before the blowup operation. There are several cases:

- (B1) If p lies in a fixed surface at the minimum of J , then performing the blowup reduces the normalized symplectic area label on the fat vertex corresponding to $\Sigma = J^{-1}(j_{\min})$ by λ and produces a new isolated fixed point with J -value $J = j_{\min} + \lambda$. Other than this the graph is unchanged, and in particular there are no new edges. The case in which $p \in J^{-1}(j_{\max})$ is similar, except that the new vertex has J -value $J = j_{\max} - \lambda$.
- (B2) If p is an isolated fixed point at the minimum of J with weights both equal to 1, then the vertex corresponding to p is removed from the graph and there is a fat vertex of normalized area $A = \lambda$ added with genus zero and J -value $J = j_{\min} + \lambda$. The case in which p is an isolated fixed point at the maximum with weights both equal to -1 is similar, except that the new fat vertex is at J -value $J = j_{\max} - \lambda$. In each case there is no change in the edge set of the graph.
- (B3) If p is an isolated fixed point at the minimum of J with weights n, m satisfying $n < m$, then the vertex associated to p is removed from the graph and replaced by two new vertices. The edge with label n that was attached to the vertex corresponding to p is now attached to a new vertex with J -value $J = j_{\min} + m\lambda$ and the edge labeled m is now attached to a new vertex with J -value $J = j_{\min} + n\lambda$. The two new vertices are attached to each other with an edge labeled $m - n$. Note that if any of these edge labels are equal to 1 then that edge is not drawn in the graph. The case that p is a maximum is a reflection of this case.
- (B4) If p is an isolated fixed point which is not at the minimum or maximum of J with weights $-n, m$ (with $n, m \geq 0$) then the vertex corresponding to p is removed from

the graph and replaced with two new vertices. The edge with label n that was attached to the vertex corresponding to p is now attached to a new vertex with J -value $J = J(p) - n\lambda$ and the edge labeled m is now attached to a new vertex with J -value $J = J(p) + m\lambda$. The two new vertices are attached to each other with an edge labeled $m + n$. Note that if any of these edge labels are equal to 1 then that edge is not drawn in the graph.

The resulting graph corresponds to the \mathbb{S}^1 -equivariant blowup at p of size λ of the original \mathbb{S}^1 -space if and only if it satisfies:

- (1) the J -value labels along each chain of edges are strictly monotone,
- (2) all new vertices added to the graph have J -value in the interval (j_{\min}, j_{\max}) ,
- (3) the area label associated to each fat vertex is strictly positive.

The resulting graph satisfies those properties for a given value $\lambda > 0$ if and only if an \mathbb{S}^1 -equivariant blowup at p of size λ is possible in the corresponding \mathbb{S}^1 -space.

Remark 2.8. A situation which will be particularly relevant in this paper is Case (B4) with $m = n = 1$, in which a single vertex with J -value $J = j$ which is not connected to any edges is replaced by a pair of vertices with J -value labels $J = j - \lambda$ and $J = j + \lambda$ connected by an edge labeled 2.

2.4.4. *Toric blowups.* Blowups can also be performed on completely elliptic rank zero singular points of integrable systems, these are known as *toric blowups* of integrable systems. Given a completely elliptic rank zero point p of an n -dimensional integrable system the coordinates of the local normal form give an identification to \mathbb{C}^n and this identification can be used to define the blowup of the system. The details and proof that the resulting system is independent of all choices can be found for instance in [LFP19, Section 4.2].

2.4.5. *The minimal \mathbb{S}^1 -spaces.* This section follows Karshon [Kar99, Section 6]. We say that an \mathbb{S}^1 -space is *minimal* if it does not admit an \mathbb{S}^1 -equivariant blowdown. Given a toric system (M, ω, F) , any homomorphism $\mathbb{S}^1 \hookrightarrow \mathbb{T}^2$ induces a Hamiltonian action of \mathbb{S}^1 on (M, ω) .

Theorem 2.9 (Karshon [Kar99, Theorem 6.3]). *An \mathbb{S}^1 -space (M, ω, J) is minimal if and only if either:*

- (1) (M, ω, J) is induced by the standard toric system on $\mathbb{C}\mathbb{P}^2$ with some multiple of the Fubini-Study form by a homomorphism $\mathbb{S}^1 \hookrightarrow \mathbb{T}^2$;
- (2) (M, ω, J) is induced by the standard toric system on one of the scaled Hirzebruch surfaces by a homomorphism $\mathbb{S}^1 \hookrightarrow \mathbb{T}^2$;
- (3) it has two fixed surfaces and no other fixed points.

The minimal models from Case (3) turn out to be *ruled surfaces*, which in this context we define as \mathbb{S}^2 -bundles over a surface Σ . The \mathbb{S}^1 -action rotates the spheres leaving Σ invariant, and the two fixed surfaces are each diffeomorphic to Σ . The Hamiltonian function J is the standard height function on the sphere component. The Karshon graph of a ruled surface consists of two fat vertices, with J -values j_0 and $j_0 + s$, area labels $a > 0$ and $a + ns$ (for some $n \in \mathbb{Z}$), and both labeled with the same genus $g \in \mathbb{Z}_{\geq 0}$. In the case that $n = 0$ the ruled surface is simply the product $\mathbb{S}^2 \times \Sigma_g$, where Σ_g is a surface of genus g .

The minimal Karshon graphs are those that can be obtained from these minimal \mathbb{S}^1 -spaces. Thus, this completes Karshon's classification of \mathbb{S}^1 -spaces by describing the set of

graphs which are obtained from \mathbb{S}^1 -spaces, which is the set of all graphs that can be obtained from the minimal graphs by a finite sequence of blowups.

2.5. Singularities of integrable systems. Let $(M, \omega, F = (f_1, \dots, f_n))$ be an n -dimensional integrable system. A point $p \in M$ is a *singular point* if $\text{rank}(p) := \text{rank}(dF_p) < n$. The rank of a singular point coincides with the dimension of the space spanned by the vectors $\mathcal{X}_p^{f_1}, \dots, \mathcal{X}_p^{f_n} \in T_p M$. The space of quadratic forms on $T_p M$ has a Lie algebra structure making it isomorphic to $\mathfrak{sp}(2n, \mathbb{R})$, and a rank zero singular point p is *non-degenerate* if and only if the Hessians of f_1, \dots, f_n span a Cartan subalgebra. If p is singular with $\text{rank}(p) > 0$ there is a similar definition of non-degenerate after taking the symplectic quotient by what can roughly be thought of as the non-singular part of the momentum map. We refer to [BF04] for the details.

Cartan subalgebras of $T_p M \cong \mathfrak{sp}(2n, \mathbb{R})$ were classified by Williamson [Wil36], and that pointwise classification was extended to a local classification by a series of papers, such as Colin de Verdière-Vey [CdVV79], Rüssmann [Rüs64], Vey [Vey78], Eliasson [Eli84] published partially in [Eli90], Dufour & Molino [DM91], Miranda & Vũ Ngọc [MVuN05], Vũ Ngọc & Wacheux [VuNW13], Chaperon [Cha13], and Miranda & Zung [MZ04].

Theorem 2.10 (local normal form for non-degenerate singularities). *Let $p \in M$ be a non-degenerate singular point of an integrable system $(M, \omega, F = (f_1, \dots, f_n))$. Then:*

- (1) *there exists local symplectic coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on an open neighborhood $U \subset M$ and smooth functions $q_1, \dots, q_n: U \rightarrow \mathbb{R}$ where we have the following possibilities for the form of each q_j :*
 - *Elliptic component: $q_j = (x_j^2 + \xi_j^2)/2$;*
 - *Hyperbolic component: $q_j = x_j \xi_j$;*
 - *Focus-focus component: $q_j = x_j \xi_{j+1} - x_{j+1} \xi_j$ and $q_{j+1} = x_j \xi_j + x_{j+1} \xi_{j+1}$;*
 - *Regular component: $q_j = \xi_j$,**such that $\{q_i, f_j\} = 0$ for all $i, j \in \{0, \dots, n\}$ and p corresponds to the origin in these coordinates;*
- (2) *if there is no hyperbolic component then the system of equations $\{q_i, f_j\} = 0$ for all possible i, j is equivalent to the existence of a local diffeomorphism $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$g \circ F = (q_1, \dots, q_n) \circ (x_1, \dots, x_n, \xi_1, \dots, \xi_n).$$

Non-degenerate singular points are thus classified into their *Williamson type* by this theorem. For instance, in the case that $\dim(M) = 4$ each non-degenerate singular point is of exactly one of the following six types:

- rank 0: elliptic-elliptic, focus-focus, hyperbolic-hyperbolic, hyperbolic-elliptic;
- rank 1: elliptic-regular, hyperbolic-regular.

Remark 2.11. Suppose that $M = \Sigma_1 \times \Sigma_2$ where Σ_1 and Σ_2 are surfaces. For $i \in \{1, 2\}$ let $\pi_i: M \rightarrow \Sigma_i$ be the projection map, let ω_i be a symplectic form on Σ_i , and let $f_i: \Sigma_i \rightarrow \mathbb{R}$ be a Morse function. Then $(M, \omega_1 \oplus \omega_2, F = (f_1 \circ \pi_1, f_2 \circ \pi_2))$ is an integrable system and all singular points of this system are non-degenerate. Moreover, $p = (p_1, p_2)$ is a singular point of F if and only if at least one of the p_i is a critical point of the corresponding f_i , and the Williamson type of p is determined by the Morse indices of p_1 and p_2 , since the product of the Morse charts forms one of the charts as discussed in Theorem 2.10 (equivalently one

can compare the eigenvalues of the block diagonal Hessians of the Morse functions with the Hessian of their sum). Regular points of the Morse function correspond to regular blocks in the local normal form, index 1 critical points correspond to hyperbolic blocks, and index 0 or 2 critical points correspond to elliptic blocks. For instance, if $p_1 \in \Sigma_1$ is an index 1 critical point with Morse coordinates a, b then locally $f_1 = a^2 - b^2 = (a - b)(a + b)$ so taking $x = a - b$ and $\xi = a + b$ gives the coordinates for a hyperbolic block from Theorem 2.10.

The next result follows from [BF04, Proposition 1.16].

Lemma 2.12. *Let $(M, \omega, F = (J, H))$ be a four dimensional integrable system and let $M^{\text{HR}} \subset M$ be the set of hyperbolic-regular singular points. If $C \subset M$ is a connected component of M^{HR} then $F(C)$ is the image of an immersion from a one-dimensional manifold into \mathbb{R}^2 and thus for any $p \in C$ there exists a set $U \subset C$ which is an open (as a subset of C) neighborhood of p such that $F(U)$ is a one-dimensional submanifold of \mathbb{R}^2 .*

2.6. Integrable systems and singular Lagrangian fibrations. Let (M, ω, F) be an n -dimensional integrable system. A connected component of a fiber of $F: M \rightarrow \mathbb{R}^n$ is called *singular* if it contains a singular point of the integrable system and called *regular* otherwise. It is easy to see that every regular component of a fiber of F is a Lagrangian submanifold of M , which are all diffeomorphic to n -tori in the case that M is compact. Let B be the topological space obtained as the quotient space of M by the equivalence relation relating two points if and only if they are in the same component of the same level set of F , and let $\pi: M \rightarrow B$ be the associated quotient map. Notice that the union of the fibers of π which are Lagrangian submanifolds forms an open dense set in M . Thus $\pi: M \rightarrow B$ is a *singular Lagrangian fibration* of M and B is known as the *base*. We say that this singular Lagrangian fibration is *induced* by the momentum map F .

In the case that all fibers of F are connected, the base B can be naturally identified with the image of the momentum map $F(M)$. This is what occurs for toric and semitoric integrable systems. For more general classes of integrable systems, such as hypersemitoric systems, there is a natural surjection $B \rightarrow F(M)$ but this is not necessarily a bijection.

Remark 2.13. Concerning Lemma 2.12, notice that the connected components of $F(M^{\text{HR}})$ are not always embedded curves in \mathbb{R}^2 . This is because given two components C and C' of M^{HR} the curves $F(C)$ and $F(C')$ may pass through each other in \mathbb{R}^2 , though their images $\pi(C)$ and $\pi(C')$ cannot intersect in the base of the fibration induced by F .

2.7. Semitoric systems and marked polygons. Due to Theorem 2.10, semitoric systems (as in Definition 1.2) can have singular points of three types: elliptic-elliptic, focus-focus, and elliptic-regular. A semitoric system $(M, \omega, F = (J, H))$ is *simple* if there is at most one focus-focus point in each fiber of J . Simple semitoric systems are classified by Pelayo & Vũ Ngọc in terms of five invariants: the number of focus-focus points, the semitoric polygon, the height invariant, the Taylor series invariant, and the twisting index invariant. This classification was extended to non-simple systems in [PPT19]. We will focus our attention on the first three invariants, which exist nearly unchanged in the non-simple case and were already developed in [VN07] before the full classification. As in [LFP19] we will package these three invariants together into a single invariant called the *marked semitoric polygon invariant*.

Given a semitoric system $(M, \omega, F = (J, H))$ there are finitely many singular points of focus-focus type [VN07, Corollary 5.10], so the first invariant of the system is their number

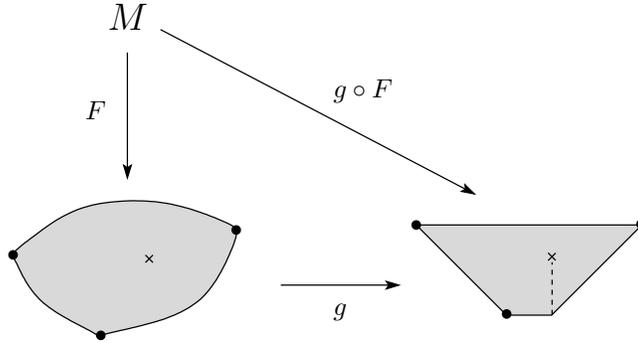


FIGURE 6. The momentum map image of a semitoric system can be “straightened out” to recover a polygon.

$m_f \in \mathbb{Z}_{\geq 0}$. Denote the set of focus-focus singular points by $M^{\text{FF}} = \{p_1, \dots, p_{m_f}\} \subset M$. The images $F(p_1), \dots, F(p_{m_f}) \in \mathbb{R}^2$ of these points are in the interior of the momentum map image $F(M)$ and we may assume that their images are in lexicographic order, i.e. that $J(p_1) \leq \dots \leq J(p_{m_f})$ and if $J(p_\ell) = J(p_{\ell'})$ with $\ell < \ell'$ then $H(p_\ell) \leq H(p_{\ell'})$. Given a point $c \in \mathbb{R}^2$ and $\epsilon = \pm 1$ let $\ell_c^\epsilon \subset \mathbb{R}^2$ be the closed ray starting at c directed along a vector in the positive J direction if $\epsilon = 1$ and the negative J direction if $\epsilon = -1$. Given $\vec{\epsilon} \in \{\pm 1\}^{m_f}$ let $\vec{\ell}^{\vec{\epsilon}} = \cup_{j=1}^{m_f} \ell_{F(p_j)}^{\epsilon_j}$. Let $M' = M \setminus F^{-1}(\vec{\ell}^{\vec{\epsilon}})$. Then, as in [VN07, Theorem 3.8], there exists a homeomorphism $g: F(M) \rightarrow \mathbb{R}^2$ preserving the first component which is smooth on $F(M) \setminus \vec{\ell}^{\vec{\epsilon}}$ such that each component of $g \circ F|_{M'}: M' \rightarrow \mathbb{R}^2$ generates an effective \mathbb{S}^1 -action. We will call such a g a *straightening map*. The closure of the image of this toric momentum map is a polygon as sketched in Figure 6. It is unique up to the freedom in the choices of $\vec{\epsilon}$ and g , which we will now encode in a group action on the triple

$$(1) \quad (\Delta := g \circ F(M), g \circ (F(p_1), \dots, F(p_{m_f})), \vec{\epsilon}).$$

Remark 2.14. Notice that we have allowed for the case of multiple focus-focus points in the same fiber of F , in which case the ordering of the labeling of the focus-focus points is not unique. This will not cause any problems in constructing a unique invariant because changing the order of points in the same fiber does not change the resulting marked semitoric polygon. However, the non-uniqueness of the ordering of the labels does cause complications if the Taylor series labels are included (which are not relevant in the present paper), in which case the focus-focus points in the same fiber have a cyclic ordering as in [PPT19].

Remark 2.15. A similar construction in the context of almost toric manifolds appeared in [Sym03].

For $k = 1, 2$ let $\pi_k: \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the projection onto the k^{th} coordinate. For $s \in \mathbb{Z}_{\geq 0}$ we call a triple

$$(\Delta, \vec{c} = (c_1, \dots, c_s), \vec{\epsilon} = (\epsilon_1, \dots, \epsilon_s))$$

a *marked weighted polygon* if all $c_i \in \text{int}(\Delta)$ and $\pi_1(c_1) \leq \dots \leq \pi_1(c_s)$. For $j \in \mathbb{R}$, let

$$(2) \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{t}_j: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto \begin{cases} (x, y + x - j) & \text{if } x \geq j, \\ (x, y) & \text{otherwise.} \end{cases}$$

Let \mathcal{T} be the group generated by powers of T and vertical translations (this is the subgroup of the group of integral affine maps on the plane which preserve the first component) and let $G_s = \{\pm 1\}^s$. Then $\mathcal{T} \times G_s$ acts on the set of marked weighted polygons by

$$(\tau, \vec{\epsilon}') \cdot (\Delta, \vec{c}, \vec{\epsilon}) = (\sigma(\Delta), (\sigma(c_1), \dots, \sigma(c_s)), (\epsilon'_1 \epsilon_1, \dots, \epsilon'_s \epsilon_s))$$

where $\sigma = \tau \circ \mathfrak{t}_{\pi_1(c_1)}^{u_1} \circ \dots \circ \mathfrak{t}_{\pi_1(c_s)}^{u_s}$ and $u_k = \epsilon_k(1 - \epsilon'_k)/2$. Let $[\Delta, \vec{c}, \vec{\epsilon}]$ denote the orbit of $(\Delta, \vec{c}, \vec{\epsilon})$ under this action.

This group action represents exactly the effect of the choice of straightening map g and cut directions \vec{c} on the triple in Equation (1) above: if $(\Delta, \vec{c}, \vec{\epsilon})$ is the result of that construction for one choice of g and \vec{c} then the set of all possible triples produced in this way is exactly the orbit $[\Delta, \vec{c}, \vec{\epsilon}]$. This orbit is the *marked semitoric polygon invariant* of the system (M, ω, F) .

Remark 2.16. Note that the marked semitoric polygon invariant contains more information than the semitoric polygon invariant introduced in [VN03] and used in the classification [PVuN09, PVuN11], since it also includes the marked points corresponding to the focus-focus values of the system. The *height invariant* of the semitoric system is encoded in the marked semitoric polygon as the vertical distance from c_k to the bottom of Δ which can be seen to not depend on the choice of representative.

The marked semitoric polygon is not a complete invariant of semitoric systems since there are many distinct systems which have the same marked semitoric polygon. This is because the Taylor series and twisting index invariants are not encoded in the marked semitoric polygon, so systems with the same marked semitoric polygon may have different Taylor series or twisting index invariants. The Taylor series, developed in [VN03], and extended to the case of multiple focus-focus points in the same fiber in [PT19], describes the semi-local (i.e. neighborhood of the fiber) structure around a focus-focus singular point, and the twisting index, introduced in [PVuN09], roughly, takes into account an additional degree of freedom when gluing this local model into the global system. These invariants will not be as important as the others for this paper, since they are not related to the structure of the underlying \mathbb{S}^1 -space.

Now we will describe exactly which marked weighted polygons can be obtained in this way from a simple semitoric system. The remainder of this section summarizes results from [PVuN11] and adapts them to the case of marked polygons that may have multiple marked points in the same vertical line. We say that a polygon is *rational* if the slope of each non-vertical edge is rational. Let q be a vertex of a rational polygon and let $v, w \in \mathbb{Z}^2$ be the primitive vectors directing the edges adjacent to q . Then we say that q satisfies:

- the *Delzant condition* if $\det(v, w) = 1$;
- the *hidden Delzant condition for m cuts* if $\det(v, T^m w) = 1$;
- the *fake condition for m cuts* if $\det(v, T^m w) = 0$,

where in each case $\det(v, w)$ denotes the determinant of the matrix with first column v and second column w . The orbit $[\Delta, \vec{c}, \vec{\epsilon}]$ can be obtained from a simple semitoric system if and only if one, and hence all, representatives $(\Delta, \vec{c}, \vec{\epsilon})$ satisfy the following three conditions:

- (1) Δ is rational and convex;
- (2) each point of $\partial\Delta \cap (\cup_k \ell_{c_k}^{\epsilon_k})$ is a vertex of Δ and satisfies either the fake or hidden Delzant condition for m cuts (in which case it is known as a *fake corner* or *hidden*

corner respectively) where $m > 0$ is the number of distinct k such that the vertex in question lies on a ray $\ell_{c_k}^{\epsilon_k}$;

- (3) each other vertex of Δ satisfies the Delzant condition (and is known as a *Delzant corner*).

Such marked semitoric polygons are known as *marked Delzant semitoric polygons*. We will represent these by drawing each ray $\ell_{c_k}^{\epsilon_k}$ as a dotted line (known as a *cut*) and the image under $g \circ F$ of each focus-focus point will be indicated by a “ \times ”. See Figure 7b for an example.

2.8. \mathbb{S}^1 -spaces and semitoric systems. In this section we briefly recall the results of [HSS15]. Given a compact simple semitoric system $(M, \omega, F = (J, H))$ the Karshon graph of the underlying \mathbb{S}^1 -space (M, ω, J) can be obtained from any representative of the associated marked semitoric polygon $[\Delta, \vec{c}, \vec{\epsilon}]$ much in the same way as obtaining the Karshon graph of a toric integrable system from the associated Delzant polygon. Let $(\Delta, \vec{c}, \vec{\epsilon})$ be any representative. The fixed surfaces are the preimages of the closure of the vertical edges of Δ , if any, which have normalized symplectic area equal to the length of the edge and are always genus zero. The isolated fixed points are the focus-focus points and also the preimages of any vertices of Δ which are not on vertical edges and are not fake corners. Let e_1, \dots, e_m , possibly with $m = 1$, be a collection of adjacent edges of Δ such that the vertex joining e_ℓ to $e_{\ell+1}$, for $\ell = 1, \dots, m-1$, is a fake vertex and the remaining two endpoints (of e_1 and e_m) are not fake. Then, due to the conditions on fake vertices, each of these edges has slope b_ℓ/k for distinct integers b_ℓ , $\ell = 1, \dots, m$, and a common integer $k > 0$. The closure of the preimage of the union $e_1 \cup \dots \cup e_m$ is a \mathbb{Z}_k -sphere, which is represented by an edge in the Karshon graph between the vertices corresponding to the endpoints of the piecewise linear curve $e_1 \cup \dots \cup e_m$ if $k > 1$.

In summary, to construct the Karshon graph from $(\Delta, \vec{c}, \vec{\epsilon})$ draw a fat vertex for each vertical edge of Δ labeled by $g = 0$ and normalized area equal to the length of the edge, draw a regular vertex for each Delzant corner, each hidden Delzant corner, and each focus-focus singular point. Finally, any time a chain of edges (connected by fake vertices) connects two hidden or Delzant vertices they each have slope of the form b/k for various b , and if $k > 1$ draw an edge between the corresponding vertices in the graph and label it with k .

Example 2.17. In this example we will produce an \mathbb{S}^1 -space on $\mathbb{C}\mathbb{P}^2$ blown up five times which cannot be extended to a toric system but can be extended to a semitoric system. This construction will depend on two parameters, $\lambda_1 \in]0, 1/3[$ and $\lambda_2 \in]0, \lambda_1[$. Starting with $\mathbb{C}\mathbb{P}^2$ with the Fubini-Study symplectic form and usual \mathbb{S}^1 -action (as in Example 2.2), perform two \mathbb{S}^1 -equivariant blowups of the same size λ_1 on the fixed surface. This produces two new fixed points, and next we perform one blowup of size λ_2 at each of these fixed points. This gives us the Karshon graph as in Figure 7c, which can be extended to the toric system corresponding to the Delzant polygon in Figure 7a. Now, performing another blowup of size λ_1 on the fixed surface produces the \mathbb{S}^1 -space corresponding to the Karshon graph in Figure 7d which can not be extended to a toric system by Lemma 2.6, but can be extended to a semitoric system. A representative of the semitoric polygon of such a semitoric system is shown in Figure 7b.

2.9. Blowups of semitoric systems. We will now describe two types of blowups of semitoric systems.

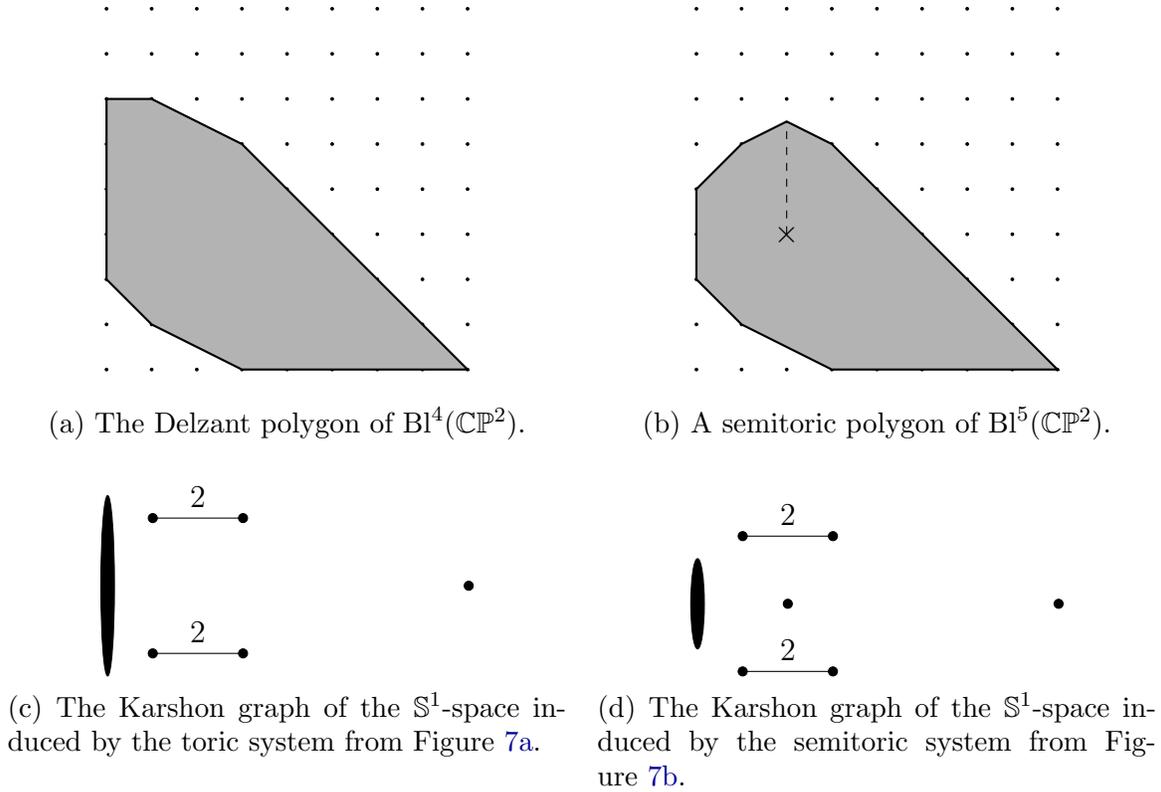


FIGURE 7. The Delzant polygons, semitoric polygons, and Karshon graphs of $\mathbb{C}\mathbb{P}^2$ blown up four and five times as in Example 2.17 with parameters $\lambda_1 = 1/4$ and $\lambda_2 = 1/8$. The Delzant and semitoric polygons are drawn over the lattice $\frac{1}{8}\mathbb{Z}^2 \subset \mathbb{R}^2$. Since the Delzant polygon for $\mathbb{C}\mathbb{P}^2$ is the convex hull of $(0, 0)$, $(1, 0)$, and $(0, 1)$, performing the four blowups of the specified sizes produces the Delzant polygon with vertices at $(1/8, 1/8)$, $(0, 2/8)$, $(0, 6/8)$, $(1/8, 6/8)$, $(3/8, 5/8)$, $(1, 0)$, and $(3/8, 0)$ shown in Figure 7a, and performing one more blowup produces a semitoric system with the semitoric polygon shown in Figure 7b.

2.9.1. *Toric blowups.* Let $(M, \omega, F = (J, H))$ be a semitoric system and let $p \in M$ be an elliptic-elliptic singular point. Then a *toric blowup* of $(M, \omega, F = (J, H))$ at p of size $\lambda > 0$ can be performed if there exists a straightening map $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (as in Section 2.7) such that $g \circ F(p)$ is a Delzant corner of $\Delta := g \circ F(M)$, $g \circ F(p)$ is the unique vertex of Δ contained in the simplex $\text{Simp}_{F(p)}^\lambda(v_1, v_2)$, where v_1, v_2 are primitive vectors directing the edges adjacent to $g \circ F(p)$, and $\text{Simp}_{F(p)}^\lambda(v_1, v_2)$ does not intersect any of the cuts or marked points in Δ . In which case the result of the blowup is the semitoric system which has all of the same invariants as the original system except that its marked polygon is the result of performing a corner chop on the marked polygon invariant of the original system. This process amounts to performing a usual \mathbb{T}^2 -equivariant blowup with respect to the toric momentum map $g \circ F$. It can be shown that the result of this operation does not depend on any of the choices made, see for instance [LFP19, Section 4.3].

Remark 2.18. Note that we only need to find *one* representative of the marked semitoric polygon invariant which admits a corner chop at the vertex corresponding to p in order to perform a toric blowup. Also note that given any elliptic-elliptic point there is at least one representative such that this point corresponds to a Delzant corner and thus for sufficiently small size $\lambda > 0$ a toric blowup can always be performed.

2.9.2. *Semitoric blowups of marked polygons.* In a semitoric system $(M, \omega, F = (J, H))$ a blowup can also be performed at an elliptic-regular point p if it lies in a surface Σ which is fixed by the \mathbb{S}^1 -action generated by J . Such blowups, and their effect on the semitoric system, are described in detail in the upcoming [HSSS], but here we will simply describe their effect on the marked semitoric polygon invariant, and then use the result which states there is a system (actually many) associated to the resulting marked semitoric polygon. In the context of almost toric manifolds, this operation already essentially appeared in the work of Symington [Sym03, Section 5.4].

For any set $B \subset \mathbb{R}^2$ let $\partial^+ B$ be the *upper boundary* of B and $\partial^- B$ the *lower boundary*. We define $\partial^+ B = \{(x, y) \in \partial B \mid y \geq y' \text{ for all } y' \text{ such that } (x, y') \in \partial B\}$, the definition of $\partial^- B$ is similar. Note that $\partial^+ B \cup \partial^- B$ is not in general the entire boundary of B , but nevertheless in the case that B is convex specifying the $\partial^+ B$ and $\partial^- B$ completely determines B . Specifically, if $B \subset \mathbb{R}^2$ is convex then B is equal to the convex hull of $\partial^+ B \cup \partial^- B$.

Definition 2.19. Suppose that $[\Delta, \vec{c}, \vec{\epsilon}]$ is the marked semitoric polygon invariant for (M, ω, F) and suppose that $\Sigma = J^{-1}(j_{\max})$ is a fixed surface of the \mathbb{S}^1 -action generated by J which occurs at the maximum value of J . Let A be the normalized symplectic area of Σ , which is equal to the vertical height of the edge of Δ corresponding to Σ , and let $\lambda > 0$ be any real number such that $\lambda < A$ and $\lambda < j_{\max} - j_{\min}$.

We now define $(\Delta', \vec{c}', \vec{\epsilon}')$ by

- (1) Δ' is the unique convex polygon with $\partial^- \Delta' = \partial^- \Delta$ and $\partial^+ \Delta' = \mathbf{t}_{j_{\max} - \lambda}^{-1}(\partial^+ \Delta)$, where \mathbf{t}_j is as in Equation (2).
- (2) \vec{c}' is equal to the set of marked points \vec{c} with one additional point added anywhere in $\text{int}(\Delta')$ on the line $J = j_{\max} - \lambda$ inserted into the list keeping the points in lexicographical order, suppose that the new point is the ℓ^{th} entry in the list.
- (3) $\vec{\epsilon}' = (\epsilon_1, \dots, \epsilon_{\ell-1}, 1, \epsilon_{\ell}, \dots, \epsilon_s)$.

If any of the points c_k lies outside of the polygon after this transformation, change its y -coordinate while keeping the x -coordinate the same to move it back into the polygon. The process for $p \in \Sigma = J^{-1}(j_{\min})$ is similar. The resulting marked Delzant semitoric polygon $[\Delta', \vec{c}', \vec{\epsilon}']$ is known as a *semitoric blowup of $[\Delta, \vec{c}, \vec{\epsilon}]$ of size λ on Σ* . Any semitoric system (M', ω', F') whose marked Delzant semitoric polygon invariant is $[\Delta', \vec{c}', \vec{\epsilon}']$ is known as a *weak semitoric blowup of (M, ω, F)* , and is independent of all choices except the placement of c_{ℓ} .

Remark 2.20. Note that in Definition 2.19 the change in the top boundary of Δ' is designed so that the new upwards cut intersects the top boundary at either a fake corner or a hidden Delzant corner, so the orbit $[\Delta', \vec{c}', \vec{\epsilon}']$ is indeed a marked Delzant semitoric polygon, and thus there exists at least one system with this as its marked semitoric polygon invariant. The result does not depend on the choice of representative for $[\Delta, \vec{c}, \vec{\epsilon}]$, since different choices produce different representatives of the same class $[\Delta', \vec{c}', \vec{\epsilon}']$ because \mathbf{t}_j , \mathbf{t}_i , and T all commute.

As an example, performing a semitoric blowup of size $\lambda = 1/4$ on the fixed surface of the polygon in Figure 7a results in the polygon in Figure 7b. Any system associated to the marked semitoric polygon described above is called a *weak* semitoric blowup of (M, ω, F) , since we are only keeping track of the marked semitoric polygon invariant and not the twisting index or Taylor series invariants.

From [HSS15] we can deduce the Karshon graph of $[\Delta, \vec{c}, \vec{e}]$ and $[\Delta', \vec{c}', \vec{e}']$ and we see that the Karshon graph of the new marked semitoric polygon is related to the original graph by reducing the area label on the fat vertex corresponding to Σ by λ and adding a new isolated fixed point with J -value label given by

$$J = \begin{cases} j_{\max} - \lambda, & \text{if } \Sigma = J^{-1}(j_{\max}), \\ j_{\min} + \lambda, & \text{if } \Sigma = J^{-1}(j_{\min}). \end{cases}$$

This is exactly Case (B1) from Section 2.4.3 describing the effect of a blowup at a point in a fixed surface on the Karshon graph. Thus, any system (M', ω', F') associated to the marked Delzant semitoric polygon $[\Delta', \vec{c}', \vec{e}']$ is \mathbb{S}^1 -equivariantly symplectomorphic to the \mathbb{S}^1 -equivariant blowup of (M, ω, F) , since they have the same Karshon graph. Moreover, the conditions for a semitoric system to admit a semitoric blowup of size λ at p are exactly the same as the conditions for the associated \mathbb{S}^1 -space to admit a blowup of size λ at p . Thus we have established:

Lemma 2.21. *Let $(M, \omega, F = (J, H))$ be a semitoric system and let Σ be a fixed surface of the \mathbb{S}^1 -action generated by J . Let (M', ω', J') be the \mathbb{S}^1 -equivariant blowup of (M, ω, J) at a point $p \in \Sigma$ of size $\lambda > 0$. Then there exists an $H': M' \rightarrow \mathbb{R}$ such that $(M', \omega', F' = (J', H'))$ is a semitoric system with one more focus-focus singular point than (M, ω, F) and the marked semitoric polygon invariant of (M', ω', F') can be obtained from the marked semitoric polygon invariant of (M, ω, F) as described in Definition 2.19.*

Remark 2.22. We have only specified the marked semitoric polygon invariant of the resulting system, and even then there is a choice of where to put the new marked point. Thus the resulting system of the semitoric blowup discussed in Lemma 2.21 is not unique, and furthermore we make no attempt to describe this blowup as an operation on the semitoric system itself, instead depending on a description of its effect on the invariants. Lemma 2.21 is exactly what we will need for the present paper.

Remark 2.23. In [KPP18] the authors define an invariant of a compact semitoric system called the *semitoric helix* which generalizes the fan of a smooth toric surface. The semitoric helix is a collection of integral vectors which takes into account the effect of the monodromy of the focus-focus points of the semitoric system. The construction of the helix only depends on the structure of the system in a neighborhood of the boundary of the momentum map image and the monodromy of the focus-focus points in the interior. Thus, in many cases, for instance if there are no hyperbolic-elliptic singular points, nearly the same construction could probably also be used to construct such an invariant for hypersemitoric systems, once the monodromy of any singular points in the interior of the momentum map image (such as the points on a flap) is understood. Corollary 5.10 states that all \mathbb{S}^1 -spaces can be extended to a system with no hyperbolic-elliptic points, so it is likely that a helix-like invariant can be developed for such systems.

2.10. Symplectic reduction and integrable systems. Given a Hamiltonian action of an n -dimensional Lie group G on a symplectic manifold (M, ω) with momentum map $\mu: M \rightarrow \mathfrak{g}^*$, where \mathfrak{g}^* is the dual Lie algebra of G , the *symplectic reduction at level $c \in \mu(M)$* is defined to be

$$(M//G)_c := \mu^{-1}(c)/G.$$

Let $\pi_c: \mu^{-1}(c) \rightarrow (M//G)_c$ be the projection on the quotient. If G acts freely on $\mu^{-1}(c)$ then $(M//G)_c$ is a smooth manifold which inherits a symplectic form ω_c from (M, ω) satisfying $\pi_c^* \omega_c = \omega|_{\mu^{-1}(c)}$, as described in the Marsden-Weinstein-Meyer theorem [MW74, Mey73]. More generally, if G does not act freely on $\mu^{-1}(c)$ then $(M//G)_c$ is a type of singular space called a stratified symplectic space, but it still inherits smooth and symplectic structures on the set of points $[x] \in (M//G)_c$ such that G acts freely on $\pi_c^{-1}([x])$. For a detailed study of singular reduction and stratified symplectic spaces, see [CB97, SL91, Alo19].

Now let (M, ω, J) be an \mathbb{S}^1 -space. Let $\hat{M} := M/\mathbb{S}^1$, let $\pi: M \rightarrow \hat{M}$ be the quotient map, and let $\hat{M}_j := (M//\mathbb{S}^1)_j$ for $j \in J(M)$. Then \hat{M} inherits a smooth structure away from the set $\text{sing}(\hat{M}) := \pi(\text{non-free}(J))$, where $\text{non-free}(J) \subset M$ denotes the set of points on which the \mathbb{S}^1 -action generated by J does not act freely (i.e. those points are fixed by some non-identity element of \mathbb{S}^1). Let $\text{smooth}(\hat{M}) := \hat{M} \setminus \text{sing}(\hat{M})$.

Note that $\hat{M}_j = \pi(J^{-1}(j)) \subset \hat{M}$ and let $\hat{H}_j := \hat{H}|_{\hat{M}_j}$.

Lemma 2.24. *Let $(M, \omega, (J, H))$ be an integrable system such that J generates an \mathbb{S}^1 -action, let $p \in M$ be such that \mathbb{S}^1 acts freely on p , and let $c = \pi(p) \in \hat{M}_j$ where $j = J(p)$. Then $c \in \text{smooth}(\hat{M})$. Furthermore:*

- (a) c is a regular point of \hat{H}_j if and only if p is a regular point of (J, H) ,
- (b) c is a non-degenerate critical point of \hat{H}_j with index 0 or 2 if and only if p is an elliptic-regular critical point of (J, H) ,
- (c) c is a non-degenerate critical point of \hat{H}_j with index 1 if and only if p is a hyperbolic-regular critical point of (J, H) .

Proof. The fact that $c \in \text{smooth}(\hat{M})$ is standard from the theory of group actions, since the \mathbb{S}^1 -action in this case is proper, smooth, and free at p .

To prove item (a), notice that c is a regular point of \hat{H}_j if and only if $d\hat{H}_j(c) \neq 0$ which is equivalent to $dH(p)$ and $dJ(p)$ being linearly independent, which is the definition of p being a regular point of (J, H) .

Items (b) and (c) follow from the fact that in this case p is a non-degenerate critical point of (J, H) if and only if c is a non-degenerate (i.e. Morse) point of \hat{H}_j , this is well known and proved for instance in [HP18, Lemma 2.4]². Since \mathbb{S}^1 acts freely on p we know $dJ(p) \neq 0$, so if p is a non-degenerate critical point of (J, H) then there are only two possibilities from Theorem 2.10, p is either elliptic-regular or hyperbolic regular. On the other hand, in this case c must be a non-degenerate critical point of \hat{H}_j , and so it must have index 0, 1, or 2. By matching up the behavior of the level sets of \hat{H}_j , it can be verified that p is elliptic-regular if and only if c has index 0 or 2, and p is hyperbolic regular if and only if c has index 1, as claimed. \square

²In the published version of [HP18] there is a small error in Lemma 2.4, it requires that $dJ(p) \neq 0$ but it should actually require that the \mathbb{S}^1 -action generated by J acts freely on p .

2.11. Parabolic degenerate points. Hypersemitoric systems can also include a certain type of degenerate point, called a *parabolic singular point*, which Colin De Verdiere [CDV03] calls the “the simplest non-Morse [i.e. non-degenerate] example”. Parabolic singular points are also sometimes called cuspidal singular points. These points are studied for instance in [EG12, BGK18, BF04], and we will follow the definition of Bolsinov & Guglielmi & Kudryavtseva [BGK18] since it is the best adapted to our situation. Before stating the definition, we note that any parabolic point p satisfies $F(p) \in \text{interior}(F(M))$ (this follows from [BGK18, Proposition 2.1]), and thus the following lemma implies that we may assume that $dJ(p) \neq 0$ at parabolic points when we define them below.

Lemma 2.25. *Let $p \in M$ be a rank 1 singular point of a compact integrable system $(M, \omega, F = (J, H))$ for which the flow of J generates a effective \mathbb{S}^1 -action. If $J(p) \in \text{interior}(J(M))$ then $dJ(p) \neq 0$, and therefore any rank 1 singular point p with $F(p) \in \text{interior}(F(M))$ has $dJ(p) \neq 0$.*

Proof. Assume that $dJ(p) = 0$, so that p is a fixed point of the \mathbb{S}^1 -action generated by J . Then all points in the orbit of p under the flow generated by H are also fixed points of the flow generated by J , and since p is a rank 1 point it is not fixed by the flow generated by H , so p is a non-isolated fixed point of the \mathbb{S}^1 -action. By Lemma 2.1, this means that p belongs to a fixed surface of the \mathbb{S}^1 -action and $J(p)$ is in the boundary of the interval $J(M)$, and thus $F(p)$ is in the boundary of $F(M)$. \square

If $p \in M$ is a singular point of rank 1, i.e. if $\text{rank}(dF_p) = 1$, then the Hessian $d^2F(p)$ is well defined on the kernel of dF_p , and can have rank 0, 1, or 2. The case of $\text{rank}(d^2F_p) = 2$ is that of non-degenerate singularities, so the next simplest case is when $\text{rank}(d^2F_p) = 1$. The degenerate points we will consider, and which are the ones we will see naturally occur in systems with hyperbolic points, are as follows:

Definition 2.26 ([BGK18, Definition 2.1]). Suppose that $p \in M$ is a singular point of the integrable system $F = (J, H): M \rightarrow \mathbb{R}^2$ such that $dJ(p) \neq 0$. Define

$$\tilde{H} := \tilde{H}_p: J^{-1}(J(p)) \rightarrow \mathbb{R}, \quad \tilde{H} := H|_{J^{-1}(J(p))}.$$

The point p is a *parabolic degenerate singular point* if:

- (1) p is a critical point of \tilde{H} ,
- (2) $\text{rank}(d^2\tilde{H}(p)) = 1$,
- (3) there exists $v \in \ker(d^2\tilde{H}(p))$ such that the third derivative of \tilde{H} in the direction determined by v at p is nonzero (this is well-defined in this case, see Remark 2.27).
- (4) $\text{rank}(d^2(H - kJ)(p)) = 3$, where $k \in \mathbb{R}$ is determined by $dH(p) = kdJ(p)$.

We call the image of a parabolic critical point a *parabolic critical value* of F .

Informally, a parabolic degenerate point can be thought of as a singular point where the rank of all relevant operators is as maximal as possible without the point being non-degenerate.

Remark 2.27. In general, the third derivative of a function is not well defined, but it is well-defined when it is evaluated at a critical point and taken in the direction of a vector in the kernel of the Hessian of the function, as in item (3) of the above definition. We define

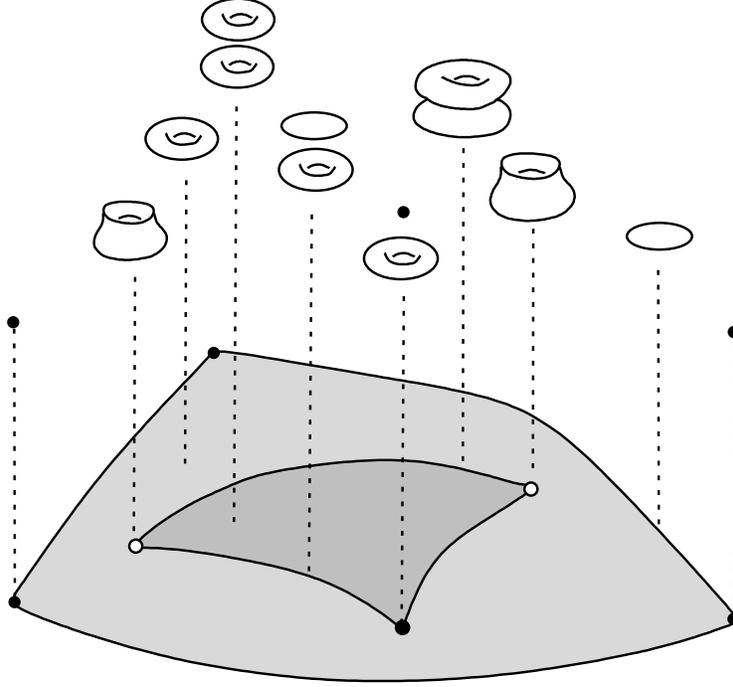


FIGURE 8. A flap in the momentum map image $F(M)$. It is a triangular region in the interior of $F(M)$ containing three families of rank 1 singular values, two of which correspond to families of elliptic-regular singular points and one of which to a family of hyperbolic-regular singular points. The point where the two families of elliptic-regular values meet is the image of an elliptic-elliptic singular point and the other two corners of the triangle are the image of parabolic singular points. The fibers above the interior points of the triangle are the disjoint union of two Lagrangian tori, outside of the triangle they are a single torus, above the elliptic-regular values they are the disjoint union of a torus and a circle, above the elliptic-elliptic value it is the disjoint union of a torus and a point, above the hyperbolic-regular values they are double tori (as in Figure 15a), and above the two degenerate values they are cuspidal tori (as in Figure 15c).

the third derivative of \tilde{H} in the direction of $v \in \ker(d^2\tilde{H}(p))$ by

$$v^3(\tilde{H}) = \frac{d^3}{dt^3}\tilde{H}(\gamma(t))|_{t=0}$$

where $\gamma:]-\varepsilon, \varepsilon[\rightarrow J^{-1}(J(p))$ is a curve satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. It can be shown that this definition does not depend on the choice of such a curve γ by a short calculation, see [BGK18, Remark 2.1].

These points do not admit a *symplectic* normal form, but they do admit a *smooth* normal form.

Proposition 2.28 ([BGK18, Proposition 2.1]). *Let $p \in M$ be a parabolic singular point of an analytic integrable system $(M, \omega, F = (J, H))$ for which the flow of J generates an*

effective \mathbb{S}^1 -action. Then there exists a neighborhood U of p equipped with coordinates (x, y, t, θ) centered at p , and a local diffeomorphism $g = (g_1, g_2)$ of \mathbb{R}^2 around the origin with $g_1(x_1, x_2) = \pm x_1 + \text{const}$ and $\frac{d}{dx_2}(g_2) \neq 0$ such that

$$g \circ F|_U = (t, x^3 + tx + y^2).$$

For an example of a parabolic point, simply take the local normal form $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$, $(x, y, t, \theta) \mapsto (t, x^3 + tx + y^2)$ with symplectic form $\omega = dx \wedge dy + dt \wedge d\theta$, but it is important to keep in mind that the coordinates in Proposition 2.28 are in general not canonical (so the symplectic form does not always take the form described here).

Remark 2.29. By construction, the parabolic points that appear in the present paper will all have the local normal form described in Proposition 2.28, even when the integrable systems are non-analytic (in our situation – only smooth). Non-analytic parabolic points do not necessarily take this form in general, because for $t < 0$ the fibers of the integrable system can be disconnected, in which case a flat function may be added on one of the branches of the fibers but not the other.

Parabolic points come in one parameter families in M , parameterized by θ in the above Proposition, which project to a single point in $F(M)$. Furthermore, in a neighborhood of a parabolic point there are two surfaces of non-degenerate singular points which meet at the parabolic point, one of hyperbolic-regular type points and one of elliptic-regular type points. In the momentum map image this appears as a curve of images of elliptic-regular points which meets with a curve of images of hyperbolic-regular points. This behavior can be seen for example in the neighborhood of the images of the parabolic degenerate points in Figure 8, which has exactly two parabolic singular values.

A detailed discussion of the topological properties of parabolic singularities can be found in [EG12] (where they are called *cuspidal* singular points), especially for the case that the parabolic points appear in a configuration known as a *flap*. A flap is one of the typical situations in which such parabolic points occur. The image of a flap in the momentum map image is shown in Figure 8. These are called flaps because of their topological form in the base space B of the singular Lagrangian fibration induced by F (as discussed in Section 2.6). The region around this triangle in B can be obtained by gluing the triangle to the rest of the base space along the curve of hyperbolic-regular points and parabolic degenerate points, so it is like a flap glued onto the momentum map image. The examples discussed in Section 3.1 have such flaps, as does the system described in Example 3.2.

In [EG12] the authors also discuss *pleats*, which are sometimes also called swallow tails, and these swallowtails are also studied from a global viewpoint by Efstathiou & Sugny [ES10]. Such configurations are also possible in hypersemitoric systems, see for instance Figure 9 and the example in Section 6.6 of [LFP19]. Parabolic points are also very common in physical systems, for instance see [BRF00] and see the references in [BGK18]. Recently, the complete symplectic invariants of parabolic points and parabolic orbits were described in [BGK18]; these invariants are non-trivial and were found to be encoded in the affine structure of the base of the Lagrangian fibration near the parabolic values.

2.12. Parabolic points and reduction. Parabolic points admit a natural \mathbb{S}^1 -action, and the coordinates from Proposition 2.28 can actually be extended to a tubular neighborhood of the entire orbit of the parabolic point, taking $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. In these coordinates, the

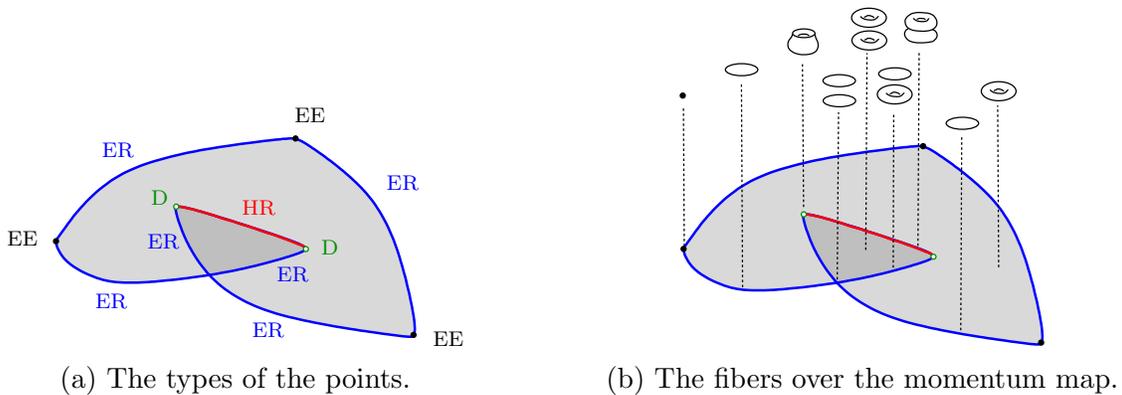


FIGURE 9. A momentum map image of a hypersemitoric system which includes a pleat (or “swallow tail”) configuration of singular points, such as the system from [LFP19, Section 6.6].

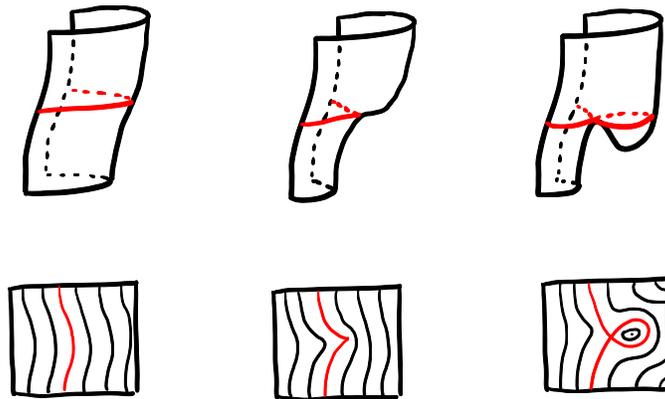


FIGURE 10. The function $f_t(x) = x^3 + tx + y^2$ and its level sets for various choices of t . The left figure is $t = 1$, the middle is $t = 0$, and the right is $t = -1$.

Hamiltonian of the \mathbb{S}^1 -action is given by $J(x, y, t, \theta) = t$, and the Hamiltonian vector field of J is $\frac{\partial}{\partial \theta}$, see [BGK18, Proposition 3.1]. Notice that this \mathbb{S}^1 -action is free in a neighborhood of the parabolic orbit. Locally, performing symplectic reduction with respect to this action at some level t yields a disk and the other Hamiltonian reduces to a function

$$(3) \quad f_t(x, y) = x^3 + tx + y^2.$$

We can think of this as a family of functions on the disk parameterized by t . The graph of f_t and its level sets for various values of t can be seen in Figure 10. Notice:

- if $t < 0$ then f_t has two non-degenerate critical points (of index 1 and 0);
- if $t > 0$ then f_t has no critical points;
- if $t = 0$ then f_t has exactly one critical point, which is degenerate.

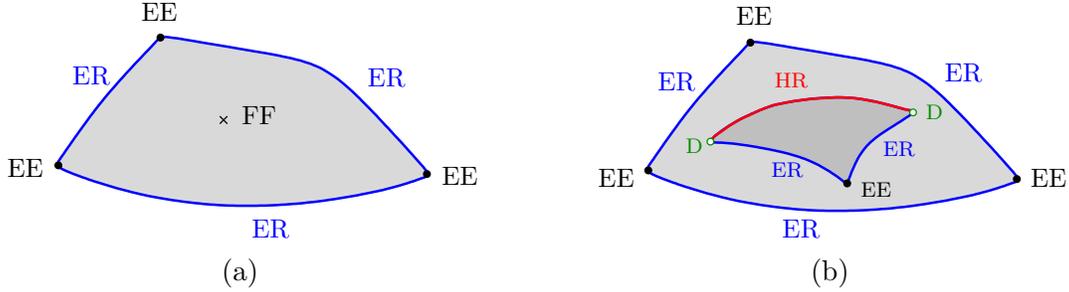


FIGURE 11. Using the technique of [DP16] a focus-focus singular point as in Figure (a) can be replaced with a triangle of singular points as in Figure (b). The configuration of singular values in Figure (b) is called a flap.

This family parameterizes the process of two non-degenerate critical points coming together and annihilating as t increases, or being born as t decreases, and thus it is called a *birth-death singularity*. In fact, Equation (3) is the typical such bifurcation in Morse theory [Cer70].

For $t < 0$ the point $(x, y) = (\sqrt{-t}/3, 0)$ is an index 1 critical point, and the level set that it lies on, $f_t^{-1}\left(-2\left(\frac{-t}{3}\right)^{3/2}\right)$, traces out a curve with a loop, as in the right hand figure in Figure 10. We will call the region enclosed by this loop the *teardrop region*. Notice that in a system with a flap, the one-parameter family of functions on the reduced space at level $J^{-1}(j)$, produces a teardrop region which appears, grows, shrinks, and disappears as j increases.

Finally, notice that if a function takes the form of Equation (3) on the reduced space (at a smooth point) then it extends to the typical example of a parabolic point given in Section 2.11.

3. EXAMPLES

Hypersemitoric systems generalize semitoric systems and are defined in Definition 1.5. These hypersemitoric systems in principle admit any type of non-degenerate singularity (though we will see in Proposition 4.1 that hyperbolic-hyperbolic singular points cannot occur due to the presence of the global \mathbb{S}^1 -action) and additionally may have degenerate singular points of parabolic type.

3.1. The Dullin & Pelayo technique and first examples. One way to obtain examples of hypersemitoric systems is by a technique of Dullin & Pelayo [DP16]. They describe how to start with any semitoric system $(M, \omega, (J, H))$ and perturb H near a focus-focus point to produce a flap of singularities, as shown in Figure 11. This technique does not change J or (M, ω) and thus does not change the underlying \mathbb{S}^1 -space. Moreover, if the original system was hypersemitoric then this operation produces a hypersemitoric system, since the \mathbb{S}^1 -action is preserved and the only new singular points introduced are either non-degenerate or parabolic (see [DP16, Remark 6.4]).

The idea in [DP16] is to define a function $G: M \rightarrow \mathbb{R}$ supported in a neighborhood of a given focus-focus point using the coordinates of the local normal form. Then they replace the original integrable system (J, H) by $(J, \tilde{H} = H + G)$ and obtain a new system with a triangle of singular points as desired.

In [DP16, Section 8] using a version of their technique the authors give the following example. Consider $M = \mathbb{S}^2 \times \mathbb{R}^2$ taking ω to be the product of the standard symplectic forms. Taking coordinates (x, y, z, u, v) where the coordinates on \mathbb{S}^2 are inherited from the inclusion $\mathbb{S}^2 \subset \mathbb{R}^3$, let

$$J = \frac{u^2 + v^2}{2} + z, \quad H = \frac{xu + yv}{2}, \quad \text{and} \quad G_\gamma = \gamma z^2,$$

where $0 \leq \gamma \leq 1$. Here they use a globally defined G_γ since this is easier to work with in practice. Let $\tilde{H}_\gamma = H + G_\gamma$, so $\tilde{H}_0 = H$. Then $(M, \omega, (J, H))$ is the *coupled spin oscillator* as in [VN07], and $(M, \omega, (J, \tilde{H}_\gamma))$ transitions from the coupled spin oscillator (when $\gamma = 0$) into a system which has hyperbolic singularities (when $\gamma > \frac{1}{2}$).

In [LFP19, Section 6.6] the authors give an explicit example of a parameter-dependent integrable system (J, H_t) , $0 \leq t \leq 1$ on the first Hirzebruch surface which transitions from being toric for $t = 0$, to having a flap of singular values for $t \approx \frac{1}{2}$, to having a pleat of singular values for $t \approx 1$.

Remark 3.1. The Dullin & Pelayo technique can produce many examples of hypersemitoric systems from semitoric systems, but not all hypersemitoric systems can be formed this way. For instance hypersemitoric systems can have fixed surfaces which are not spheres, in which case they could never come from a semitoric system via this technique.

3.2. Representative examples. Next, we give two examples which are important in their own right and also illustrate the idea of our proof of Theorem 1.6 in Section 5.

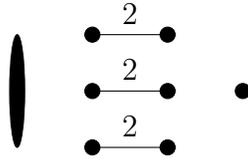
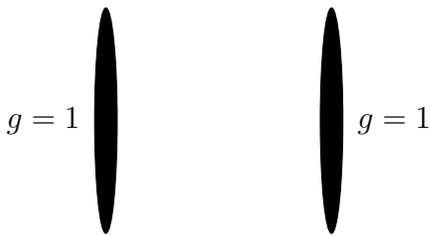


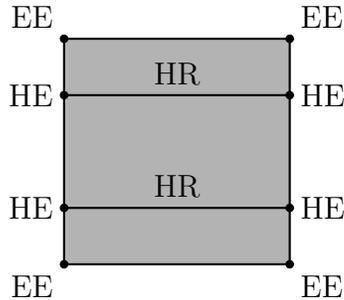
FIGURE 12. An \mathbb{S}^1 -space which cannot be obtained from a toric or semitoric system.

Example 3.2. In this example we will produce an \mathbb{S}^1 -space on $\mathbb{C}\mathbb{P}^2$ blown up six times which cannot be extended to a semitoric system but can be extended to a hypersemitoric system. We start with the \mathbb{S}^1 -space on $\text{Bl}^5(\mathbb{C}\mathbb{P}^2)$ described in Example 2.17 whose Karshon graph is shown in Figure 7d. Now we perform another blowup of size λ_2 (using the notation from Example 2.17) at the isolated fixed point which is not at the maximum value of J , which produces the Karshon graph shown in Figure 12. This system cannot be extended to a semitoric system since each edge in the Karshon graph must be associated to a unique edge (or chain of edges) from the top or bottom boundary of the semitoric polygon, so there cannot be three edges in the Karshon graph which intersect in a single level set $J^{-1}(j)$, but we can describe how to produce a hypersemitoric system that this system extends to.

Start with the semitoric system (J, H) on $\text{Bl}^5(\mathbb{C}\mathbb{P}^2)$ and use the Dullin & Pelayo technique described in Section 3.1 around the unique focus-focus point to create a new integrable system (J, \tilde{H}) . The point $p \in \text{Bl}^5(\mathbb{C}\mathbb{P}^2)$ that was focus-focus in (J, H) is now an elliptic-elliptic singular point and the image of (J, \tilde{H}) now contains a triangle of singular values, as in Figure 11. Notice that (J, H) and (J, \tilde{H}) have the same underlying \mathbb{S}^1 -space since the manifold,



(a) The Karshon graph. There are no \mathbb{Z}_k -spheres with $k > 1$ and the only fixed points occur in the fixed surfaces, which are both tori.



(b) The momentum map image. The points are labeled HR (hyperbolic-regular), HH (hyperbolic-hyperbolic), and EE (elliptic-elliptic). The boundary is entirely elliptic-regular points, except for the marked rank zero points.

FIGURE 13. The Karshon graph and momentum map image for the system on $\mathbb{S}^2 \times \mathbb{T}^2$ from Example 3.3.

symplectic form, and J are all unchanged. Since p is an elliptic-elliptic singular point of (J, H) we can perform a toric-type blowdown of size λ_1 on p (assuming λ_1 is sufficiently small), producing a new integrable system on $\text{Bl}^6(\mathbb{C}\mathbb{P}^2)$ which is of hypersemitoric type and has the desired underlying \mathbb{S}^1 -space. The idea of this trick is shown in Figure 2.

Example 3.3. Consider the \mathbb{S}^1 -space with $M = \mathbb{S}^2 \times \mathbb{T}^2$ (with the direct sum of the usual symplectic forms) where the Hamiltonian J is the usual height function on the sphere. Thus, the \mathbb{S}^1 -action rotates the sphere component and does not effect the torus component. Then $M^{\mathbb{S}^1}$ is the disjoint union of two copies of \mathbb{T}^2 and there are no \mathbb{Z}_k spheres for $k > 1$, so the Karshon graph is as in Figure 13a. In toric or semitoric systems the components of the fixed point set are always isolated points or embedded spheres, so this cannot be extended to a semitoric system. Consider the torus presented as the surface of revolution of the circle $x^2 + (y - 2)^2 = 1$ around the x -axis, and consider the function $h(x, y, z) = z$ restricted to this surface. Then h is the usual Morse function on \mathbb{T}^2 , i.e. the (unperturbed) height function when the torus is standing “on its end” which has four critical points: one of index 0, two of index 1, and one of index 2. Now let $\pi_{\mathbb{T}^2}: M \rightarrow \mathbb{T}^2$ be projection and let $H = h \circ \pi_{\mathbb{T}^2}: M \rightarrow \mathbb{R}$.

Then, as discussed in Remark 2.11, $(M, \omega, (J, H))$ is an integrable system with no degenerate points. By considering the index of the critical points of the Morse functions, this integrable system has elliptic-elliptic, elliptic-regular, hyperbolic-regular, and hyperbolic-elliptic points. Since J is proper and generates an effective \mathbb{S}^1 -action (J, H) is thus a hypersemitoric system. The image of (J, H) with the types of points labeled is shown in Figure 13b.

4. PROPERTIES OF INTEGRABLE SYSTEMS WITH \mathbb{S}^1 -ACTIONS

In this section we will make use of the local normal form theorem, Theorem 2.10, which implies that in a neighborhood U of a rank 0 singular point p of an integrable system $(M, \omega, F = (J, H))$ there are coordinates $\psi: U \rightarrow \mathbb{R}^4$, with coordinate functions $\psi = (x, \xi, y, \eta)$, such that there are functions $f_1, f_2: U \rightarrow \mathbb{R}$ of a specific type (depending on the type of p) satisfying that $\{f_i, J\} = \{f_i, H\} = 0$ for $i \in \{1, 2\}$. Note that the presence of hyperbolic blocks

in the singularities means that in general we do not have a local diffeomorphism g of \mathbb{R}^2 such that $g \circ (J, H) = (f_1, f_2)$, but the fact that J Poisson commutes with each f_i will be sufficient for these proofs, since this implies that f_1 and f_2 are invariant under the flow of \mathcal{X}^J , so the flow of \mathcal{X}^J must stay on the joint level sets $(f_1, f_2) = (c_1, c_2)$. See item (2) of Theorem 2.10.

4.1. Systems with \mathbb{S}^1 -actions. In this section we prove some results which apply to any integrable system in which one of the integrals generates an \mathbb{S}^1 -action.

The following proposition is probably well-known to experts, but for the convenience of the reader we include a short proof here. Part 2 also follows from the work of Zung [Zun96, Zun03], in which he classifies the local symmetries of non-degenerate singular points without depending on Eliasson's theorem, and finds in particular that hyperbolic-hyperbolic singularities do not admit a local \mathbb{S}^1 -action, see [Zun96, Theorem 6.1].

Proposition 4.1. *Let $(M, \omega, F = (J, H))$ be an integrable system such that J generates an effective \mathbb{S}^1 -action. Then:*

- (1) *If $p \in M$ is a singular point of hyperbolic-elliptic type then p is a non-isolated fixed point of the \mathbb{S}^1 -action, and thus it lies in a fixed surface of the \mathbb{S}^1 -action;*
- (2) *(M, ω, F) has no singular points of hyperbolic-hyperbolic type.*

Proof. Assume that $p \in M$ is a singular point and $J: M \rightarrow \mathbb{R}$ generates an effective global \mathbb{S}^1 -action. Eliasson's theorem (cf. Theorem 2.10) around p implies that in a neighborhood U of p there are symplectic coordinates

$$\psi = (x, \xi, y, \eta): U \rightarrow \mathbb{R}^4$$

such that $\psi(p) = (0, 0, 0, 0)$ and functions $f_1, f_2: U \rightarrow \mathbb{R}$ of a certain form (depending on the type of p) such that $\{J|_U, f_i\} = 0$ for $i \in \{1, 2\}$ in U . In both cases since p is a rank 0 singular point of (f_1, f_2) we have that $dJ(p) = 0$, so p is a fixed point for the associated \mathbb{S}^1 -action.

- (1) If $p \in M$ is of hyperbolic-elliptic type then we may take

$$f_1 = (x^2 + \xi^2)/2 \quad \text{and} \quad f_2 = y\eta.$$

Consider the point $\tilde{p} = \psi^{-1}(0, 0, y, 0) \in U$ for some sufficiently small y and let ϕ_t be the time- t flow of \mathcal{X}^J . Since ϕ_t preserves f_1 and f_2 we see that $\phi_t(\tilde{p}) = \psi^{-1}(0, 0, \phi_t^1(y), 0)$ for small enough t , and thus if the action on \tilde{p} is non-trivial then either the forwards or backwards flow has to approach the fixed point p contradicting the fact that the flow is periodic. Thus, for all sufficiently small y the point $\psi^{-1}(0, 0, y, 0)$ is fixed by the \mathbb{S}^1 -action so p is not an isolated fixed point of the \mathbb{S}^1 -action, and so by Lemma 2.1 it must lie on a fixed surface.

- (2) If $p \in M$ is of hyperbolic-hyperbolic type then we may take

$$f_1 = x\xi \quad \text{and} \quad f_2 = y\eta.$$

Let $p' = \psi^{-1}(x, 0, 0, 0) \in U$ for some sufficiently small x . Then

$$dJ(p') \in \text{span}\{df_1(p'), df_2(p')\} = \text{span}\{d\xi\}$$

so $\mathcal{X}^J \in \text{span}\{\partial_x\}$. If p' is not fixed by the \mathbb{S}^1 -action then either forwards or backwards flow of \mathcal{X}^J must approach the fixed point p . This contradicts the fact that the flow of \mathcal{X}^J is periodic, so p' must be a fixed point of the \mathbb{S}^1 -action. Thus, following similar reasoning we see that for all sufficiently small $x, \xi, y, \eta \in \mathbb{R}$ the points $\psi^{-1}(x, 0, 0, 0)$, $\psi^{-1}(0, \xi, 0, 0)$, $\psi^{-1}(0, 0, y, 0)$,

and $\psi^{-1}(0, 0, 0, \eta)$ are all fixed by the \mathbb{S}^1 -action, so p is not an isolated fixed point and also the component of $M^{\mathbb{S}^1}$ containing p is not a surface. This contradicts Lemma 2.1. \square

Let $M^{\text{HR}}, M^{\text{HE}}, M^{\text{D}} \subset M$ be the set of hyperbolic-regular singular points, hyperbolic-elliptic singular points, and degenerate singular points, respectively.

Lemma 4.2. *Let $(M, \omega, (J, H))$ be an integrable system such that J generates an effective \mathbb{S}^1 -action and let $C \subset M$ be a connected component of M^{HR} . Then $F(C) \subset \mathbb{R}^2$ does not have any vertical tangencies.*

Proof. Suppose that $p \in M^{\text{HR}}$ and let $C \subset M^{\text{HR}}$ be the connected component of M^{HR} containing p . By Lemma 2.12 $F(C) \subset \mathbb{R}^2$ is a one-dimensional immersed submanifold. According to [BF04, Proposition 1.16], the tangent vector to the curve $F(C)$ is given by (a, b) , where $a, b \in \mathbb{R}$ satisfy $b\mathcal{X}^J(p) - a\mathcal{X}^H(p) = 0$. Such a, b must exist since $\text{rank}(p) = 1$. If $F(C)$ has a vertical tangent we have that $(a, b) = (0, 1)$ and therefore $\mathcal{X}^J(p) = 0$ so $dJ(p) = 0$. This implies that $dJ(p') = 0$ as well for any p' obtained by flowing along \mathcal{X}^H for a sufficiently short amount of time, and note that $\mathcal{X}^H(p) \neq 0$ since the rank of p is 1, so $p \neq p'$. Therefore p is a non-isolated fixed point of the \mathbb{S}^1 -action, so by Lemma 2.1 it lies in a fixed surface $\Sigma \subset M^{\text{HR}}$ where $\Sigma = J^{-1}(j_{\min})$ or $\Sigma = J^{-1}(j_{\max})$. This implies that $F(p) \in \partial(F(M))$, but the image of hyperbolic-regular points must lie in the interior of the momentum map image, giving a contradiction. \square

We know that the image of an connected component of M^{HR} is an immersed curve by Lemma 2.12, and Lemma 4.2 implies that it is actually embedded (and is a graph over J).

Corollary 4.3. *If $C \subset M$ is a connected component of M^{HR} then $F(C)$ is homeomorphic to an open interval, the endpoints of $F(C)$ are distinct, and each endpoint is either an element of $F(M^{\text{HE}})$ or $F(M^{\text{D}})$. In particular, $F(C)$ is not a curve which connects the image of a fixed surface back to itself and it is not homeomorphic to a loop.*

Proof. By Lemma 4.2 $F(C)$ does not have any vertical tangent so it cannot include a self-intersection and cannot form a loop. Thus, $F(C)$ is homeomorphic to an interval. Due to the local normal forms result, Theorem 2.10, given any point $p \in M$ which is regular or singular of type elliptic-elliptic, elliptic-regular, or focus-focus, there exists a neighborhood of p which does not include any hyperbolic-regular points. Thus, the end points of the interval $F(C)$ must be the image of the only other possible points in M , either hyperbolic-elliptic or degenerate. \square

Corollary 4.3 is illustrated in Figure 14.

Corollary 4.4. *Let $(M, \omega, (J, H))$ be an integrable system such that J generates an effective \mathbb{S}^1 -action. If the \mathbb{S}^1 -action has strictly less than two fixed surfaces and (J, H) has a singularity of hyperbolic-regular type, then (J, H) has at least one degenerate singular point.*

Proof. Recall hyperbolic-regular values come in one-parameter families. By Corollary 4.3 this family terminates in two points, each of which are either the image of a degenerate or hyperbolic-elliptic point. It is impossible for both endpoints to be the images of hyperbolic-elliptic points, since hyperbolic-elliptic points always lie in a fixed surface by Lemma 4.1, a single family of hyperbolic-regular points cannot connect the same fixed surface to itself by Corollary 4.3, and we have assumed that there are strictly less than two fixed surfaces. \square

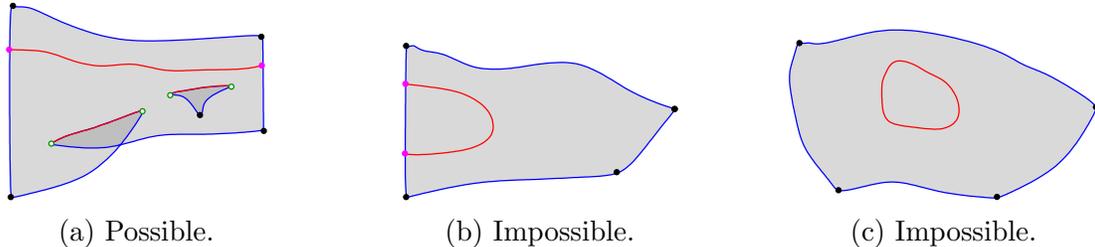


FIGURE 14. One possible momentum map image of a integrable system with a global \mathbb{S}^1 -action and two impossible ones: (a) is a possible momentum map image. It include a flap, a pleat, and a curve of hyperbolic-regular points whose two endpoints are both hyperbolic-elliptic points. We have seen such behaviors in the examples discussed in Section 3. (b) is not possible since a family of hyperbolic-regular points cannot connect a fixed surface to itself and (c) is not possible because such a system cannot produce a loop of hyperbolic-regular values.

Remark 4.5. By Corollary 4.4, if a compact integrable system with a Hamiltonian \mathbb{S}^1 -symmetry has a hyperbolic-regular point and has less than two fixed surfaces then it must have a degenerate point, which is why we need to allow certain degenerate singular points in order to be able to extend all possible \mathbb{S}^1 -spaces. On the other hand, we conjecture that if a \mathbb{S}^1 -space has two fixed surfaces (in which case it is called *tall* as in [KT14]) then it can always be extended to a hypersemitoric system which has no degenerate points.

Thus, any \mathbb{S}^1 -action which has less than two fixed surfaces and cannot be extended to a semitoric system must have at least one degenerate point in any extension:

Corollary 4.6. *Let (M, ω, J) be an \mathbb{S}^1 -space which has strictly less than two fixed surfaces and has three or more \mathbb{Z}_k -spheres passing through a single level set of J . If $H: M \rightarrow \mathbb{R}$ is such that $(M, \omega, (J, H))$ is an integrable system, then $(M, \omega, (J, H))$ has at least one degenerate singular point.*

In particular, this implies that the \mathbb{S}^1 -space on $\mathbb{C}\mathbb{P}^2$ blown up six times from Example 3.2, sketched in Figure 12, cannot be extended to an integrable system with no degenerate points. This justifies why we need to allow degenerate points in hypersemitoric systems, since our goal is to show that all \mathbb{S}^1 -spaces can be extended to such a system.

Remark 4.7. (*A question of Dullin & Pelayo*). The nature of the relationship between the existence of hyperbolic-regular singularities and degenerate singularities in the presence of a global \mathbb{S}^1 -action has long been considered³. The heart of this question is whether a loop of hyperbolic-regular values is possible in the presence of a global \mathbb{S}^1 -action. If such a loop is not possible, then the endpoint of a one-parameter family of hyperbolic-regular values will often be forced to be the image of a degenerate singular point (except in the cases that it can instead be the image of a hyperbolic-elliptic point, which is what happens in Example 3.3). In Corollary 4.3 we have shown that, as Dullin & Pelayo guessed, it is *not possible to have a loop of hyperbolic-regular values in the presence of a global \mathbb{S}^1 -action*, and in Corollary 4.4

³for instance see Question 5.2 and the following paragraph in the arXiv v1 version of [DP16], which can be found at <https://arxiv.org/abs/1503.01534v1.pdf>.

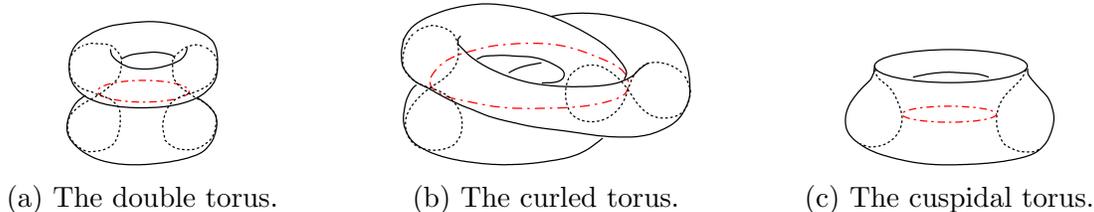


FIGURE 15. Some fibers which include hyperbolic-regular singular points.

we have shown that if the \mathbb{S}^1 -action has less than two fixed surfaces then the presence of a hyperbolic-elliptic singular point forces the presence of a degenerate singular point.

4.2. Fibers of hypersemitoric systems and isotropy. In this section we will discuss how the topology of the fibers which contain hyperbolic-regular points is related to the isotropy of the \mathbb{S}^1 -action in certain cases. A common fiber which contains hyperbolic-regular points is the *double torus*, which is homeomorphic to two tori glued along an \mathbb{S}^1 , or equivalently as a figure eight (an immersion of \mathbb{S}^1 with a single transverse self-intersection) crossed with \mathbb{S}^1 , as in Figure 15a. This fiber occurs in every example in Section 3, and in particular this is the type of fiber that contains the hyperbolic-regular points produced by the technique of Dullin & Pelayo [DP16]. The fibers of integrable systems for which one of the integrals generates an \mathbb{S}^1 -action can be more complicated than this, though. Another possibility, for instance, is the *curled torus*, see Figure 15b. A curled torus fiber is homeomorphic to a figure eight crossed with the interval $[0, 1]$ modulo the relation $(x, 0) \sim (\phi(x), 1)$, where ϕ is a map from the figure eight to itself which switches the top and bottom teardrops (for instance rotation by π).

Proposition 4.8. *Let $(M, \omega, F = (J, H))$ be an integrable system such that J generates an effective \mathbb{S}^1 -action and consider a fiber $F^{-1}(c)$. Then*

- (1) *if $F^{-1}(c)$ is a double torus then the \mathbb{S}^1 -action generated by J acts freely on $F^{-1}(c)$;*
- (2) *if $F^{-1}(c)$ is a curled torus then the \mathbb{S}^1 -action generated by J acts freely on the regular points of $F^{-1}(c)$ and acts with isotropy subgroup \mathbb{Z}_2 on the hyperbolic-regular points of $F^{-1}(c)$.*

Proof. First suppose that $F^{-1}(c)$ is a double torus fiber. Using the local normal form around any of the hyperbolic-regular points we can see that the period of the flow of \mathcal{X}^J is equal for all points in a neighborhood of the hyperbolic-regular point in question. Similarly, the flow of a hyperbolic-regular point in a curled torus fiber will have exactly half of the period of the flow of the regular points in an open set around that point. Since an effective Hamiltonian \mathbb{S}^1 -action is free on a dense set this proves the claim. \square

More complicated fibers are also possible. In the following Example 4.9 there are fibers which contain multiple curves of hyperbolic-regular points. Note that there are also degenerate points that are not parabolic and thus the system is not hypersemitoric.

Example 4.9 (Martynchuk & Efstathiou [ME17], Section 4.2). Consider $M = \mathbb{S}^2 \times \mathbb{S}^2$ with coordinates $((x_1, y_1, z_1), (x_2, y_2, z_2))$ obtained from inclusion as the product of unit spheres in $\mathbb{R}^3 \times \mathbb{R}^3$ and the standard product symplectic form $\omega = \omega_{\mathbb{S}^2} \oplus \omega_{\mathbb{S}^2}$ and integrable system

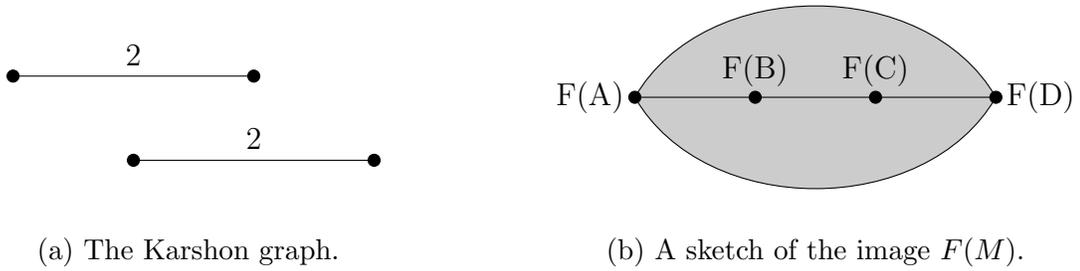


FIGURE 16. The Karshon graph and momentum map image for the system described in Example 4.9. In the momentum map image, the images of the fixed points are indicated with dots, and the curves inside of the interior of the image are the image of hyperbolic-regular points.

$F = (J, H): \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{R}^2$ given by

$$J = z_1 + 2z_2, \quad H = \operatorname{Re}((x_1 + iy_1)^2(x_2 - iy_2)).$$

Let $N = (0, 0, 1)$ denote the north pole and $S = (0, 0, -1)$ denote the south pole of each sphere. J generates an effective \mathbb{S}^1 -action with four fixed points given by

$$A = (S, S), \quad B = (N, S), \quad C = (S, N), \quad \text{and} \quad D = (N, N),$$

and two \mathbb{Z}_2 -spheres: one connecting A to C , consisting of points of the form $(S, (x_2, y_2, z_2))$, and one connecting B to D , consisting of points of the form $(N, (x_2, y_2, z_2))$. Thus the Karshon graph of (M, ω, J) is as shown in Figure 16a. The integrable system $(M, \omega, (J, H))$ is not hypersemitoric since the points A and D are degenerate points that are not parabolic since parabolic points are always in the interior of the momentum map. The image of F is shown in Figure 16b. Given any $c \in \mathbb{R}^2$ on the open interval connecting $F(A)$ to $F(B)$ or $F(C)$ to $F(D)$ the fiber $F^{-1}(c)$ is a curled torus, and given any $c \in \mathbb{R}^2$ on the open interval connecting $F(B)$ to $F(C)$ the fiber $F^{-1}(c)$ is two curled torus fibers glued along a regular orbit. Comparing Figure 16a to Figure 16b we see that the two \mathbb{Z}_2 -spheres get mapped to the same points in the image in the region between $F(B)$ and $F(C)$.

5. ALL \mathbb{S}^1 -SPACES CAN BE EXTENDED TO HYPERSEMITORIC SYSTEMS

In this section we prove Theorem 1.6, which states that any \mathbb{S}^1 -space can be extended to a hypersemitoric system. We prove this making use of Karshon's classification of minimal models.

5.1. Preparations for the proof. First we will show that all of the minimal \mathbb{S}^1 -spaces described by Karshon admit an extension to a hypersemitoric system.

Proposition 5.1. *If (M, ω, J) is a four-dimensional compact Hamiltonian \mathbb{S}^1 -space which does not admit an \mathbb{S}^1 -equivariant blowdown then there exists an $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is a hypersemitoric system with no degenerate singular points.*

Proof. Due to Theorem 2.9 there are three classes of minimal \mathbb{S}^1 -spaces. The first two classes are induced by toric actions, so by definition these extend to toric integrable systems which are in particular hypersemitoric. The remaining minimal models are spaces with two fixed

surfaces and no interior points, which are thus \mathbb{S}^2 -bundles over a closed surface Σ with an \mathbb{S}^1 -action that fixes Σ and rotates each fiber (i.e. ruled surfaces).

Given such a space (M, ω, J) let $f: \Sigma \rightarrow \mathbb{R}$ be any Morse function on Σ , let $\pi_\Sigma: M \rightarrow \Sigma$ be the projection map, and define $H: M \rightarrow \mathbb{R}$ by $H = f \circ \pi_\Sigma$. We claim that $(M, \omega, F = (J, H))$ is hypersemitoric. Indeed, locally around any point $p \in M$ there is a neighborhood $U = \mathbb{S}^2 \times U_\Sigma$ where $U_\Sigma \subset \Sigma$ is an open set and $J|_U = g \circ \pi_{\mathbb{S}^2}$, where g is the usual height function on \mathbb{S}^2 and $\pi_{\mathbb{S}^2}: U \rightarrow \mathbb{S}^2$ is the projection map. The function J locally has this form because, as discussed in Theorem 2.9, the \mathbb{S}^1 -action on the ruled surfaces is rotation of the \mathbb{S}^2 , which locally is thus generated by the height function on \mathbb{S}^2 . This can also be seen in the explicit realization of these minimal models described in [Kar99, proof of Lemma 6.15]. Thus, $F|_U = (g \circ \pi_{\mathbb{S}^2}, f \circ \pi_\Sigma)$. Since g and f are both Morse as in Remark 2.11 this implies that all singular points of $F = (J, H)$ in U are non-degenerate, and thus all singular points of F are non-degenerate, so the system is hypersemitoric. \square

Remark 5.2. The image of the hypersemitoric system constructed on a ruled surface in the proof of Proposition 5.1 will always be a rectangle, and the images of the hyperbolic-regular points of the system will be horizontal lines across the rectangle (as in Figure 13b). So there are two vertical boundary components in the image corresponding to the two fixed surfaces in these \mathbb{S}^1 -spaces, but even though these two edges of the image have the same length this does not mean that the corresponding fixed surfaces have the same symplectic area (in fact, they often do not). This is because the hypersemitoric system we construct is not toric so there is no relationship between the length of the edges and the symplectic volume of the corresponding submanifolds.

Remark 5.3. In Proposition 5.1 we show that given an \mathbb{S}^1 -space with $M = \mathbb{S}^2 \times \Sigma_g$ where J rotates the sphere, we can take a Morse function f on Σ to produce a hypersemitoric system $(M, \omega, (J, H = f \circ \pi_\Sigma))$. Notice that we may furthermore choose f so that \mathcal{X}^H has 2π -periodic flow in a preimage of a neighborhood of the upper boundary of $F(M)$, and therefore on that neighborhood (J, H) forms an (open) toric integrable system.

Let Γ be a Karshon graph. We call each connected component of Γ a *component* of Γ . The *maximum component* of Γ is the component of Γ which contains the vertex corresponding to the maximum value of the momentum map and the *minimum component* of Γ is the component of Γ which contains the vertex corresponding to the minimum value of the momentum map.

Lemma 5.4. *Let Γ be the Karshon graph of an \mathbb{S}^1 -space (M, ω, J) and suppose that Γ can be obtained from a minimal Karshon graph Γ_{\min} by a sequence of \mathbb{S}^1 -equivariant blowups. Then it is possible to obtain Γ from Γ_{\min} by performing blowups in an order given by three stages:*

- **Stage 1:** *Perform a sequence of blowups on isolated fixed points in the maximum and minimum components to obtain a new graph Γ' from Γ_{\min} ;*
- **Stage 2:** *Perform a sequence of blowups on points that lie in fixed surfaces to obtain a new graph Γ'' from Γ' ;*
- **Stage 3:** *Perform a sequence of blowups on isolated fixed points that correspond to the components of Γ'' which are not the maximum or minimum components to obtain the desired graph Γ from Γ'' .*

Remark 5.5. In Section 2.4.3 we list four possible cases of the effect of \mathbb{S}^1 -equivariant blowups on a Karshon graph, labeled (B1)–(B4). In *Stage 1* from the above lemma all

blowups will be of types (B2), (B3), and (B4). In *Stage 2* all blowups will be of type (B1). In *Stage 3* all blowups will be of type (B4). An example of performing blowups in this order is described in Section 5.2.

Proof of Lemma 5.4. This lemma follows from the fact that blowups in different components of Karshon graph do not interact with each other. In Stage 1 the blowups needed so that the maximum and minimum components of Γ' agree with those of Γ are performed. Then each remaining component of Γ has to be produced by a blowup on a point in a fixed surface which produces a new component of the Karshon graph (these blowups are performed in Stage 2) followed by a sequence of blowups on the new component (these blowups are performed in Stage 3). \square

5.2. Following the proof on an example. In this section we will sketch the proof of Theorem 1.6 by applying it to a specific example. Let Γ denote the graph shown in Figure 17i. We will show how a hypersemitoric system can be constructed which has Γ as the Karshon graph of its underlying \mathbb{S}^1 -space, and in the proof of Theorem 1.6 we will follow this same construction and show that it works for all possible Karshon graphs Γ .

The entire process is shown in Figure 17. The Karshon graphs in Figures 17a, 17c, 17e, 17g, and 17i correspond to the systems shown in Figures 17b, 17d, 17f, 17h, and 17j, respectively.

The graph Γ can be obtained from the minimal graph Γ_{\min} , shown in Figure 17a, by a sequence of blowups. The graph Γ_{\min} is the Karshon graph of the third Hirzebruch surface with the standard \mathbb{S}^1 -action (with scaling such that it has the Delzant polygon as shown in Figure 17b), and can thus be extended to a toric integrable system. The idea is to perform the sequence of blowups on Γ_{\min} in three stages as discussed in Lemma 5.4, while showing that the property of being able to be extended to a hypersemitoric system is preserved. Let $(M_{\min}, \omega_{\min}, F_{\min})$ be the toric integrable system associated with the minimal model, so the image of F is the polygon shown in Figure 17b with vertices at $(0, 0)$, $(0, 4)$, $(3, 4)$, and $(12, 0)$.

Stage 1: Performing a toric blowup at the far right fixed point of size $\lambda_1 = 1$ produces the toric integrable system corresponding to the Delzant polygon shown in Figure 17d. Denote this system by (M', ω', F') . The underlying \mathbb{S}^1 -space has Karshon graph Γ' as shown in Figure 17c.

Stage 2: Next we perform three weak semitoric blowups all of the same size $\lambda_2 = \lambda_3 = \lambda_4 = 1$ at points on the fixed surface at the minimum of the momentum map. This produces three new focus-focus points. Let (M'', ω'', F'') be the resulting semitoric system. A representative of the semitoric polygon of this system is shown on the left in Figure 17f and the Karshon graph for the underlying \mathbb{S}^1 -space is the graph Γ'' shown in Figure 17e.

Stage 3: In order to perform a toric blowup on the three new focus-focus points we will now perform a supercritical Hamiltonian-Hopf bifurcation to transform them into elliptic-elliptic points, as in Dullin & Pelayo [DP16]. The momentum map image of the resulting system $(M'', \omega'', \hat{F}'' = (J'', \hat{H}''))$ is shown on the right in Figure 17f. Notice that this bifurcation does not change the underlying \mathbb{S}^1 -space (M'', ω'', J'') and thus the associated Karshon graph shown in Figure 17g is still equal to the graph Γ'' in Figure 17e. Now two of the focus-focus points have been replaced by elliptic-elliptic points, at which we can perform toric blowups.

Now we perform a toric blowup of size $\lambda_5 = 1/2$ on the top flap and a toric blowup of size $\lambda_6 = 1/2$ on the bottom flap. Finally, we perform one more blowup of size $\lambda_7 = 1/4$ on one

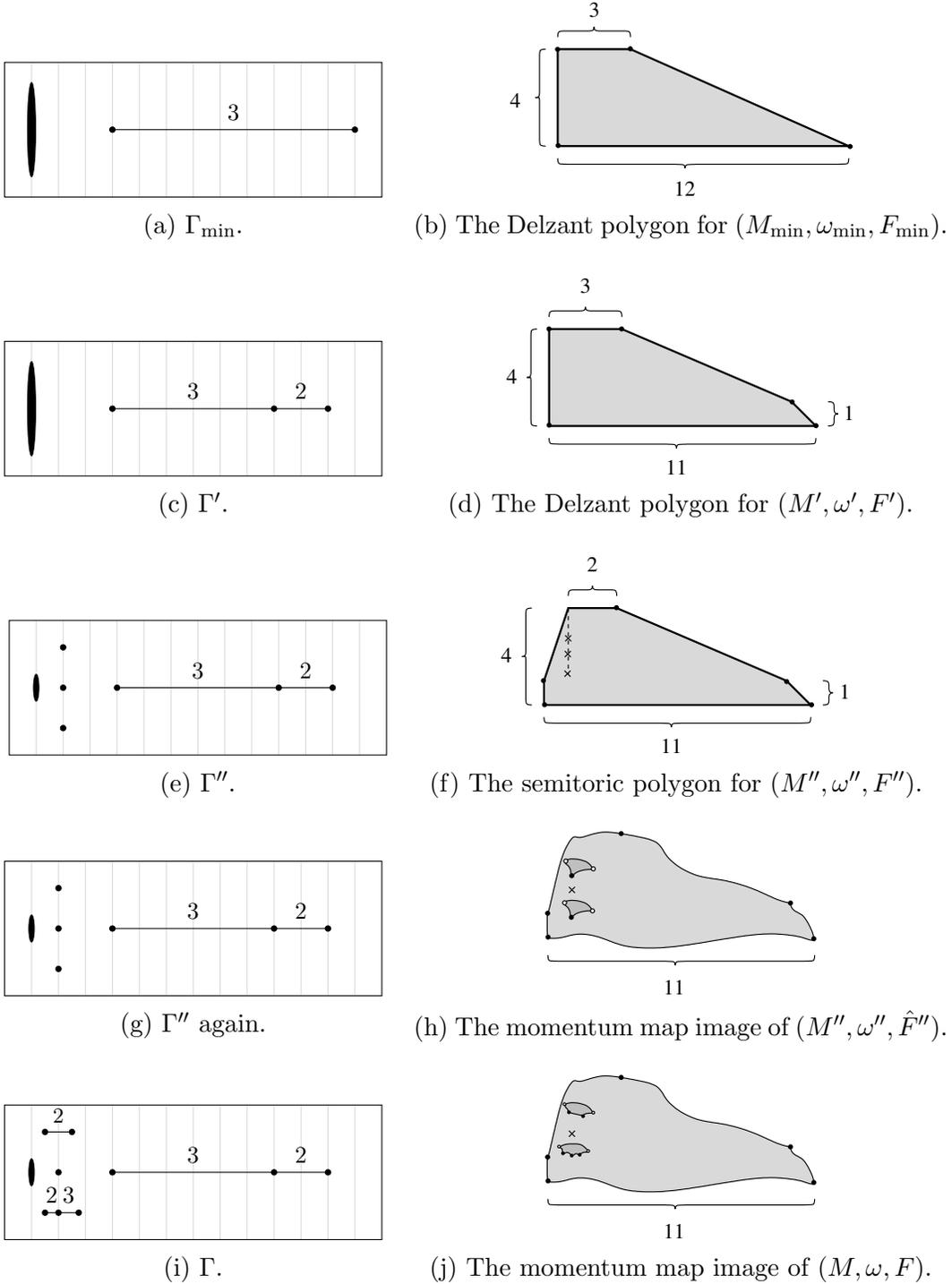


FIGURE 17. The Karshon graphs (left) and corresponding Delzant polygons, semitoric polygons, and momentum map images (right) for the systems discussed in the example from Section 5.2. The Karshon graphs are drawn over lines representing integer values of the momentum map so the momentum map labels can be easily read off of the graph.

of the two new elliptic-elliptic points formed on the bottom flap to produce a hypersemitoric system (M, ω, F) . Since we were able to make this system by performing the same blowups used to produce Γ from Γ_{\min} we see that the Karshon graph associated to (M, ω, J) is Γ , as desired. The proof that these flaps can always be made large enough to admit blowups of the desired size is the content of Section 5.4. This completes the example.

5.3. Semitoric blowups of hypersemitoric systems. In the proof of Theorem 1.6 we will need to perform what is essentially a semitoric blowup on minimal hypersemitoric system which is not semitoric. This means we cannot depend on the polygon invariant to define the weak semitoric blowup as we did in Section 2.9.2, since this system is not semitoric. The strategy we will employ follows from the observation that the minimal \mathbb{S}^1 -spaces on ruled surfaces can be obtained by gluing a Hirzebruch surface with a product manifold $\mathbb{S}^2 \times \Sigma_g$, where Σ_g is a surface of genus g . Thus, we may perform the semitoric blowups on the Hirzebruch surface, which is a toric (and hence also semitoric) integrable system, before gluing it with $\mathbb{S}^2 \times \Sigma_g$, to obtain the desired \mathbb{S}^1 -space. This technique allows us to be sure that the hyperbolic-regular points do not interact with the semitoric blowups.

Roughly, the following lemma explains that a semitoric blowup can be performed without changing the structure of the preimage of a neighborhood of the bottom boundary of the momentum map image, which is the region we will use for the gluing.

Lemma 5.6. *Let (M, ω, F) be a semitoric system and let $[\Delta, \vec{c}_i, (+1)_{j=1}^s]$ be its marked polygon invariant. Assume that Δ has no vertices on the interior of its bottom boundary. Suppose that $[\Delta', (c_1, \dots, c_{\ell-1}, c, c_\ell), (+1)_{j=1}^{s+1}]$ is a semitoric blowup of $[\Delta, \vec{c}_i, (+1)_{j=1}^s]$ and let (M', ω', F') be a semitoric system having $[\Delta', (c_1, \dots, c_{\ell-1}, c, c_\ell), (+1)_{j=1}^{s+1}]$ as its marked polygon invariant. Let g and g' be the straightening maps such that $\Delta = g \circ F$ and $\Delta' = g' \circ F'$, as in Section 2.7. Then there exist open sets $U \subset M$ and $U' \subset M'$ and a symplectomorphism $\phi: U \rightarrow U'$ such that*

- $F(U)$ is an open neighborhood of $\partial^-(F(M))$ and $F'(U')$ is an open neighborhood of $\partial^-(F'(M'))$, as subsets of $F(M)$ and $F'(M')$ respectively,
- $g \circ F(U)$ and $g' \circ F'(U')$ do not intersect the cuts or marked points in $\Delta = g \circ F(M)$ and $\Delta' = g' \circ F'(M')$,
- ϕ is equivariant with respect to the \mathbb{T}^2 -action induced by the toric momentum maps $g \circ F|_U$ and $g' \circ F'|_{U'}$.

Proof. By the description of semitoric blowups of polygons in Section 2.9.2, notice that Δ and Δ' are equal as sets in a neighborhood of their common bottom boundary. Let $\tilde{V} \subset \mathbb{R}^2$ be a convex open neighborhood of the bottom boundary (for instance, \tilde{V} could be an ε -neighborhood of the bottom boundary in \mathbb{R}^2 for sufficiently small $\varepsilon > 0$), sufficiently small such that $\tilde{V} \cap \Delta = \tilde{V} \cap \Delta'$. Since we have chosen representatives where all cuts are upwards by taking \tilde{V} small enough we may assume that V does not intersect any cuts or marked points in Δ or Δ' . Let $U = (g \circ F)^{-1}(V)$ and let $U' = (g' \circ F')^{-1}(V)$. Now $(U, \omega|_U, (g \circ F)|_U)$ and $(U', \omega'|_{U'}, (g' \circ F')|_{U'})$ are open toric integrable systems, since $(g \circ F)|_U$ and $(g' \circ F')|_{U'}$ each induce a \mathbb{T}^2 -action. By Karshon & Lerman [KL15, Proposition] open toric integrable systems are classified by their momentum map image if the momentum map is proper onto a convex open set. Since $(g \circ F)|_U$ and $(g' \circ F')|_{U'}$ s are proper onto \tilde{V} they are isomorphic as open toric systems, which means that there exists a \mathbb{T}^2 -equivariant symplectomorphism

ϕ between them, as in the statement of the lemma. The other two points of the theorem are automatic from the choice of V and the construction of U and U' . \square

The next lemma is exactly what we will need in the proof of Theorem 1.6.

Lemma 5.7. *Let $(M_{\min}, \omega_{\min}, J_{\min})$ be a minimal \mathbb{S}^1 -space which is a ruled surface (as in Theorem 2.9), and let (M, ω, J) be an \mathbb{S}^1 -space obtained by taking a sequence of $k \in \mathbb{Z}_{>0}$ blowups of $(M_{\min}, \omega_{\min}, J_{\min})$ on fixed surfaces (i.e. those of type (B1)). Then there exists $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, (J, H))$ is hypersemitoric with exactly k focus-focus points and no degenerate points.*

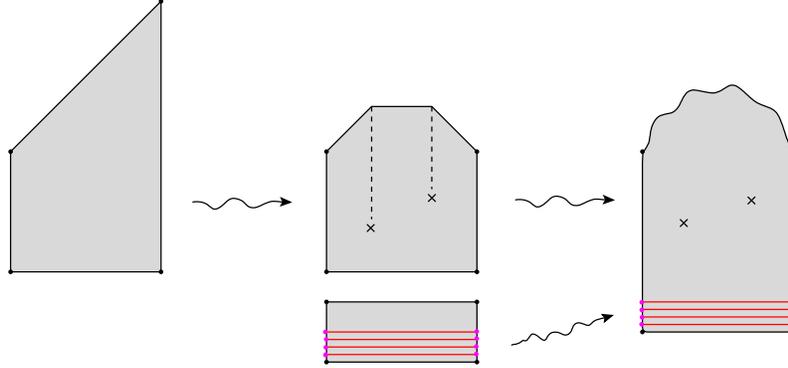


FIGURE 18. The strategy to obtain a hypersemitoric system with the desired underlying \mathbb{S}^1 -space in the proof of Lemma 5.7. We start with a Hirzebruch surface, perform the desired semitoric blowups on the Hirzebruch surface, and then glue to a system on $\mathbb{S}^2 \times \Sigma_g$ to obtain the desired genus of the fixed surfaces.

Proof. Let Γ denote the Karshon graph of (M, ω, J) . We will construct a hypersemitoric system whose underlying \mathbb{S}^1 -space also has Karshon graph Γ , and the result will follow. The Karshon graph for $(M_{\min}, \omega_{\min}, J_{\min})$ is two fat vertices a distance of $s > 0$ apart (we assume that they have J -values 0 and s), with genus $g > 0$ (they both have the same genus), and area labels $a > 0$ for the vertex at $J = 0$ and $a + ns > 0$ for the vertex at $J = s$, for some $n \in \mathbb{Z}$. On the other hand, for some small $\varepsilon > 0$ consider the n^{th} Hirzebruch surface with Delzant polygon given by the convex hull of $(0, 0)$, $(s, 0)$, $(s, a - \varepsilon + ns)$, $(0, a - \varepsilon)$. The underlying \mathbb{S}^1 -space of this toric system has Karshon graph given by two vertices at J -values 0 and s , which have area labels $a - \varepsilon$ and $a - \varepsilon + ns$, and are both labeled with genus 0.

By assumption, (M, ω, J) can be obtained from $(M_{\min}, \omega_{\min}, J_{\min})$ by a sequence of blowups of type (B1) of sizes $\lambda_1^{\text{left}}, \dots, \lambda_{k_{\text{left}}}^{\text{left}}$ on the left fixed surface and $\lambda_1^{\text{right}}, \dots, \lambda_{k_{\text{right}}}^{\text{right}}$ on the right fixed surface. Notice this implies that $\sum_{i=1}^{k_{\text{left}}} \lambda_i^{\text{left}} - a > 0$ and $\sum_{i=1}^{k_{\text{right}}} \lambda_i^{\text{right}} - (a + ns) > 0$, we assume that $\varepsilon > 0$ is smaller than both of these values. Now, perform semitoric blowups of sizes $\lambda_1^{\text{left}}, \dots, \lambda_{k_{\text{left}}}^{\text{left}}$ on the left fixed surface of the Hirzebruch surface and sizes $\lambda_1^{\text{right}}, \dots, \lambda_{k_{\text{right}}}^{\text{right}}$ on the right fixed surface of the Hirzebruch surface. This yields a semitoric system (M', ω', F') whose Karshon graph Γ' is exactly the same as Γ except for the labels on the fat vertices: the area labels on Γ' are both too small by ε , and both have genus 0 instead of the desired genus g . We will change this system by gluing another system to the

bottom to obtain the desired Karshon graph. Notice that by Lemma 5.6 this system is still toric in a neighborhood of the bottom boundary.

By Proposition 5.1, there exists a hypersemitoric system on $(M_\varepsilon, \omega_\varepsilon, F_\varepsilon)$ with $M_\varepsilon = \mathbb{S}^2 \times \Sigma_g$, whose Karshon graph consists of two fat vertices at J -values 0 and s , which both are labeled with area ε and genus g . Moreover, as in Remark 5.3 we may assume that the system is toric in the preimage under F_ε of a neighborhood of the top boundary of $F(M)$. To complete the proof, we will glue this system to the bottom of the system described above, as in Figure 18. This is essentially the simplest case of symplectic gluing, we will describe it briefly here, but the full details, and the general situation, can be found in [Gom95].

Since (M', ω', F') is toric on the preimage $U' \subset M'$ of a neighborhood of the bottom boundary of $F'(M')$, whose momentum map image we may assume is $F'(U') = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < s, 0 < y < R\}$, we know that we may take $U' = \mathbb{S}^2 \times \mathbb{D}_R^2$ with $F' = (z, \rho)$ where z is the height coordinate on \mathbb{S}^2 (appropriately scaled) and ρ is the radial coordinate on the disk of radius R , \mathbb{D}_R^2 . Similarly, the local model for the preimage of the top boundary of $F_\varepsilon(M_\varepsilon)$ is $U_\varepsilon = \mathbb{S}^2 \times \mathbb{D}_r^2$ with $F_\varepsilon = (z_\varepsilon, -\rho_\varepsilon)$ where z_ε is the height coordinate on \mathbb{S}^2 (appropriately scaled) and ρ_ε is the radial coordinate on \mathbb{D}_r . These two regions, after removing the center of each disk, may be embedded into a region modeled by $(\mathbb{S}^1 \times]-r, R[) \times \mathbb{S}^2$ with momentum map $F = (z, \rho)$ where z is as before and $\rho \in]-r, R[$. This can be used to glue the regions together, and therefore glue together the integrable systems (M', ω', F') and $(M_\varepsilon, \omega_\varepsilon, F_\varepsilon)$, to obtain a new integrable system which is also of hypersemitoric type. This process does not introduce any new \mathbb{Z}_k -spheres, and the fixed surfaces in the new system are the connected sum of the fixed surfaces of the original two systems, and thus they are surfaces of genus g with areas a and $a + ns$. Therefore, this system has the desired Karshon graph. \square

5.4. The size of flaps from Hamiltonian-Hopf bifurcations. There is one more result needed to show that the algorithm from Section 5.2 works in full generality, which we establish in this section. We will show that the flaps produced by Hamiltonian-Hopf bifurcations can be made large enough to contain the required blowups in the last step of the algorithm. We will do this by first producing small flaps using the Pelayo-Dullin technique, and then proving that we can enlarge each small flap to contain the desired set. On the reduced space, the flap will correspond to the connected region below (or above) the level set of a Morse function at the level of an index 1 singular point, see Section 2.12 and in particular Figure 10.

We say that a vertex of a Karshon graph is *isolated* if is a non-fat vertex which is not connected to any edges, which means the weights of the fixed point it corresponds to both have absolute value 1. Suppose that (M, ω, J) is an \mathbb{S}^1 -space, let Γ be the associated Karshon graph, and let v_0 be an isolated vertex in Γ which is not at the maximum or minimum value of J . Let $\Gamma_0 := \Gamma$ and suppose that Γ_1 is a Karshon graph obtained from Γ_0 by performing a blowup at v_0 . Similarly, let $\Gamma_2, \dots, \Gamma_k$ be Karshon graphs such that for each $1 \leq \ell \leq k$, Γ_ℓ is obtained from $\Gamma_{\ell-1}$ by a blowup, and further suppose that at each step the blowup is performed on one of the new vertices created by this process, so all blowups will effect the same connected component of the graph. Therefore, Γ_k is equal to Γ_0 except that the vertex v_0 is replaced by a more complicated component. In such a situation we say that Γ_k is obtained from Γ by a *sequence of blowups at v_0* . For instance, vertices associated to focus-focus points are always isolated.

For $\ell \in \{0, \dots, k-1\}$ let $(M_\ell, \omega_\ell, J_\ell)$ be the \mathbb{S}^1 -space associated to Γ_ℓ . Performing a blowup on Γ_ℓ corresponds to an \mathbb{S}^1 -equivariant blowup on (M_ℓ, ω_ℓ) which corresponds to

removing an \mathbb{S}^1 -equivariantly embedded ball from (M_ℓ, ω_ℓ) and collapsing the boundary. Let $S_\ell \subset M_\ell$ be the image of this ball and let $\pi_\ell: M_\ell \rightarrow M_{\ell-1}$ be the blowup map. Let S be the union of the images of these balls projected to M . That is

$$S = S_0 \cup \left(\bigcup_{\ell=1}^k (\pi_1 \circ \dots \circ \pi_\ell)(S_\ell) \right).$$

We say that $S \subset M$ is the *set associated to the sequence of blowups used to obtain Γ_k from Γ* . We call such a set a *blowup set at p* . The following Proposition depends on two technical lemmas, Lemma A.1 and Lemma A.2, which we prove in Appendix A. Let $\mathbb{D}_r^2 = \{z \in \mathbb{C} \mid |z|^2 < r^2\}$ and let $\mathbb{D}^2 = \mathbb{D}_1^2$.

The aim of the following proposition is to turn blowups happening around a focus-focus point into blowups happening around an elliptic-elliptic point by transforming the original system into a system which includes a flap. The main difficulty is making sure that the flap is large enough to include the entire set on which the blowups occur. Item (3) of Proposition 5.8 roughly states that $\tilde{\pi}(S)$ lies entirely in the new flap created when transforming F into \tilde{F} .

Proposition 5.8. *Suppose that $(M, \omega, F = (J, H))$ is a compact integrable system such that J generates an effective \mathbb{S}^1 -action and let Γ be the Karshon graph of the \mathbb{S}^1 -space (M, ω, J) . Let $p \in M$ be a focus-focus singular point of (M, ω, F) and let v_0 be the corresponding isolated vertex in Γ . Suppose that a Karshon graph Γ_k can be obtained from Γ by a sequence of $k > 0$ blowups at v_0 , as described above. Let $S \subset M$ be the set associated to this sequence of blowups. Then for any open neighborhood U of \bar{S} there exists an integrable system $(M, \omega, \tilde{F} = (J, \tilde{H}))$ satisfying*

- (1) \tilde{H} and H are equal outside of U ;
- (2) p is an elliptic-elliptic critical point of (M, ω, \tilde{F}) ;
- (3) let $\tilde{\pi}: M \rightarrow B$ be the singular Lagrangian fibration induced by \tilde{F} (see Section 2.6). Then $\tilde{\pi}(S)$ is in a single component of $B \setminus \overline{\tilde{\pi}(M^{\text{HR}})}$ where $M^{\text{HR}} \subset M$ is the set of hyperbolic-regular points of M , and moreover all points in the preimage under $\tilde{\pi}$ of this component are of either regular-regular or elliptic-regular type, with the exception of p .

Furthermore, if (M, ω, F) was of hypersemitoric type, then (M, ω, \tilde{F}) is also of hypersemitoric type.

Proof. Let $p \in M$ be the focus-focus point in question, and let $j_0 = J(p)$. Suppose that the blowups used to produce the set S have sizes $\lambda_1, \dots, \lambda_k$. Then there exists an $\varepsilon > 0$ such that the blowup set for a sequence of blowups of sizes $\lambda_1 + \varepsilon, \dots, \lambda_k + \varepsilon$ is contained in the set U . Without loss of generality we assume that U is the interior of this blowup set, and note that this means that U is closed under the \mathbb{S}^1 -action.

The function $H: M \rightarrow \mathbb{R}$ induces a function $\hat{H}_j: \hat{M}_j \rightarrow \mathbb{R}$ on each reduced space $\hat{M}_j := (M//\mathbb{S}^1)_j$ satisfying $\hat{H}_j \circ \pi_j = H$ (see Lemma A.1). Let $\hat{U}_j = (J^{-1}(j) \cap U)/\mathbb{S}^1 \subset \hat{M}_j$. Since U is the interior of a blowup set, there are no \mathbb{Z}_k -spheres which intersect U and there are no fixed points of the \mathbb{S}^1 -action in U , with the exception of the focus-focus point p . Thus, for $j \neq j_0$ we have that $\hat{U}_j \subset \text{smooth}(\hat{M}_j)$ and the functions $\hat{H}_j|_{\hat{U}_j}$ are Morse. Let $[p] \in \hat{M}_{j_0}$ be the image of the focus-focus point p . Then $[p] \notin \text{smooth}(\hat{M}_j)$, $\hat{U}_{j_0} \setminus \{[p]\} \subset \text{smooth}(\hat{M}_{j_0})$,

and $\hat{H}_{j_0}|_{\hat{U}_{j_0} \setminus \{[p]\}}$ is Morse. Next we will show that we may assume that they have no singular points restricted to \hat{U}_j , for $j \neq j_0$, or $\hat{U}_{j_0} \setminus \{[p]\}$, for $j = j_0$.

Step 1: *We may assume the Morse functions have no critical points.* Due to the fact that U is the interior of a blowup set, each \hat{U}_j is homeomorphic to an open disk by a homeomorphism which is furthermore a diffeomorphism for all points except $[p] \in \hat{U}_{j_0}$ (we verify this in Lemma A.2). So we may view \hat{H} as a one parameter family of Morse functions on the disk, except for at the point $[p]$. If U intersects any elliptic-regular or hyperbolic-regular points of F then for some values of j there will be a non-degenerate critical point of \hat{H}_j inside \hat{U}_j , but it will be away from a neighborhood of $[p]$. Thus, we can change \hat{H} to move these critical points into $\hat{U}_j \setminus \hat{S}_j$ without changing anything in a neighborhood $[p]$. For instance, we can compose \hat{H} with a smooth automorphism of \hat{U}_j which is the identity near $[p]$ and near the boundary, and which moves all critical points to be close to the boundary. Call the new function \hat{H}' . Now there exists a set V such that $S \subset V \subset U$ which is also the interior of a blowup set such that for each j the function $\hat{H}'_j|_{\hat{V}_j}$ is a Morse function with zero critical points for all $j \in J(V) \setminus \{j_0\}$, and $\hat{H}'_j|_{\hat{V}_{j_0} \setminus \{[p]\}}$ is also a Morse function with no critical points. As in Lemma A.1, $\{\hat{H}'_j\}_j$ extends to an integrable system $F' = (J, H')$ which still has a focus-focus point at p since H and H' are equal in a neighborhood of p .

Step 2: *Creating a flap with the Dullin-Pelayo technique.* Next we apply the technique of [DP16] to the focus-focus point p in the integrable system F' and obtain a new integrable system $F'' = (J, H'')$ in which p is an elliptic-elliptic point. The point $[p]$ is either a local max or local min of F'' on the reduced space, assume it is a local min. Furthermore, we may assume that F'' is equal to F' , and hence to F , outside of the set V . This new integrable system satisfies all of the desired properties from the statement of the theorem except that the new flap produced around the image of p is not necessarily large enough to contain all of S .

Step 3: *A family of Morse functions on the disk.* As with the set U , the image of the blowup set V in each reduced space is diffeomorphic to a disk (except for one point, at which it is still homeomorphic to a disk but not diffeomorphic, this is at the focus-focus point), so we can parameterize $\hat{U} \subset \hat{M}$ as a disk times an interval, where \hat{S} is a smaller (sub-)disk times a smaller (sub-)interval. This is described more concretely (and proven) in Lemma A.2. Thus, there exists a continuous map $\phi: \mathbb{D}^2 \times]a, b[\rightarrow \hat{U}$ such that

- $\phi_j := \phi(\cdot, j)$ is a diffeomorphism for all $j \neq j_0$;
- ϕ_{j_0} is a homeomorphism which is a diffeomorphism when restricted to $\mathbb{D}^2 \setminus \{0\}$.

Furthermore, by Lemma A.2, we may assume that for $j \in]\tilde{a}, \tilde{b}[\subset]a, b[$ we have

$$\phi_j(\hat{S}_j) = \mathbb{D}_{r_j}^2 \subset \mathbb{D}^2$$

where $0 < r_j < 1$, the map $j \mapsto r_j$ is continuous, and

$$\lim_{j \rightarrow \tilde{a}^+} r_j = \lim_{j \rightarrow \tilde{b}^-} r_j = 0.$$

Step 4: *Enlarging the flap.* Let \hat{H}'' be defined by $\hat{H}'' \circ \pi = H''$ and let

$$f: \mathbb{D}^2 \times]a, b[\rightarrow \mathbb{R}$$

be given by $f = \hat{H}'' \circ \phi$, and let $f_j := f(\cdot, j)$.

Since we have produced a flap around $[p]$ using the Dullin-Pelayo technique, the function $z \mapsto f_j(z)$ is Morse with two critical points (of index 1 and 0) for all j in a neighborhood of j_0 , not including $j = j_0$, and $z \mapsto f_j(z)$ is Morse with no critical points outside of the closure of that neighborhood. This is the behavior of flaps on the reduced space, as discussed in Section 2.12. The level set containing the index 1 point is in the shape of a curve with one self-intersection point, forming a loop which encloses a teardrop shaped region (see Figure 10). Now we want to edit $j \mapsto f_j$ so that the function has precisely two critical points for a larger interval of j -values, including interval $j \in]\tilde{a}, \tilde{b}[$, therefore producing a larger flap in the integrable system. We can essentially achieve this by reparameterizing the j parameter of f_j , except that we have to be careful not to change any of the f_j in a neighborhood of the boundary of the disk. Let $\chi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ be a smooth function which is 0 in the neighborhood of the boundary of the disk, but is identically 1 in a neighborhood of the origin of the disk large enough to contain the teardrop shaped region for all f_j . Now let

$$\tilde{f}_j = (1 - \chi)f_j + \chi f_{\Psi(j)},$$

where $\Psi:]a, b[\rightarrow]a, b[$ is a bijection such that the j -values for which \tilde{f}_j has two critical points has been extended to contain the interval $]\tilde{a}, \tilde{b}[$. We denote by $]\alpha, \beta[$ the interval of j -values for which \tilde{f}_j has two critical points. Notice that $\tilde{f}_j = f_j$ near the boundary of the disk, and we may assume that Ψ is the identity in a neighborhood of j_0 so that $\tilde{f} = f$ in a neighborhood of $(0, j_0) \subset \mathbb{D}^2 \times]a, b[$ as well. Then f is of the form described in Section 2.12, that is, it satisfies:

- \tilde{f}_j is a Morse function with no critical points when $a < j < \alpha$ or $\beta < j < b$;
- \tilde{f}_j is a Morse function with one critical point of index 1 and one critical point of index 0 when $\alpha < j < \beta$ with $j \neq j_0$;
- \tilde{f}_α and f_β are smooth functions with exactly one critical point;
- $\tilde{f}_{j_0}|_{\mathbb{D}^2 \setminus \{0\}}$ is Morse with one critical point of index 1 and $0 \in \mathbb{D}$ is a local minimum of $z \mapsto \tilde{f}_{j_0}(z)$;
- there is a neighborhood \mathcal{O} of the boundary of \mathbb{D} such that $\tilde{f}_j|_{\mathcal{O}}$ is Morse with zero critical points for all $j \in]a, b[$.

We may assume that the origin is contained in the teardrop shaped region mentioned above and drawn in Figure 10. For each $j \in]a, b[$ let ψ_j be a smooth automorphism of \mathbb{D}^2 shrinking $\mathbb{D}_{(2r_j+1)/3}^2$ and acting as the identity outside of $\mathbb{D}_{(r_j+2)/3}^2$. Then $\tilde{f}_j \circ \psi_t$ has the set \mathbb{D}_{r_j} contained in the teardrop shaped region enclosed by the level set of \tilde{f}_j containing the index 1 point, as desired. Finally, as in the work of Cerf [Cer70], we may perturb $\tilde{f}_j \circ \psi_j$ to make the degenerate points of the form $(x, y) \mapsto x^3 + jx + y^2$ for some local coordinates (x, y) on the disk. Let

$$g(z, j) = \tilde{f}(\psi_t(z), j): \mathbb{D}^2 \times]a, b[\rightarrow \mathbb{R}.$$

Let $g_j := g(\cdot, j)$.

Then g has the following properties:

- $g = \tilde{f} = f$ in a neighborhood of $(0, j_0) \in \mathbb{D}^2 \times]a, b[$;
- $g = \tilde{f} = f$ in a neighborhood of the boundary $(\mathbb{S}^1 \times]a, b]) \cup (\mathbb{D}^2 \times \{a, b\})$;
- g_j is a Morse function with no critical points for j satisfying $a < j < \alpha$ or $\beta < j < b$;

- g_j is a Morse function with one critical point of index 1 and one critical point of index 0 for j satisfying $\alpha < j < \beta$ with $j \neq j_0$;
- all degenerate points can be locally modeled by $(x, y) \mapsto x^3 \pm (j - \gamma)x + y^2$,
- for each j satisfying $\tilde{a} < j < \tilde{b}$ the function g_j has exactly one critical point p_1 of index 1, and $g_j^{-1}(p_1) \cap \mathbb{D}_{r_j}^2 = \emptyset$.

Step 5: Completing the proof. Let $g: \mathbb{D}^2 \times]a, b[\rightarrow \mathbb{R}$ be as described in the previous step, and define $\hat{H}''' = g \circ \phi^{-1}$, where $\phi: \mathbb{D}^2 \times]a, b[\rightarrow \hat{U}$ is as defined in Step 3. Notice that \hat{H}'' and \hat{H}''' are equal in a neighborhood of $[p] \in \hat{M}$, so in particular \hat{H}'' and \hat{H}''' only differ at points of the reduced spaces which correspond to free orbits of \mathbb{S}^1 , and therefore by Lemma A.1 we can extend \hat{H}''' to an integrable system $(J, \tilde{H} = \hat{H}''' \circ \pi)$ on M . Since the only degenerate points of \hat{H}'''_j for any j were of the form $x^3 + tx \pm y^2$, the degenerate points of (J, \tilde{H}) are parabolic, and by Lemma 2.24 all of the non-degenerate singular points of (J, \tilde{H}) are of the desired type from the statement of the proposition. Finally, since the set \hat{S}_j lies below the level of the index 1 critical point of \hat{H}'''_j (i.e. in the teardrop shaped region) for all relevant j , item (3) of the statement of the proposition is also satisfied. \square

Remark 5.9. Notice that Proposition 5.8 can be applied to several focus-focus points simultaneously. This is because the change in the momentum map is only in a small neighborhood of the embedded ball related to the blowup, and if the given \mathbb{S}^1 -space admits blowups at several different fixed points then the related embedded balls are disjoint, and furthermore each such embedded ball admits a neighborhood such that all of these neighborhoods are also disjoint.

5.5. **Proof of Theorem 1.6.** Now we are prepared to prove Theorem 1.6.

Proof of Theorem 1.6. Let (M, ω, J) be any \mathbb{S}^1 -space and let Γ be the associated Karshon graph. If we can find any hypersemitoric system which induces the same Karshon graph Γ then the proof is complete, since in that case there would exist an \mathbb{S}^1 -equivariant symplectomorphism Φ from the (M, ω, J) to the hypersemitoric system [Kar99] and the structure of the hypersemitoric system (i.e. the new integral H) can be pulled back by Φ . First of all, notice that if Γ has no fat vertex then all fixed points are isolated and thus by Lemma 2.7 the \mathbb{S}^1 -space (M, ω, J) can be extended to a toric integrable system, which is in particular hypersemitoric, and so we are done. Now assume that Γ has at least one fat vertex. From Karshon's result Theorem 2.9 we know Γ can be obtained from a minimal Karshon graph, Γ_{\min} , by a finite sequence of \mathbb{S}^1 -equivariant blowups and by Lemma 5.4 we know that these blowups can be performed in three stages. By Karshon's classification of minimal \mathbb{S}^1 -spaces (Theorem 2.9), Γ_{\min} can either be extended to a toric system (on $\mathbb{C}\mathbb{P}^2$ or a Hirzebruch surface) or corresponds to a ruled surface.

Case 1: Γ_{\min} can be extended to a toric system. If Γ_{\min} is not a ruled surface then it can be extended to a toric system $(M_{\min}, \omega_{\min}, F_{\min})$. We will perform the required blowups to this toric system in stages, as described in Lemma 5.4, to obtain a hypersemitoric system with the required Karshon graph Γ . We will first perform blowups on the components of Γ which are connected to the vertices with maximal and minimal J -value. Then, making use of the fixed surface, we will perform blowups which produce one isolated fixed point (which will correspond to a focus-focus point in the associated integrable system) for each

remaining component of the graph Γ . Finally, we perform a sequence of blowups on each of these isolated points to obtain the remaining components of Γ .

Stage 1: *The connected components of the extreme vertices.* First we perform blowups on the graph Γ_{\min} to obtain a new graph Γ' so that the connected components of the extreme vertices of Γ' are equal to those of Γ , with the possible exception of the normalized area labels on the fat vertices. This produces at most two non-trivial chains of \mathbb{Z}_k -spheres and thus, by Lemma 2.6, the graph Γ' is the Karshon graph of an \mathbb{S}^1 -space which can be extended to a toric system (M', ω', F') . This toric system has at least one fixed surface since the graph Γ has at least one fat vertex.

Stage 2: *Producing the focus-focus points.* Now $\Gamma' \subset \Gamma$ and let $k \in \mathbb{Z}$ be the number of connected components of $\Gamma \setminus \Gamma'$. We will call each connected component of $\Gamma \setminus \Gamma'$ an *island*. We want to perform blowups on the fixed surface(s) of (M', ω', F') to produce one focus-focus point corresponding to each island, and then in the next stage we will perform a sequence of blowups on those focus-focus points to produce the desired island, but first we have to work backwards to determine the J -value that these focus-focus points should have. Each island in Γ admits a blowdown since the minimal models do not have islands. For each island, perform as many blowdowns as possible until it is an isolated vertex and let Γ'' be the graph obtained in this way. So Γ'' is obtained from Γ by performing a sequence of blowdowns.

On the other hand, Γ'' can be obtained from Γ' by performing blowups on the fixed surface(s) of the \mathbb{S}^1 -space (M', ω', J') , as in Case (B1) from Section 2.4.3. Indeed, if there is only one fixed surface $\Sigma = J^{-1}(j_{\min})$ (respectively $\Sigma = J^{-1}(j_{\max})$) then suppose that the isolated points in Γ'' have J -values $j_{\min} + \lambda_\ell$ (respectively $j_{\max} - \lambda_\ell$) for $\ell = 1, \dots, k$. Then Γ'' is obtained from Γ' by performing k blowups on points in the fixed surface Σ of sizes $\lambda_1, \dots, \lambda_k$. If (M', ω', J') has two fixed surfaces $\Sigma_{\min} = J^{-1}(j_{\min})$ and $\Sigma_{\max} = J^{-1}(j_{\max})$ then there is a choice m such that blowups of size $\lambda_1, \dots, \lambda_m$ on Σ_{\min} and blowups of size $\lambda_{m+1}, \dots, \lambda_k$ on Σ_{\max} produce Γ'' from Γ' , where the J -values of the isolated points in Γ'' are $j_{\min} + \lambda_1, \dots, j_{\min} + \lambda_m, j_{\max} - \lambda_{m+1}, \dots, j_{\max} - \lambda_k$. Such blowups are possible because Γ can be obtained from Γ_{\min} by a sequence of blowups performed in the order specified in Lemma 5.4, which can only occur if the blowups discussed in this paragraph are possible.

Since the integrable system (M', ω', F') is toric, and in particular semitoric, Lemma 2.21 implies that the blowups used to obtain Γ'' from Γ' can be realized by performing weak semitoric blowups on (M', ω', F') . Let (M'', ω'', F'') be the resulting system, which is thus semitoric and has associated Karshon graph Γ'' . Let $p_1, \dots, p_k \in M''$ be the focus-focus points of this system. Note that we may, and do, choose to perform the weak semitoric blowups in such a manner that each level set of $F'': M'' \rightarrow \mathbb{R}^2$ contains at most one focus-focus point (so the fibers of F'' which contain focus-focus points are all single-pinched tori).

Stage 3: *Constructing the islands.* Now, for $\ell = 1, \dots, k$, we will start with the focus-focus point p_ℓ , corresponding to a vertex v_ℓ of Γ'' , and construct the ℓ^{th} island of Γ . Since each island is connected in Γ , they can be obtained from the corresponding focus-focus point (i.e. isolated vertex in Γ'') by a sequence of blowups on that point (the inverses of the blowdowns from the previous stage), and on the new fixed points produced from that blowup, and so on. This is exactly the setting of Proposition 5.8, so by applying Proposition 5.8 on each focus-focus point p_1, \dots, p_k simultaneously there exists a function \hat{H}'' such that the system $(M'', \omega'', \hat{F}'' = (J'', \hat{H}''))$ is a hypersemitoric system, and notice that this new system

still has the same underlying \mathbb{S}^1 -space as (M'', ω'', F'') so the associated Karshon graph is still Γ'' . There are no issues applying Proposition 5.8 on all focus-focus points simultaneously as discussed in Remark 5.9. Now each p_ℓ is an elliptic-elliptic critical point of $(M'', \omega'', \hat{F}'')$ and by Proposition 5.8, p_ℓ is lying in a flap which is large enough so that the blowup of the desired size can be achieved by performing a toric blowup on this flap. The new elliptic-elliptic singular points formed by this blowup also admit the required blowups (again as toric blowups), and so on. The resulting integrable system after these operations is hypersemitoric and has the desired Karshon graph Γ . This completes the proof in this case.

Case 2: The minimal model is a ruled surface. In this case the minimal graph Γ_{\min} can be obtained from a ruled surface which is a sphere bundle over a surface Σ of genus g . We may assume that $g > 0$ since in the case that $g = 0$ the minimal model extends to a toric integrable system and we are back in Case 1. Let Γ_{\min} denote the Karshon graph of this minimal model. We will follow nearly the same stages as in Case 1. The minimal model has two fixed surfaces, and blowups cannot remove fixed surfaces completely, so Γ necessarily has two fixed surfaces. Thus, there are no edges which connect to the maximal or minimal vertices of Γ , so **Stage 1** is trivial in this case. That is, the Karshon graph Γ' produced in Stage 1 satisfies $\Gamma' = \Gamma_{\min}$. For **Stage 2**, as in the previous case we perform a series of blowups of type (B1) on Γ' to produce a new Karshon graph Γ'' . By Lemma 5.7, there exists a hypersemitoric system (M'', ω'', F'') which has Γ'' as its Karshon graph. Finally, **Stage 3** works exactly the same as in the previous case, and the proof is complete. \square

From the algorithm in the above proof, we automatically have the following refined version of Theorem 1.6.

Corollary 5.10. *Let (M, ω, J) be an \mathbb{S}^1 -space. Then there exists a smooth function $H: M \rightarrow \mathbb{R}^2$ such that $(M, \omega, F = (J, H))$ is a hypersemitoric system such that:*

- every degenerate orbit of (M, ω, F) lies in a cuspidal torus (Figure 15c), and every hyperbolic-regular point lies in a double torus (Figure 15a);
- there is at most one focus-focus point in each fiber of F ;
- all \mathbb{Z}_k spheres, $k \neq 0, \pm 1$, of (M, ω, J) consist entirely of elliptic-regular points of (M, ω, F) ;
- if (M, ω, J) has less than two fixed surfaces, or if it has two fixed surfaces which are both diffeomorphic to spheres, then (M, ω, F) has no singular points of hyperbolic-elliptic type.
- (M, ω, F) has no swallowtails (see Section 2.11);

5.6. **Extending to semitoric systems.** The techniques developed in this paper also provide a method to easily obtain a result originally announced by Hohloch & Sabatini & Sepe & Symington [HSSS]. Specifically, we can now reprove their result which is stated in the present paper as Theorem 1.7 part (2).

Corollary 5.11. *If (M, ω, J) is a compact \mathbb{S}^1 -space such that each non-extremal level set of J intersects at most two non-free orbits of the \mathbb{S}^1 -action which are not fixed points, and all fixed surfaces of the \mathbb{S}^1 -action are genus zero, then it can be extended to a semitoric system.*

Proof. If the fixed points of the \mathbb{S}^1 -action are all isolated then (M, ω, J) extends to a toric system by Lemma 2.7 and we are done. Otherwise, the system has a fixed surface and thus if the Karshon graph has any isolated vertices we may remove them by performing a blowdown

of type (B1), which increases the volume of the fixed surface and removes an isolated vertex of the Karshon graph. This can be repeated to remove as many isolated vertices as needed (possibly zero) so that the system has only two non-free orbits in each non-extremal level set of the periodic momentum map; call the resulting system (M', ω', J') . Thus (M', ω', J') satisfies the requirement of Lemma 2.6 to ensure that there exists a function $H': M' \rightarrow \mathbb{R}$ such that $(M', \omega', F' = (J', H'))$ is a toric system, and in particular it is semitoric. Now using Lemma 2.21 we can perform a sequence of semitoric blowups on (M', ω', F') which invert all of the blowdowns used to obtain (M', ω', J') from (M, ω, J) while preserving the fact that the system is semitoric. Thus, there exists a function $H: M \rightarrow \mathbb{R}$ such that $(M, \omega, F = (J, H))$ is semitoric, as desired. The reverse direction follows quickly from [HSS15], and thus the result is established. \square

APPENDIX A. TECHNICAL LEMMAS

The following lemmas are unsurprising, but technical to state, results which were needed in Proposition 5.8.

Lemma A.1. *Suppose that $(M, \omega, (J, H))$ is a completely integrable system such that J generates an effective \mathbb{S}^1 -action. Let $\hat{M} = M/\mathbb{S}^1$ and let $\text{smooth}(\hat{M})$ and $\text{sing}(\hat{M})$ be as above. Then there exists a function $\hat{H}: \hat{M} \rightarrow \mathbb{R}$ such that $\hat{H} \circ \pi = H$ and which is smooth when restricted to $\text{smooth}(\hat{M})$. Furthermore, let $\hat{U} \subset \hat{M}$ be an open neighborhood of $\text{sing}(\hat{M})$ and let $\hat{H}': \hat{M} \rightarrow \mathbb{R}$ be a function which is smooth on $\text{smooth}(\hat{M})$ and equal to \hat{H} on \hat{U} , and assume that $d\hat{H}'$ is non-zero almost everywhere on \hat{M} . Let $H' = \hat{H}' \circ \pi$. Then $(M, \omega, (J, H'))$ is a completely integrable system.*

Proof. The existence of \hat{H} is immediate since H is constant under the flow of J , which generates the \mathbb{S}^1 -action. Next suppose that \hat{U} and \hat{H}' are as in the statement. The conditions for the function H' to form an integrable system with J are all local, so there are no obstructions in $\pi^{-1}(\hat{U})$. The function H' is constant on the orbits of the \mathbb{S}^1 -action since they are the orbits of the flow of J , so H' and J Poisson commute. Finally, dH' and dJ are linearly independent almost everywhere in M since the fibers of π are measure zero, $dH' \wedge dJ(p) = 0$ implies that $d\hat{H}'(\pi(p)) = 0$, and $d\hat{H}'$ is non-zero almost everywhere on \hat{M} . \square

Let $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z|^2 < 1\}$ be the standard 2-disk. The following lemma essentially explains how the blowup set of a sequence of blowups descends to the reduced spaces. This is needed in the proof of Proposition 5.8 when arguing that such a set can be put onto a flap of the system.

Lemma A.2. *Let (M, ω, J) be an \mathbb{S}^1 -space, let $p \in M$ be a fixed point with weights $m, -n$ with $m, n \in \mathbb{Z}_{>0}$, and let $j_0 = J(p)$. Let $S \subset M$ be a blowup set at p (as described in Section 5.4). Let $\hat{S} = S/\mathbb{S}^1$ and $\hat{M} = M/\mathbb{S}^1$. Then $J(S) =]a, b[$ for $a, b \in \mathbb{R}$, and \hat{S} is the image of a continuous map $\rho: \mathbb{D}^2 \times]a, b[\rightarrow \hat{M}$ satisfying:*

- $\rho(z, j) \in J^{-1}(j)$ for all $z \in \mathbb{D}^2$ and $j \in]a, b[$;
- for $j \in]a, b[$ the map $\rho(\cdot, j): \mathbb{D}^2 \rightarrow \hat{M}_j$ is a homeomorphism onto its image. If $j \neq j_0$ then $\rho(\cdot, j)$ is a diffeomorphism, and the map $\rho(\cdot, j_0)|_{\mathbb{D}^2 \setminus \{0\}}$ is a diffeomorphism;
- there exist points $p_a \in \hat{M}_a$ and $p_b \in \hat{M}_b$ such that for all $z \in \mathbb{D}^2$ we have

$$\lim_{j \rightarrow a^+} \rho(j)(z) = p_a \quad \text{and} \quad \lim_{j \rightarrow b^-} \rho_j(z) = p_b.$$

Proof. Suppose that S is the blowup set of a single blowup of size r at a fixed point of the \mathbb{S}^1 -action with weights m and $-n$ where $m, n \in \mathbb{Z}_{>0}$. Then S is the image of an \mathbb{S}^1 -equivariantly embedded ball and we may choose local coordinates (z, w) on $U \subset M$ such that $S \subset U$, and the \mathbb{S}^1 action is given by $\lambda \cdot (z, w) = (\lambda^m z, \lambda^{-n} w)$ for $\lambda \in \mathbb{S}^1$ with momentum map $J(z, w) = \frac{1}{2}(m|z|^2 - n|w|^2)$. We may assume that we have taken the coordinates (z, w) to be adapted to this ball, that is, we may assume $S = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 < r^2\}$. Taking $H(z, w) = \frac{1}{2}|w|^2$ we have a toric integrable system (J, H) on U . Take some level set $J^{-1}(j)$ and we will compute that $\hat{S}_j := (J^{-1}(j) \cap S)/\mathbb{S}^1 \subset J^{-1}(j)/\mathbb{S}^1$ is a disk. Assume that $j > 0$ (the case of $j \leq 0$ is similar). Then $m|z|^2 - n|w|^2 = 2j$, and in particular $m|z|^2 - n|w|^2 > 0$, so $z \neq 0$. Using the \mathbb{S}^1 -action we may assume that $z \in \mathbb{R}_{>0}$, which yields coordinates $z \in \mathbb{R}_{>0}$ and $w \in \mathbb{C}$ on $J^{-1}(j)/\mathbb{S}^1$. These coordinates satisfy $z = \sqrt{\frac{2j+n|w|^2}{m}}$ from the fact that $J(z, w) = j$. Using the fact that $|z|^2 + |w|^2 < r^2$ we obtain that

$$0 \leq |w|^2 < \frac{r^2 - \frac{2j}{m}}{\frac{n}{m} + 1}.$$

Therefore, the image of S in the reduced space can be naturally identified with its $w \in \mathbb{C}$ coordinate, for the above bound on the magnitude of w , and therefore \hat{S}_j is a disk. Moreover, this disk will vary continuously with j , and shrinks to nothing as $j \rightarrow \frac{1}{2}mr^2$. Therefore, the lemma is proved in the case that S is the blowup set of a single blowup.

Now suppose that \tilde{S} is a blowup set obtained from taking $k > 0$ blowups and has the properties described in the lemma, and assume that S is obtained by taking the same sequence of blowups and then one more. After performing the first k blowups, the next blowup is obtained by removing an \mathbb{S}^1 -equivariantly embedded ball, whose image S_{k+1} again has the desired properties in the theorem. The set S is obtained as the union of \tilde{S} with $\tilde{S}_{k+1} := \pi(S_{k+1})$, where π is the projection M blown up $k+1$ times onto the original manifold M . For any value of j such that $\tilde{S} \cap J^{-1}(j) = \emptyset$ or $\tilde{S}_{k+1} \cap J^{-1}(j) = \emptyset$ the result clearly holds. Suppose that $J^{-1}(j)$ intersects both \tilde{S} and \tilde{S}_{k+1} . Then the $(k+1)^{\text{st}}$ blowup must have occurred at a point for which one of the weights is not 0 or 1, and therefore the \mathbb{Z}^k -spheres coming from that point must include the exceptional divisor from one of the earlier blowups, and therefore the S_{k+1} blowup must include this exceptional divisor in its interior (at least at the level $J = j$). Taking the image in the reduced space at level j , this implies that the image of \tilde{S}_{k+1} in the reduced space at level j is an annulus which is perfectly filled in by the image of \tilde{S} in the reduced space. The result now follows. \square

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