Integrable Hamiltonian systems

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Dankwoord

Ik bedank alle studenten, in het bijzonder Annelies De Meulenaere, voor hun waardevolle opmerkingen en suggesties. Dit heeft de cursus immens verbeterd.

Contents

CHAPTER 1

Integrability

'Integrability' of an equation means roughly that one is able to solve it explicitly. Since many early examples were solved by 'integrating the differential equation' such equations were (and are still) referred to as *integrable* equations.

There are various notions of integrability which fit different settings to different degrees. In this chapter, we will first briefly study *Frobenius integrability* before we shift our focus to *Liouville integrability* which suits Hamiltonian dynamics better. Frobenius integrability is mainly used in differential geometry when dealing with distributions ('plane fields'). The notion of Liouville integrability roughly describes a collection of Hamiltonian systems that are 'mutually energy conserving'.
For other types and notions of integrability, we refer the interested reader to the literature: For algebraic integrability in the sense of *Integrable systems and differential Galois theory*, see for instance [**Bolsinov** & **Morales-Ruiz** & **Zung**, Part I]. For integrable PDEs and their interaction with dynamics, see for example [**Kappeler** & **Pöschel**] and [**Kuksin**].

1.1. Hamiltonian systems in $\mathbb{R}^{2n}$

In this introductory section, we define Hamiltonian systems in local coordinates in $\mathbb{R}^{2n}$ in order to provide intuition and motivation without using too many new notions. In the following section, we will then provide the coordinate free definition of Hamiltonian systems on symplectic manifolds. Hamiltonian systems are named after the the Irish mathematician *William Rowan Hamilton* (1805 – 1865) and are of considerable interest since many physically relevant systems belong to this class.

For the definition and basic properties of a flow of an ordinary differential equation, we refer the reader to the appendix, more precisely to Definition and Proposition A.14.

Definition 1.1 (Hamiltonian system in standard coordinates). *Consider $\mathbb{R}^{2n}$ with coordinates*

$$z := (q, p) := (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$$

and let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. The vector field given by

$$X^H(z) = X^H(q, p) := \begin{pmatrix} \partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} := \begin{pmatrix} \partial_{p_1} H(q,p) \\ \vdots \\ \partial_{p_1} H(q,p) \\ -\partial_{q_1} H(q,p) \\ \vdots \\ -\partial_{q_n} H(q,p) \end{pmatrix}$$

is called **Hamiltonian vector field of** *H. The associated ordinary differential equation*

$$z' = X^H(z),$$

reads in (q, p) coordinates

$$\begin{cases} q_i' = \partial_{p_i} H(q,p), \\ p_i' = -\partial_{q_i} H(q,p), \end{cases} \quad \forall\, 1 \le i \le n, \qquad \text{briefly} \quad \begin{cases} q' = \partial_p H(q,p), \\ p' = -\partial_q H(q,p), \end{cases}$$

and is called **Hamiltonian equation** *or* **Hamiltonian system** *and its solutions* **Hamiltonian solutions**. *The associated flow is called* **Hamiltonian flow** *and usually denoted by Φ^H. In this context, H is usually called* **Hamiltonian function** *or briefly* **Hamiltonian**.

Later in Definition 1.35, we will define Hamiltonian systems in a coordinate free way on arbitrary symplectic manifolds. Easy systems like rotating around the origin are Hamiltonian:

Example 1.2. *The Hamiltonian vector field and Hamiltonian system of $H : \mathbb{R}^2 \to \mathbb{R}$, $H(q, p) := \frac{1}{2}(q^2 + p^2)$ are given by*

$$X^H(q,p) = \begin{pmatrix} p \\ -q \end{pmatrix} \qquad \text{and} \qquad \begin{cases} q' = p, \\ p' = -q \end{cases}$$

which is in fact equivalent to $q'' = -q$. The flow is given by

$$\Phi_t^H(q,p) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

i.e., the solutions form concentric circles centered at the origin, parametrised counterclockwise.

Hamiltonian systems have interesting geometric properties. Let us first consider the relation between a Hamiltonian vector field in local standard coordinates and the gradient of the Hamiltonian function induced by the Euclidean metric.

LEMMA 1.3. *The Hamiltonian vector field and the gradient vector field of the Hamiltonian function interact as follows:*

1) *The Hamiltonian vector field and the gradient of the Hamiltonian function H are perpendicular, i.e., $X^H \perp \operatorname{grad} H$.*
2) *X^H vanishes if and only if $\operatorname{grad}_{eu} H$ vanishes if and only DH vanishes. Thus a Hamiltonian solution $z(t) \equiv z_0 \in \mathbb{R}^{2n}$ is constant if and only if $0 = X^H(z_0) = \operatorname{grad}_{eu} H|_{z_0} = DH|_{z_0}$.*

PROOF. **1)** Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian with usual coordinates $(q, p) \in \mathbb{R}^{2n}$. Its derivative is given by $DH = (\partial_q H, \partial_p H)$ and its gradient $\operatorname{grad}_{eu} H$ w.r.t. the Euclidean metric $\langle \cdot, \cdot \rangle_{eu}$ defined by $\langle \operatorname{grad}_{eu} H, v \rangle = DH.v$ for all vectors $v \in \mathbb{R}^{2n}$, i.e., we find $\operatorname{grad}_{eu} H = (\partial_q H, \partial_p H)^T$. Thus we calculate

$$\langle X^H, \operatorname{grad}_{eu} H \rangle_{eu} = \left\langle \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix}, \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} \right\rangle_{eu} = 0$$

2) This follows from the definition of X^H and $\operatorname{grad}_{eu} H$ and DH in local coordinates and existence and uniqueness of solutions of ODEs. □

Constant Hamiltonian solutions are often called **singular** or **stationary** and nonconstant ones **regular**.

Due to the identity

$$X^H = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathrm{Id} \\ -\mathrm{Id} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathrm{Id} \\ -\mathrm{Id} & \mathbf{0} \end{pmatrix} \operatorname{grad}_{eu} H$$

where Id is the $(n \times n)$-identity matrix and $\mathbf{0}$ the $(n \times n)$ zero matrix the Hamiltonian vector field is sometimes regarded as **skewgradient**.

DEFINITION 1.4. *Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be smooth and $r \in \mathbb{R}^l$.*

1) *$f^{-1}(r) := \{x \in \mathbb{R}^k \mid f(x) = r\}$ is called* **level set of** *$r \in \mathbb{R}^l$ or* **fiber over** *$r \in \mathbb{R}^l$.*
2) *$x \in \mathbb{R}^k$ is* **regular** *if $\operatorname{rk} Df|_x = l$. The set of regular points is denoted by $\mathbb{R}^k_{reg}$.*
3) *$x \in \mathbb{R}^k$ is called a* **singular point** *or* **critical point of rank** *s if $\operatorname{rk} Df|_x = s < l$. The set of critical points of f is denoted by $\operatorname{Crit}(f)$ or $\mathbb{R}^k_{sing}$.*
4) *The set $f(\operatorname{Crit}(f))$ is called* **bifurcation diagram** *of f.*
5) *$r \in \mathbb{R}^l$ is a* **regular value of** *f and $f^{-1}(r)$ a* **regular fiber** *if $\operatorname{rk} Df|_x = l$ for all $x \in f^{-1}(r)$. The set of regular values is denoted by $\mathbb{R}^l_{reg}$. Otherwise $r \in \mathbb{R}^l$ is a* **singular value** *or a* **critical value**

and $f^{-1}(r)$ is a **singular** *or* **critical fiber**. *The set of singular or critical values is denoted by $\mathbb{R}^l_{sing}$.*

6) *We set $\mathbb{R}^l_{reg,\emptyset} := \mathbb{R}^l \setminus \mathbb{R}^l_{sing} = \mathbb{R}^l_{reg} \cup (\mathbb{R}^l \setminus f(\mathbb{R}^k))$.*

Concerning the dimension of level sets, we find

Lemma 1.5. *Let $f : \mathbb{R}^k \to \mathbb{R}^l$ with $k \geq l$ and $r \in \mathbb{R}^l$ a regular value. Then*

$$\dim\left(f^{-1}(r)\right) = k - l.$$

If r is a singular value then $\dim\left(f^{-1}(r)\right) \leq k - l$ whenever defined.

Proof. This follows from Theorem A.26 (Implicite functions). □

Note that, according to Theorem A.29 (Sard), being a regular value is the typical case, i.e., a so-called 'generic property'.

The relation between gradients, Hamiltonian vector fields, and level sets is described by the following statement.

Lemma 1.6.

1) *Let $F : \mathbb{R}^k \to \mathbb{R}$ be smooth. Then $\operatorname{grad}_{eu} F$ is perpendicular to the level sets of F and the solutions of the* **gradient system**

$$x' = \operatorname{grad}_{eu} F|_x$$

cross the level sets of F perpendicularly.

2) *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth. Then X^H is tangent to the level sets of H and a solution of $z' = X^H(z)$ stays within one and the same level set for all times.*

Proof. **1)** Let $r \in \mathbb{R}$ with $\emptyset \neq f^{-1}(r)$ and let $c :]-\varepsilon, \varepsilon[\to f^{-1}(r)$ be a smooth curve with $c' \neq 0$ unless the connected component of $f^{-1}(r)$ containing $c(]-\varepsilon, \varepsilon[)$ consists only of one point (which then must be a critical point having vanishing gradient). The concatenation $f \circ c \equiv r$ is constant so that

$$0 = (f \circ c)' = Df|_c.c' = \langle \operatorname{grad}_{eu} f, c' \rangle_{eu}. \tag{1.7}$$

Since c lies in $f^{-1}(r)$, its tangent vector c' is tangent to $f^{-1}(r)$. According to equation (1.7), $\operatorname{grad} f$ and c' are perpendicular. $\operatorname{grad}_{eu} f$ does not vanish if and only if Df does not vanish, i.e., the gradient is nonzero along regular

level sets and stands perpendicular on the level set, i.e., the gradient solutions are regular and cross the level set perpendicularly. If $\mathrm{grad}_{eu} f$ vanishes, the associated solution is constant.
2) Consider a solution z of the Hamiltonian equation $z' = X^H(z)$. Differentiating the concatenation $H \circ z$ yields

$$(H \circ z)' = DH|_z.z' = DH|_z.X^H(z) = \langle \mathrm{grad}_{eu} H|_z, X^H(z)\rangle_{eu} = 0.$$

Therefore the function $H \circ z$ is constant, i.e., z stays in the same level set of H for all times. Since the tangent vector of z is given by $z' = X^H(z)$ the Hamiltonian vector field X^H lies in the tangent space of the level set. □

Example 1.8. *According to Example 1.2, the Hamiltonian solutions of $H : \mathbb{R}^2 \to \mathbb{R}$, $H(q, p) := \frac{1}{2}(q^2 + p^2)$ are concentric circles around the origin. The solutions of the gradient system $z' = \mathrm{grad}_{eu} H|_z$ satisfy $q' = q$ and $p' = p$ and are thus rays emanating form the origin. Thus the Hamiltonian solutions and gradient solutions cross each other perpendicularly.*

Given a Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$, Lemma 1.6 means in particular that a given Hamiltonian solution does not roam unconstraint through $\mathbb{R}^{2n}$ but is confined to a level set of H, i.e., a subset of dimension $\leq 2n-1$. Hamiltonian solutions are therefore subject to 'geometric' restrictions purely by *being a Hamiltonian solution*. In terms of physics, staying in one and the same level set can be interpreted as follows:

Corollary 1.9 (**Energy conservation**). *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth and consider H as 'its own energy function', i.e., given $r \in \mathbb{R}$, consider $H^{-1}(r)$ as* **set of energy level** *r. Then Hamiltonian solutions are* **energy conserving** *since they stay within one and the same level set.*

Lemma 1.6 implies that the regular solutions of a 2-dimensional Hamiltonian system associated to a smooth $H : \mathbb{R}^2 \to \mathbb{R}$ are up to parametrization completely determined by the connected components of the regular level sets: The dimension of the trajectory of a nonconstant solution and of a regular level set both are one. Therefore, given the graph of a Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$, we can deduce the 'location' of all regular solutions up to parametrization. Singular level sets may contain singular and regular points. Here, singular point satisfy $0 = DH = X^H$, i.e., this point corresponds to a constant solution. Such constant solutions are linked by so-called *homoclinic* or *heteroclinic* solutions (zie [**Hohloch2**]).

Question 1.10. *Can we get even more control over the whereabouts of a Hamiltonian solution beyond the fact that it is staying within a certain energy level set of its associated Hamiltonian function? If yes, are there necessary and/or sufficient conditions?*

Since Hamiltonian solutions preserve level sets, a reasonable idea is to look for other Hamiltonian functions of which the level sets are 'compatible' in the following sense: given $H : \mathbb{R}^{2n} \to \mathbb{R}$, set $H =: h_1$ and look for a $h_2 : \mathbb{R}^{2n} \to \mathbb{R}$ such that the solutions of h_1 also stay within the level sets of h_2 and vice versa. If h_1 and h_2 satisfy this property then the solutions of both systems live within the intersection of level sets $h_1^{-1}(r_1) \cap h_2^{-1}(r_2)$ for some values $r_1, r_2 \in \mathbb{R}$. Since according to Lemma 1.5,

$$\dim(h_1^{-1}(r_1) \cap h_2^{-1}(r_2)) = \dim(h^{-1}(r_1, r_2)) \leq 2n - 2$$

the solutions of h_1 and h_2 are confined to $(2n-2)$-dimensional sets within the $2n$-dimensional space $\mathbb{R}^{2n}$. Iterating this idea by looking for k such functions $h_1, \dots, h_k$ yields solutions staying in level sets of dimension

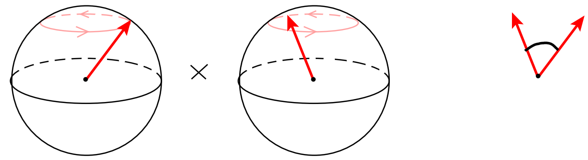
$$\dim(h_1^{-1}(r_1) \cap \dots \cap h_k^{-1}(r_k)) = \dim(h^{-1}(r_1, \dots, r_k)) \leq 2n - k.$$

The situations $k < n$ and $k = n$ and $k > n$ correspond to so-called *subintegrable* and *completely integrable* and *superintegrable* situations respectively which we will adress later in (more) detail.

Question 1.11. *Given a Hamiltonian $H =: h_1$, how do we find such functions h_2, h_3, ..., meaning, are there natural candidates for the functions h_2, h_3, ... we should look at first?*

Since the whole approach is based on the property of energy conservation, it makes sense to look for other preserved quantities of the system and try to express them by means of a Hamiltonian function. Examples of such preserved quantities are for instance preserved angles, rotational invariance, 'symmetries of the system' etc.

Example 1.12. *The* **coupled angular momenta** *system lives on $\mathbb{S}^2 \times \mathbb{S}^2$ and consists of rotation around the vertical axes on both spheres with coupled speed, see Figure 1.1. The coupled rotation preserves the angle between the rotating vectors in each of the two spheres. Expressed by means of a scalar product, this preserved angle can be seen as Hamiltonian function which has 'compatible level sets' in the sense described above. For details see Example 1.56.*



(a) (b)

FIGURE 1.1. Coupled angular momenta system: (a) Rotation with same speed on both spheres. (b) The preserved angle between two rotating vectors.

Once we found a suitable candidate like the preserved angle in Example 1.12, the essential question is:

QUESTION 1.13. *How do we verify that a geometrically or dynamically promising candidate really has the desired properties? How does the 'compatible level set property' translate into mathematical formulas?*

Given Hamiltonian functions $h_1, h_2, \ldots, h_k : \mathbb{R}^{2n} \to \mathbb{R}$ of which the solutions mutually stay in each other's level sets it is reasonable to suspect that very likely their flows $\Phi^{h_1}, \ldots, \Phi^{h_k}$ have to commute mutually in order to make such compatible level sets possible. This would then require mutually vanishing Lie brackets of $X^{h_1}, \ldots, X^{h_k}$. But passing from the Hamiltonian functions to the Hamiltonian vector field means differentiating the Hamiltonian functions, i.e., information gets lost during this transition. So it is somewhat doubtful if the Lie bracket – which applies to vectorfields only and not to functions – is a strong enough notion to track level sets. But there is a notion of 'Lie bracket for functions', namely the so-called Poisson bracket. We will see in Definition 1.50 that this is indeed the mathematical notion we are looking for.

1.2. Frobenius integrability

In the discussion after Question 1.13 we suspected some involvement of the Lie bracket in quest of generalizing the idea of 'energy conservation' in order to confine solutions to lower dimensional sets. In this section, we study the notion of 'integrability' related to the Lie bracket. We will see that this notion applies to 'general' vector fields on manifolds, not to the specific class of Hamiltonian vector fields – although the may be used to obtain nice examples.

Recall that each vector field X gives, via the ordinary differential equation $x' = X(x)$, rise to a flow Φ_t^X. Vice versa, each flow Φ_t gives rise to a vector field via $X^\Phi := \frac{d}{dt}\Phi_t$. Thus questions of flows translate into questions of vector fields and vice versa. For background details on flows of (autonomous) vector fields, we refer the reader to Section A.4 or literature like for instance [**Teschl**]. In the following, we will be working on general smooth manifolds instead of $\mathbb{R}^m$. For the definition of manifolds and submanifolds, see for instance Definition A.1 and Definition A.2 or literature like [**Warner**] and [**Petersen**].

Let us start with a notion motivated that can be seen as 'generalized span of vector fields'.

Definition 1.14. *Let M be a smooth m-dimensional manifold and let $n \in \mathbb{N}$ with $1 \leq n \leq m$.*

1) *An n-**dimensional distribution** $\mathscr{D}$ on M is given by $\mathscr{D} = \bigcup_{x\in M} \mathscr{D}_x$ where $\mathscr{D}_x \subseteq T_xM$ is an n-dimensional subspace of the tangent space T_xM for all $x \in M$.*
2) *An n-dimensional distribution $\mathscr{D}$ is* **smooth** *if, for each $x \in M$, there exists a neighbourhood U of x and smooth, linearly independent vector fields $X_1, \ldots, X_n$ on U such that*
$$\mathscr{D}_y = \mathrm{Span}\{X_1|_y, \ldots, X_n|_y\} \qquad \forall\, y \in U.$$

Distributions that admit only subspaces of the same dimension are also called *regular* or *of constant rank* whereas distributions where the dimension of the subspaces varies are often referred to as *singular*. Intuitively, a smooth distribution $\mathscr{D}$ is a distribution where the map $x \mapsto \mathscr{D}_x$ varies smoothly.

Example 1.15. *Recall that $T_x\mathbb{R}^m \simeq \{x\}x\mathbb{R}^m \simeq \mathbb{R}^m$ for all $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and stel $e_1, \ldots, e_m$ the standard basis of $\mathbb{R}^m$.*

1) *Stel $m = 3$. The distribution given by $\mathscr{D}_x := \mathrm{Span}_{\mathbb{R}}\{e_1\} \subseteq T_x\mathbb{R}^3$ for all $x \in \mathbb{R}^3$ with $x_1 > 0$ and $\mathscr{D}_x := \mathrm{Span}_{\mathbb{R}}\{e_2\} \subseteq T_x\mathbb{R}^3$ for all $x \in \mathbb{R}^3$ with $x_1 \leq 0$ is 1-dimensional. It is not smooth on the plane $\{x \in \mathbb{R}^3 \mid x_1 = 0\}$.*
2) *Stel $m = 4$. Then $\mathscr{D}_x := \mathrm{Span}_{\mathbb{R}}\{(1 + x_4^2)e_1, e_2 + x_2e_3\} \subseteq T_x\mathbb{R}^4$ is a 2-dimensional, smooth distribution.*

Now we come to the crucial notion of this section:

DEFINITION 1.16. *A* **Lie bracket** *on an $\mathbb{R}$-vector space V is a mapping*

$$[\cdot,\cdot] : V \times V \to V, \quad (u,v) \mapsto [u,v]$$

that satisfies

(i) $[\lambda u, v] = \lambda[u,v]$ *and* $[u+\tilde{u}, v] = [u,v] + [\tilde{u}, v]$
for all $u, \tilde{u}, v \in V$ *and for all* $\lambda \in \mathbb{R}$.

(ii) $[u,v] = -[v,u]$
for all $u, v \in V$ (**skewsymmetry** *or* **anti-commutativity**).

(iii) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$
for all $u, v, w \in V$ (**Jacobi identity**).

(i) and (ii) together imply that the Lie bracket is **bilinear**. *A vector space equipped with a Lie bracket is said to be a* **Lie algebra**.

Given a smooth manifold M, a smooth function $f : M \to \mathbb{R}$ and a vector field A on M, then the **derivative of f w.r.t. the vector field** A is defined as

$$A(f)(x) := Df|_x.A|_x \quad \text{for all } x \in M \text{ and all } A|_x \in T_xM,$$

briefly $A(f) = Df.A$. It each point x, it can be seen as the directional derivative of f in direction of the vector $A|_x$.

On $\mathbb{R}^m$, there exists the following standard notion of a Lie bracket for vector fields which generalizes in the obvious way to (the tangent space of) a smooth manifold:

EXAMPLE 1.17 (**Lie bracket for vector fields on $\mathbb{R}^m$**). *Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{R}^m$ and let $A, B : \mathbb{R}^m \to \mathbb{R}^m$ be smooth vector fields given in coordinates by $A = \sum_{k=1}^m A^k e_k$ and $B = \sum_{\ell=1}^m B^\ell e_\ell$ where $A^k, B^\ell : \mathbb{R}^m \to \mathbb{R}$ are the (smooth) coefficient functions indexed by k and l. Then*

$$[A,B] := \sum_{k=1}^m \Big(A(B^k) - B(A^k)\Big) e_k = \sum_{k,\ell=1}^m \Big(A^\ell D_\ell B^k - B^\ell D_\ell A^k\Big) e_k$$

is a vector field $[A,B] : \mathbb{R}^m \to \mathbb{R}^m$, called **Lie bracket of the vector fields** *A* **and** *B. This turns the space of smooth vector fields on $\mathbb{R}^m$ into a Lie algebra.*

The Lie bracket of two vector fields A and B satisfies

$$[A,B](f) = A(B(f)) - B(A(f)),$$

briefly $[A,B] = AB - BA$, and is for this reason also called **commutator** of A and B. On can see the commutator also as differentiation of one vector field along another. This motivates the notation $[A, B] = \mathscr{L}_A B$ where $\mathscr{L}_A$ is the so-called **Lie derivative**. Moreover, one can show that

$$[A, B]|_z = \frac{d}{dt}\Big|_{t=0} D\Phi^A_{-t}|_{\Phi^A_t(z)} B|_{\Phi^A_t(z)} = \frac{d}{dt}\Big|_{t=0} ((\Phi^A_t)^* B)|_z,$$

i.e., there is a way to express the Lie bracket by means of the flows of A and B, for details see for example [**Hohloch2**].

Note that the convention for the definition of the Lie bracket varies throughout the literature up the choice of a sign, i.e., some authors define $[A, B]$ as $AB - BA$, others as $-(AB - BA) = BA - AB$.

Proposition 1.18. *Let A, B be smooth vector fields with associated flows Φ^A and Φ^B. Then*

$$\Phi^A \circ \Phi^B = \Phi^B \circ \Phi^A \quad \Leftrightarrow \quad AB = BA \quad \Leftrightarrow \quad [A, B] = 0$$

Proof. See for instance [**Hohloch2**] or [**Warner**]. □

Thus the formula of commuting flows $\Phi^A \circ \Phi^B = \Phi^B \circ \Phi^A$ can be seen as 'differentiated version' of $AB = BA$.

Definition 1.19. *Let M be a smooth manifold.*

1) *A vector field X on M* **lies in the distribution** *$\mathscr{D}$, briefly $X \in \mathscr{D}$, if $X|_x \in \mathscr{D}_x$ for all $x \in M$.*
2) *A smooth distribution is* **involutive** *or* **completely integrable** *if $[X, Y] \in \mathscr{D}$ for all vector fields $X, Y \in \mathscr{D}$.*

The seminal example for involutive distributions is

Example 1.20. *Let $A, B : \mathbb{R}^m \to \mathbb{R}^m$ be differentiable, nonvanishing vector fields with commuting flows. Then the distribution given by*

$$\mathscr{D}_x := \mathrm{Span}\{A_x, B_x\} \subseteq T_x\mathbb{R}^m \quad \forall\, x \in \mathbb{R}^m$$

is involutive.

Proof. $[A_x, B_x] = 0_x \in \mathscr{D}_x$ for all $x \in \mathbb{R}^m$ by Proposition 1.18. □

This example motivates

Definition 1.21. *Let $\mathscr{D}$ be a smooth distribution on a manifold M. A submanifold $N \subseteq M$ is an* **integral manifold** *of the distribution $\mathscr{D}$ if $T_xN = \mathscr{D}_x$ for all $x \in N$. A* **maximal integral manifold** *of a smooth* *distribution $\mathscr{D}$ on M is a connected integral manifold of $\mathscr{D}$ that is not a proper subset of any other connected integral manifold of $\mathscr{D}$.*

The relation between integral manifolds and involutive distributions is characterized by

THEOREM 1.22 (**Frobenius**). *Let M be a smooth manifold with a smooth distribution $\mathscr{D}$. Then there are equivalent:*

(i) $\mathscr{D}$ is involutive, i.e., $[X, Y] \in \mathscr{D}$ for all vector fields $X, Y \in \mathscr{D}$.
(ii) Through each point of M, there passes a unique maximal integral manifold of $\mathscr{D}$.

PROOF. See [**Warner**]. □

This version of Frobenius' theorem illuminates the relation between vector fields and integrability of distributions. There is also a 'dual' version based on differential forms instead of vector fields, see [**Warner**].

REMARK 1.23. *Integrability in the sense of Theorem 1.22 (Frobenius) is usually referred to as* **Frobenius integrability**. *It is more general than the so-called* **Liouville integrability** *that will be discussed in Remark 1.53.*

Frobenius integrability is a form of integrability tailored for vector fields and their flows. Since Hamiltonian vector fields carry additional information, namely being induced by a function, Frobenius integrability is eventually not 'fine' enough to 'keep track' of this additional information: Frobenius integrability does not 'see' the level sets of the underlying Hamiltonian functions as shown later in Example 1.41.

1.3. Hamiltonian systems on symplectic manifolds

In Section 1.1, we defined Hamiltonian vector fields on $\mathbb{R}^{2n}$ using special coordinates $z = (q, p) \in \mathbb{R}^{2n}$. In this section, we will define Hamiltonian dynamics in their natural setting, namely on so-called symplectic manifolds. We will discover in Theorem 1.34 (Darboux) that we can in fact always find local coordinates on a symplectic manifold in which the Hamiltonian system looks as previously defined in Section 1.1.

The notions of *(smooth) manifold* and *differential k-forms* used in the following are recalled in Appendix A.1 and Appendix A.3. For the definition of symplectic geometry, we need in particular

Definition 1.24. *A 2-form ω on a smooth manifold M is* **nondegenerate** *if, for all $x \in M$, $\omega_x(u, v) = 0$ for all $v \in T_xM$ implies $u = 0$.*

In local coordinates, a 2-form ω on an open $V \subseteq \mathbb{R}^m$ can be represented by a skewsymmetric $(m \times m)$-matrix Ω defined by the equation

$$\omega_x(u, v) = u^T \Omega_x v \qquad \forall\ x \in V,\ \forall\ u, v \in T_xV. \tag{1.25}$$

Being nondegenerate means in terms of linear algebra that $\det(\Omega_x) \neq 0$ for all $x \in V$.

Lemma 1.26. *If a smooth manifold admits a nondegenerate 2-form then the dimension of the manifold must be even.*

Proof. Let M be a smooth m-dimensional manifold and ω be a nondegenerate 2-form on M. Let $x \in M$ and consider the representing skewsymmetric $(m \times m)$-matrix Ω_x of ω_x on the tangent space T_xM. Then the skewsymmetry implies $\det(\Omega_x) = (-1)^m \det(\Omega_x)$. If m were odd then we would get $\det(\Omega_x) = -\det(\Omega_x)$ en thus $\det(\Omega_x) = 0$ in contradiction to the nondegeneracy of ω. Therefore m must be even. □

Definition 1.27. *A differential form is* **symplectic** *if it is a smooth, nondegenerate, closed 2-form. A smooth manifold is said to be* **symplectic** *if it carries a symplectic form.*

Lemma 1.26 implies immediately

Corollary 1.28. *Symplectic manifolds are even dimensional.*

The seminal example of a symplectic manifold is

Example 1.29. *Consider $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and endow it with the standard symplectic* *form*

$$\omega_{st} := -\sum_{i=1}^{n} dq_i \wedge dp_i = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

This form is represented by the skewsymmetric $(2n \times 2n)$*-matrix*

$$\begin{pmatrix} \mathbf{0} & -\mathrm{Id} \\ \mathrm{Id} & \mathbf{0} \end{pmatrix}$$

where Id *is the* $(n \times n)$*-identity matrix and* $\mathbf{0}$ *the* $(n \times n)$ *zero matrix.*

Other important examples are

Example 1.30.

1) *The* $2n$*-torus* $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n} \simeq (\mathbb{R}/\mathbb{Z})^{2n}$ *with the symplectic form induced by Example 1.29 is symplectic.*
2) $\mathbb{C}^n$ *with coordinates* $(z_1, \ldots, z_n)$ *and* $-\frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ *is a symplectic manifold. The identification* $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ *via* $z_k = q_k + ip_k$ *recovers Example 1.29.*
3) *Denote by* $\langle \cdot, \cdot \rangle_{eu}$ *the Euclidean scalar product in* $\mathbb{R}^3$*. The* 2*-sphere* $\mathbb{S}^2 \subset \mathbb{R}^3$ *with* $\omega_{\mathbb{S}^2}$ *given by*
$$(\omega_{\mathbb{S}^2})_x(u, v) := \langle x, u \times v \rangle_{eu}$$
for all $x \in \mathbb{S}^2$ *and all* $u, v \in T_x\mathbb{S}^2$ *is a symplectic manifold.* $\omega_{\mathbb{S}^2}$ *is usually considered as standard symplectic form on* $\mathbb{S}^2$*.*
4) *The complex projective space* $\mathbb{CP}^n$ *is symplectic for all* $n \in \mathbb{N}$*. Its standard symplectic form* ω_{FS} *is called* **Fubini-Study form***. If we identify* $\mathbb{CP}^1 \simeq \mathbb{S}^2$ *with* $\omega_{\mathbb{S}^2}$ *from Example 1.30 part 3)), we obtain*
$$\omega_{FS} = -\frac{1}{4}\omega_{\mathbb{S}^2}.$$
5) *Any volume form of a* 2*-dimensional manifold is a symplectic form, i.e., all orientable* 2*-dimensional manifolds are symplectic.*
6) *Any cotangent bundle is symplectic: let* N *be a manifold and* T^*N *its cotangent bundle and* $\tau : T^*N \to N$*,* $z = (q, p) \to q$ *the footpoint projection with* $p \in T_q^*N$*. The so-called* **tautological 1-form** θ *on* T^*N *is then defined by* $\theta_z := (D\tau|_z)^* p \in T_z^*N$ *for all* $z \in N$*. Its exterior derivative is a symplectic form* $\omega := -d\theta$ *on* T^*N*, often referred to as the* **natural** *or* **standard symplectic form** *of the cotangent bundle.*

Proof. **1)** Left to the reader.
2) Left to the reader.

3) Left to the reader or see [**Hohloch & Palmer**, Section 2.3].
4) Denote by $\mathbb{C}^\times$ the multiplicative group $(\mathbb{C} \setminus \{0\}, \cdot)$ and set

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times \simeq \mathbb{S}^{2n+1}/\mathbb{S}^1$$

with quotient map $\tau : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ where $\tau(z) =: [z]$ is the equivalence class $[z] = [z_0, \dots, z_n]$ arisen from the identification $(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ with $\lambda \in \mathbb{C}^\times$. The 2-form

$$\tilde{\omega}_{FS}|_z := \frac{i}{2\,|z|^4} \sum_{j=0}^{n} \sum_{k \neq j} |z_k|^2 \, dz_j \wedge d\bar{z}_j - \bar{z}_k z_j \, dz_k \wedge \bar{z}_j$$

on $\mathbb{C}^{n+1} \setminus \{0\}$ descends to a unique symplectic form ω_{FS} on the quotient $\mathbb{CP}^n$, satisfying $\tilde{\omega}_{FS} = \tau^* \omega_{FS}$.
5) Left to the reader.
6) Left to the reader or see [**Cushman & Bates**, Part II, Chapter 6.2] □

Attention: Whereas even dimensional tori and all complex projective spaces can be endowed with a symplectic form, this is *not* true for $2n$-spheres with $n > 1$:

Proposition 1.31. *Among all spheres $\mathbb{S}^k$ with $k \in \mathbb{N}_0$, only $\mathbb{S}^2$ is symplectic, i.e., higher dimensional spheres of even dimension are not symplectic (those of odd dimension are anyway not symplectic due to Corollary 1.28).*

Proof. We know from Example 1.30 part 3)) that $\mathbb{S}^2$ is symplectic and from Corollary 1.28 that odd dimensional spheres cannot be symplectic. Thus it remains to consider $2n$-dimensional spheres with $n > 1$. We argue by contradiction: let $n > 1$ and assume that $\mathbb{S}^{2n}$ admits a symplectic form ω. Since ω is closed we can see it as cohomology class $\omega \in H^2(\mathbb{S}^{2n})$. Because of $H^2(\mathbb{S}^{2n}) = 0$, the 2-form ω is in fact exact, i.e., there is a 1-form α with $\omega = d\alpha$. We find $0 \neq \omega^n := \omega \wedge \cdots \wedge \omega \in H^{2n}(\mathbb{S}^{2n})$, i.e., ω^n is a volume form on $\mathbb{S}^{2n}$. Moreover, using $d\omega = 0$, we compute $d(\alpha \wedge (\omega^{n-1})) = d\alpha \wedge (\omega^{n-1}) = \omega^n$. Using Stokes' theorem, we find

$$0 \neq \mathrm{vol}\left(\mathbb{S}^{2n}\right) = \int_{S^{2n}} \omega^n = \int_{\mathbb{S}^{2n}} d(\alpha \wedge (\omega^{n-1})) \overset{Stokes}{=} \int_{\partial \mathbb{S}^{2n}} \alpha \wedge (\omega^{n-1}) = 0$$

since the boundary $\partial \mathbb{S}^{2n}$ of $\mathbb{S}^{2n}$ is the empty set. This contradiction shows that $2n$-spheres with $n > 1$ cannot admit a symplectic form. □

This extends Corollary 1.28 to

Corollary 1.32. *Let M be a smooth manifold. Then*

$$M \text{ even dimensional} \quad \begin{array}{c}\not\Rightarrow \\ \Leftarrow\end{array} \quad M \text{ symplectic}$$

Let N_1 and N_2 be smooth manifolds and $f : N_1 \to N_2$ a smooth map and σ a 2-form on N_2. Then the **pullback** $f^*\sigma$ of σ under f to N_1 is defined as

$$(f^*\sigma)_x(u,v) := \sigma_{f(x)}(Df|_x.u, Df|_x.v) \qquad \forall\, x \in N_1,\ \forall\, u, v \in T_xN_1.$$

For the properties of the pullback of, see Definition and Proposition A.13.

Definition 1.33. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and $\psi : M_1 \to M_2$ a smooth map.*
1) *ψ is said to be* **symplectic** *if $\psi^*\omega_2 = \omega_1$.*
2) *A* **symplectomorphism** *is a symplectic diffeomorphism.*
3) *(M_1, ω_1) and (M_2, ω_2) are* **symplectomorphic** *if there exists a symplectomorphism $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$.*

If $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is symplectic then $\psi^*\omega_2 = \omega_1$ forces $D\psi$ to be injective since the symplectic forms are nondegenerate. In particular, we have $\dim M_1 \le \dim M_2$. If $\dim M_1 = \dim M_2$ then a symplectic ψ is a local diffeomorphism and thus a local symplectomorphism.

In general, symplectomorphisms are for symplectic geometry what isomorphisms are for linear algebra and diffeomorphisms for differential geometry and isometries for Riemannian geometry.

We will see now that *locally* all symplectic manifolds look the same:

Theorem 1.34 (**Darboux**). *Let (M, ω) be a symplectic manifold of dimension $2n$. Endow $\mathbb{R}^{2n}$ with coordinates $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$. Then, for all $x \in M$, there exists an open neighbourhood U with $x \in U$ and a chart $\psi : U \to \mathbb{R}^{2n}$ such that $\psi(x) = 0$ and*

$$(\psi^{-1})^*\omega = -\sum_{k=1}^{n} dq_k \wedge dp_k$$

i.e., $(U, \omega|_U)$ and $(\mathbb{R}^{2n}, -\sum_{k=1}^{n} dq_k \wedge dp_k)$ are symplectomorphic.

Proof. See for example [**Hofer & Zehnder**] or [**McDuff & Salamon**]. □

This means in particular that symplectic manifolds have no lokal invariants (in contrast to curvature on Riemannian manifolds). Thus, unless we add some 'local structure', symplectic manifolds are locally 'quite boring'.

In Section 1.1, we defined the Hamiltonian systems of a Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ using coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$ and the explicit expression $X^H(q, p) = \left(\partial_p H(q, p), -\partial_q H(q, p)\right)^T$. Now we give a coordinate free version:

Definition 1.35. *Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a smooth function. The equation*

$$\omega(X^H, \cdot) = -dH(\cdot) \tag{1.36}$$

defines a vector field X^H, called the **Hamiltonian vector field of** H. *The associated ODE*

$$z' = X^H(z)$$

is called **Hamiltonian equation**. *Its flow is referred to as* **Hamiltonian flow** *and usually denoted by Φ^H. In this context, the function H is usually referred to as* **Hamiltonian function**.

Denote by Ω_x a representing matrix of ω_x in $x \in M$ in some local coordinates and denote by $DH|_x$ the representation of $dH|_x$ in these local coordinates. Since $\det(\Omega_x) \neq 0$ we can in fact solve equation (1.36) explicitly for X^H via

$$(X^H(x))^T \Omega_x = -DH|_x \quad \Leftrightarrow \quad X^H(x) = -\left(DH|_x\, \Omega_x^{-1}\right)^T$$

More abstractly, the Hamiltonian vector field X^H is dual to the 1-form $-dH$ under the isomorphism between vector fields and 1-forms induced by ω via $X \mapsto \omega(X, \cdot)$.

Lemma 1.37. *If we apply Definition 1.35 to the symplectic manifold $(\mathbb{R}^{2n}, -\sum_{k=1}^n dq_k \wedge dp_k)$ and the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ then we recover $X^H(q, p) = (\partial_p H(q, p), -\partial_q H(q, p))^T$ from Section 1.1.*

Proof. Write $X^H = (X_q^H, X_p^H)^T$ in components w.r.t. the coordinate splitting (q, p). Then the equation $\omega(X^H, \cdot) = -dH(\cdot)$ transforms on the space $(\mathbb{R}^{2n}, -\sum_{k=1}^n dq_k \wedge dp_k)$ into

$$\begin{pmatrix} X_q^H, & X_p^H \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\mathrm{Id} \\ \mathrm{Id} & \mathbf{0} \end{pmatrix} = -\begin{pmatrix} \partial_q H, & \partial_p H \end{pmatrix}.$$

This implies the identity

$$X^H = \begin{pmatrix} X_q^H \\ X_p^H \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix}$$

what we used in Definition 1.1 to define the Hamiltonian vector field. □

1.4. Liouville integrability

The notion of Frobenius integrability introduced in Section 1.2 is primarily defined fo vector fields and distribution and makes use of the properties of the Lie bracket. In the present section, we are looking for a notion of integrability particularly suited for Hamiltonian systems. Since Hamiltonian dynamics do not only involve the Hamiltonian vector field, but also display a vital input by the Hamiltonian function, we are looking now for some kind of 'Lie bracket for functions' to replace the Lie bracket.

Definition 1.38. *A* **Poisson algebra** *is a triple* $(\mathscr{P}, \diamond, \{\cdot,\cdot\})$ *such that*

1) $(\mathscr{P}, \diamond) := (\mathscr{P}, \diamond, +)$ *is an associative algebra over a field* $\mathbb{K}$ *w.r.t. the (bilinear) multiplication* $\diamond$.
2) *There exists a map, referred to as* **Poisson bracket**,

$$\{\cdot,\cdot\} : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$$

satisfying

(i) $\{f+g,h\} = \{f,h\} + \{g,h\}$ *and* $\{cf,h\} = c\,\{f,h\}$ *for all* $f,g,h \in \mathscr{P}$ *and for all* $c \in \mathbb{K}$.

(ii) $\{f,g\} = -\{g,f\}$ *for all* $f,g \in \mathscr{P}$ (**skewsymmetry** *or* **anti-commutativity**).

(iii) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ *for all* $f,g,h \in \mathscr{P}$ (**Jacobi identity**).

(iv) $\{f \diamond g,h\} = \{f,h\} \diamond g + f \diamond \{g,h\}$ *for all* $f,g,h \in \mathscr{P}$ (**Leibniz rule**).

Note that (i) and (ii) together imply **bilinearity** *of the Poisson bracket.*

Thus Poisson brackets are Lie brackets that satisfy in addition the Leibniz rule. The anti-commutativity implies $\{h,h\} = 0$ for all $h \in \mathscr{P}$. We are in particular interested in

Example 1.39 (**Poisson bracket for Hamiltonian functions**). *Let (M,ω) be a symplectic manifold of dimension $2n$. Then*

$$\{\cdot,\cdot\} : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$$
$$\{f,g\} := -\omega(X^f, X^g)$$

defines a **Poisson bracket on** $C^\infty(M,\mathbb{R})$. *In local coordinates $(q,p) = (q_1,\dots,q_n,p_1,\dots,p_n) \in \mathbb{R}^{2n}$ on $(\mathbb{R}^{2n}, -\sum_{i=1}^n dq_i \wedge dp_i)$, this yields for $f,g \in C^\infty(\mathbb{R}^{2n},\mathbb{R})$*

$$\{f,g\} = \partial_q f\, \partial_p g - \partial_p f\, \partial_q g = \left(\sum_{k=1}^n \partial_{q_k} f\, \partial_{p_k} g - \partial_{p_k} f\, \partial_{q_k} g\right).$$

This Poisson bracket relates as follows to the Lie bracket of the Hamiltonian vector fields X^f and X^g:

$$[X^f, X^g] = X^{-\{f,g\}}.$$

In the literatur, the Poisson bracket is sometimes defined with the other sign, i.e., as $\{f,g\} = \omega(X^f,X^g) = \partial_p f \partial_q g - \partial_q f \partial_p g = -(\partial_q f \partial_p g - \partial_p f \partial_q g)$. In this case, we have $[X^f,X^g] = X^{\{f,g\}}$ instead of $[X^f,X^g] = X^{-\{f,g\}}$. The algebraic implication of the choice of sign is explained in the following statement.

Lemma 1.40. *Let (M,ω) be a symplectic manifold and denote the set of smooth vector fields on M by $Vect(M)$. Then*

1) *$(Vect(M), [\cdot,\cdot])$ is a Lie algebra.*
2) *$(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ is a Poisson algebra. Forgetting the Leibniz rule, $\{\cdot,\cdot\}$ induces the structure of a Lie algebra on $C^\infty(M,\mathbb{R})$.*
3) *The map $h \mapsto X^h$ is in our convention $[X^f,X^g] = X^{-\{f,g\}}$ a Lie algebra anti-homomorphism from $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ to $(Vect(M), [\cdot,\cdot])$. In the convention $[X^f,X^g] = X^{\{f,g\}}$, it is a Lie algebra homomorphism. The kernel consists in both cases of the set of constant functions.*

Proof. Left to the reader. □

Because of the formula $[X^f,X^g] = X^{-\{f,g\}}$, a vanishing Poisson bracket $\{f,g\} = 0$ leads to $0 = X^0 = X^{-\{f,g\}} = [X^f,X^g]$, i.e., a vanishing Lie bracket of the associated Hamiltonian vector fields. But is the converse true, i.e., does a vanishing Lie algebra imply a vanishing Poisson bracket?

EXAMPLE 1.41. *Consider* $\mathbb{R}^{2n}$ *with local coordinates* $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ *and* $g, h \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ *given by*

$$g(q, p) := q_1 \qquad \text{and} \qquad h(q, p) := p_1$$

Then $[X^g, X^h] = 0$*, but* $\{g, h\} = 1 \neq 0$*.*

PROOF. We compute $\{g, h\} = 1 + 0 + \cdots + 0 = 1$ which implies immediately $[X^g, X^h] = X^{-\{g,h\}} = 0$ since the Hamiltonian vector field of a constant function always vanishes. □

This leads to

COROLLARY 1.42. *Let* f, g *be Hamiltonian functions with Hamiltonian flows* Φ^f*,* Φ^g*. Then*

$$\Phi^f \circ \Phi^g = \Phi^f \circ \Phi^g \quad \Leftrightarrow \quad [X^f, X^g] = 0 \quad \overset{\Leftarrow}{\not\Rightarrow} \quad \{f, g\} = 0.$$

PROOF. Example 1.41 shows that a vanishing Lie bracket does not imply vanishing of the Poisson bracket. On the other hand, the formula $[X^f, X^g] = X^{-\{f,g\}}$ shows that a vanishing Poisson bracket inplies vanishing of the Lie bracket. Proposition 1.18 eventually shows a vanishing Lie bracket to be equivalent to commuting flows. □

The Poisson bracket is therefore a 'finer comb' than the Lie bracket. Now we will start answering the question why the Poisson bracket is the right tool to measure the 'compatibility' of Hamiltonian flows with level sets of other Hamiltonian functions.

DEFINITION 1.43. *Let* N *be a smooth manifold,* $f \in C^\infty(N, \mathbb{R})$*, and let* $\gamma :]-\varepsilon, \varepsilon[\to N$ *be a smooth curve. Then the* **evolution** *of* f *along* γ *is given by* $t \mapsto (f \circ \gamma)'(t)$*.*

The evolution has the following geometric meaning.

LEMMA 1.44. *Let* N *be a smooth manifold and* $f \in C^\infty(N, \mathbb{R})$ *and* $\gamma :]-\varepsilon, \varepsilon[\to \mathbb{R}^{2n}$ *smooth. Then the evolution of* f *along* γ *vanishes if and only if* γ *stays within a level set of* f*.*

PROOF.

The evolution of f along γ vanishes $\Leftrightarrow (f \circ \gamma)' \equiv 0$

$$\begin{aligned}&\Leftrightarrow (f \circ \gamma) \text{ is constant}\\ &\Leftrightarrow \gamma \text{ stays within a level set of } f\end{aligned}$$

□

Let us gain some intuition for the evolution in a Hamiltonian setting: First consider $\mathbb{R}^{2n}$ with local coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $h \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$. Let $\gamma :] - \varepsilon, \varepsilon[\to \mathbb{R}^{2n}$ be a smooth curve with components $\gamma = (\gamma_q, \gamma_p) = (\gamma_{q_1}, \ldots, \gamma_{q_n}, \gamma_{p_1}, \ldots, \gamma_{p_n}) \in \mathbb{R}^{2n}$. In these coordinates, the evolution of h along γ is given by

$$\begin{aligned}(h \circ \gamma)'(t) = Dh|_{\gamma(t)}.\gamma'(t) &= \sum_{k=1}^{n} \partial_{q_k} h|_{\gamma(t)} \gamma'_{q_k}(t) + \partial_{p_k} h|_{\gamma(t)} \gamma'_{p_k}(t)\\ &= \partial_q h|_{\gamma(t)} \gamma'_q(t) + \partial_p h|_{\gamma(t)} \gamma'_p(t)\end{aligned}$$

what already to some extend like the local formula of the Poisson bracket looks. If γ is in fact a Hamiltonian solution, then we regain the formula of the Poisson bracket fully:

Corollary 1.45. *Let (M, ω) be symplectic, $f, g \in C^\infty(M, \mathbb{R})$, and let $z^g :] - \varepsilon, \varepsilon[\to M$ a Hamiltonian solution of g. Then*

$$z^g \text{ stays within a level set of } f \quad \Leftrightarrow \quad \{f, g\} = 0.$$

Proof. According to Lemma 1.44, z^g stays within a level set of f precisely when the evolution of f along z^g vanishes. Thus we compute

$$0 = (f \circ z^g)' = Df.(z^g)' = Df.X^g = -\omega(X^f, X^g) = \{f, g\}.$$

□

Therefore we are interested in pairs of Hamiltonians f, g with $\{f, g\} = 0$ if we want their flows to stay within each others level sets.

Definition 1.46. *Let (M, ω) be symplectic and $f \in C^\infty(M, \mathbb{R})$. A function $g \in C^\infty(M, \mathbb{R})$ satisfying $\{f, g\} = 0$ is said to be an* **integral** *of f. We set*

$$I(f) := \{g \in C^\infty(M, \mathbb{R}) \mid g \text{ integral of } f\}.$$

Integrals have the following properties:

Lemma 1.47. *Let (M, ω) be symplectic and $f, g \in C^\infty(M, \mathbb{R})$. Then*
1) $f \in I(g) \;\Leftrightarrow\; g \in I(f)$.

2) $f \in I(f)$, *i.e., the 'energy' f is an integral of f.*
3) $(I(f), \{\cdot,\cdot\})$ *is a Lie algebra. In particular, the Poisson bracket of two integrals is again an integral.*

Proof. **1)** The anti-commutativity of the Poisson bracket implies for all smooth functions f and g that $0 = \{f, g\} = -\{g, f\}$.
2) The anti-commutativity of the Poisson bracket implies $\{f, f\} = 0$ for all smooth functions f.
3) Let g, $h \in I(f)$, i.e., $\{f, g\} = 0 = \{f, h\}$. By adding zero and using the Jacobi identity, we obtain

$$\begin{aligned}\{f,\{g,h\}\} &= \{f,\{g,h\}\} + \{g,0\} + \{h,0\}\\ &= \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\\ &= 0.\end{aligned}$$

□

Recall that the Hamiltonian vector field X^f of a smooth real valued function f transforms under a symplectomorphism ψ via

$$X^{f\circ\psi}(x) = D\psi^{-1}|_{\psi(x)}X^f(\psi(x)). \tag{1.48}$$

The Poisson bracket behaves under concatenation as follows.

Lemma 1.49. *Let (M, ω) be a symplectic manifold and $\{\cdot,\cdot\}$ the Poisson bracket induced by ω.*
1) *Let (N, σ) be a symplectic manifold, $\psi : (N, \sigma) \to (M, \omega)$ a symplectomorphism and $f, g \in C^\infty(M, \mathbb{R})$. Then*
$$\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi.$$
2) *Let $h_1, \ldots, h_n \in C^\infty(M, \mathbb{R})$ with $\{h_i, h_j\} = 0$ for all $1 \le i, j \le n$. Set $h := (h_1, \ldots, h_n)$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then*
$$\{f \circ h,\ h_i\} = 0, \quad \forall\, 1 \le i \le n,$$
i.e., $f \circ h \in I(h_i)$ for all $1 \le i \le n$.

Proof. **1)** We calculate

$$\begin{aligned}\{f\circ\psi, g\circ\psi\} &= \omega(X^{f\circ\psi}, X^{g\circ\psi}) = -d(f\circ\psi)(X^{g\circ\psi})\\ &\overset{(1.48)}{=} -Df|_\psi\, D\psi\, (D\psi)^{-1}|_\psi\, X^g|_\psi = -Df|_\psi\, X^g|_\psi = (-Df\, X^g)\circ\psi\\ &= \{f,g\}\circ\psi.\end{aligned}$$

2) We calculate

$$
\begin{aligned}
\{f\circ h,\ h_i\} &= -d(f\circ h)(X^{h_i}) = Df|_h\, Dh\, X^{h_i}\\
&= -Df|_h \begin{pmatrix} \partial_{q_1}h_1 & \dots & \partial_{q_n}h_1 & \partial_{p_1}h_1 & \dots & \partial_{p_n}h_1\\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_{q_1}h_n & \dots & \partial_{q_n}h_n & \partial_{p_1}h_n & \dots & \partial_{p_n}h_n \end{pmatrix} \begin{pmatrix} \partial_{p_1}h_i \\ \vdots \\ \partial_{p_n}h_i \\ -\partial_{q_1}h_i \\ \vdots \\ -\partial_{q_n}h_i \end{pmatrix}\\
&= Df|_h \begin{pmatrix} -dh_1(X^{h_i}) \\ \vdots \\ -dh_n(X^{h_i}) \end{pmatrix} = Df|_h \begin{pmatrix} \{h_1,h_i\} \\ \vdots \\ \{h_n,h_i\} \end{pmatrix} = Df|_h \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\\
&= 0.
\end{aligned}
$$

□

Now we are ready for

Definition 1.50. *Let (M,ω) be a $2n$-dimensional symplectic manifold. Then a smooth function $h := (h_1,\dots,h_n) : M \to \mathbb{R}^n$ is said to be a* **(momentum map of a) completely integrable (Hamiltonian) system** *if*

1) $X^{h_1},\dots,X^{h_n}$ are almost everywhere linearly independent.
2) $\{h_i,h_j\} = 0$ for all $1 \le i,j \le n$ (**Poisson commutative**).

A completely integrable system is often abbreviated by (M,ω,h).

Contrary to standard notions in physics, some mathematicians call the momentum map briefly *moment map*.

The measure theoretic notion ‘almost everywhere’ does not depend on the choice of $\omega^n := \omega\wedge\cdots\wedge\omega$, the natural n-dimensional volume on (M,ω), or the Lebesgue measure since ω^n coincides with the Lebesgue measure up to scaling by a strictly positive function.

Remark 1.51. *1) Condition 1) in Definition 1.50 is equivalent to requiring* $\operatorname{rk} Dh = n$ *almost everywhere.*
2) Due to condition 2) in Definition 1.50, the Hamiltonian flows Φ^{h_1}, $\dots$, Φ^{h_n} of a completely integrable system (M,ω,h) commute.
3) If the flows Φ^{h_1}, $\dots$, Φ^{h_n} are defined on whole $\mathbb{R}$, then we get an action of the group $(\mathbb{R}^n,+)$ on M via

$$
\begin{aligned}
&\mathbb{R}^n\times M \to M,\\
&(t,x) = (t_1,\dots,t_n,x) \mapsto \Phi^{h_1}_{t_1}\circ\cdots\circ\Phi^{h_n}_{t_n}(x) =: \Phi^h_t(x).
\end{aligned}
$$

Since the flows commute, the definition of Φ_t^h does not depend on the order of concatenation, i.e., for all permutations σ, we have

$$\Phi_{t_{\sigma(1)}}^{h_{\sigma(1)}} \circ \cdots \circ \Phi_{t_{\sigma(n)}}^{h_{\sigma(n)}}(x) = \Phi_{t_1}^{h_1} \circ \cdots \circ \Phi_{t_n}^{h_n}(x).$$

If one wants to emphasize the 'additional preserved quantity aspect' of a given Hamiltonian function, the following definition is more intuitive than Definition 1.50.

Definition 1.52. *Let (M, ω) be a $2n$-dimensional symplectic manifold. A Hamiltonian $H : M \to \mathbb{R}$ is* **completely integrable** *if H has n integrals $h_1, \ldots, h_n \in I(H)$ that form an integrable system $(M, \omega, h = (h_1, \ldots, h_n))$.*

In this situation, one often uses the 'energy' of H as integral $h_1 := H$.

Remark 1.53. *Integrability in the sense of Definition 1.50 or Definition 1.52 is usually referred to as* **Liouville integrability**. *Liouville integrability implies Frobenius integrability but not the other way around, see Corollary 1.42.*

The main difference here is that Frobenius integrability cannot 'see' the level sets of a Hamiltonian function but Liouville integrability can.

Let us now have a look at some examples from physics.

Example 1.54 (**Uncoupled harmonic oscillator**). *On $\mathbb{R}^4$ with coordinates $(q, p) = (q_1, q_2, p_1, p_2)$ and symplectic form $-\sum_{k=1}^2 dq_i \wedge dp_i$ consider $H : \mathbb{R}^4 \to \mathbb{R}$ given by*

$$H(q, p) := h_1(q, p) + h_2(q, p) := \frac{\nu_1}{2}(q_1^2 + p_1^2) + \frac{\nu_2}{2}(q_2^2 + p_2^2)$$

where $\nu_1, \nu_2 \in \mathbb{R}^{>0}$. Then

1) $h := (h_1, h_2) : \mathbb{R}^4 \to \mathbb{R}^2$ is a completely integrable system.

2) H is completely integrable in the sense of Definition 1.52.

Proof. Left as an exercise to the reader or see [**Cushman & Bates**, Part I, Chapter 1]. □

Example 1.55 (**Coupled spin oscillator**). *Let $\lambda, \mu \in \mathbb{R}^{>0}$. Endow $M := \mathbb{S}^2 \times \mathbb{R}^2$ with the symplectic form $\omega := \lambda\omega_{\mathbb{S}^2} \oplus \mu\omega_{st}$ where $\omega_{\mathbb{S}^2}$ the standard symplectic form on $\mathbb{S}^2$ is (see Example 1.30 part 3))). Let (x, y, z) be Cartesian coordinates on $\mathbb{S}^2 \subset \mathbb{R}^3$ and (u, v) Cartesian coordinates on $\mathbb{R}^2$. Then $h := (L, H) : M \to \mathbb{R}^2$ given by*

$$L(x, y, z, u, v) := \frac{\mu}{2}(u^2 + v^2) + \lambda(z - 1),$$
$$H(x, y, z, u, v) := \frac{1}{2}(xu + yv)$$

is a completely integrable system often called **coupled spin oscillator**. *L describes the coupled rotation of a vector $(x, y, z) \in \mathbb{S}^2$ and a vector $(u, v) \in \mathbb{R}^2$ and H the preserved angle between $(x, y) \in \mathbb{R}^2$, the projection of $(x, y, z) \in \mathbb{S}^2$ to $\mathbb{R}^2$, and the vector $(u, v) \in \mathbb{R}^2$, see Figure 1.2 and Figure 1.3.*

Proof. Left to the reader. $\square$

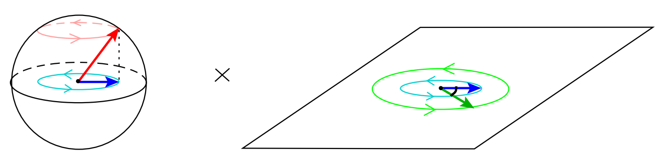


Figure 1.2. Geometric motivation for the coupled spin oscillator system: Coupled rotation on $\mathbb{S}^2$ and in $\mathbb{R}^2$ and preserved angle between the projected vector (blue) and the vector (green) in $\mathbb{R}^2$.

The *coupled spin oscillator* in Example 1.55 can be seen as a 'linearization' of the following system:

Example 1.56 (**Coupled angular momenta**). *Let $\vec{R} := (R_1, R_2) \in \mathbb{R}^2$ with $0 < R_1 < R_2 < \infty$ and $t \in [0, 1]$. Consider the product of 2-spheres $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with Cartesian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ and symplectic form $-R_1\omega_{\mathbb{S}^2} \oplus R_2\omega_{\mathbb{S}^2}$ where $\omega_{\mathbb{S}^2}$ is the standard symplectic form on the 2-sphere, see Example 1.30. Then*

$$h_{\vec{R},t} := (J_{\vec{R}}, H_t) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}^2$$

given by

$$J_{\vec{R}}(x_1, y_1, z_1, x_2, y_2, z_2) := R_1 z_1 + R_2 z_2,$$

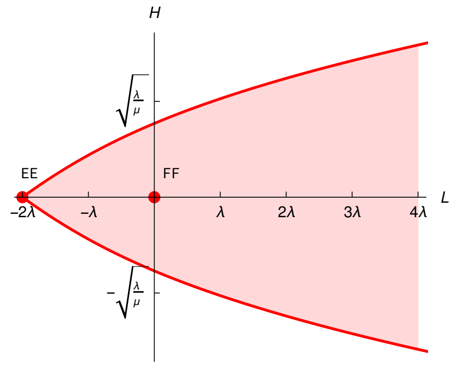


FIGURE 1.3. Image of the momentum map of the coupled spin oscillator. The singular points of rank 0 are marked as red points.

$$H_t(x_1, y_1, z_1, x_2, y_2, z_2) := (1 - t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2)$$

describes the so-called **coupled angular momenta system**. *It is a completely integrable system which describes the coupled rotation of vectors on the two spheres with angle between the rotating vectors as preserved 'symmetry', see Figure 1.1. The image $h_{\vec{R},t}(\mathbb{S}^2 \times \mathbb{S}^2) \subset \mathbb{R}^2$ of the momentum map is displayed in Figure 1.4.*

PROOF. This is a very special case of the system studied in [**Hohloch & Palmer**]. See also the earlier references therein. □

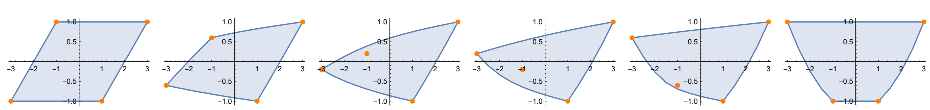


FIGURE 1.4. Image of the momentum map of the coupled angular momenta system for $\vec{R} = (1, 2)$ and t passing from $t = 0$ (at the very left) to $t = 1$ (at the very right). The singular points of rank 0 are marked as red points. When the 'coupling parameter' t changes, one of the rank zero points transitions from being elliptic-elliptic to being focus-focus and then back to elliptic-elliptic.

Example 1.57 (**Spherical pendulum**). *Consider* $\mathbb{R}^3$ *with standard coordinates* $q = (q_1, q_2, q_3)$ *and* $T^*\mathbb{R}^3 \simeq \mathbb{R}^6$ *with standard coordinates* $(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3)$ *and endow it with its standard symplectic form. Moreover, consider* $\mathbb{S}^2 \subset \mathbb{R}^3$ *with coordinates* $q = (q_1, q_2, q_3)$ *induced from* $\mathbb{R}^3$ *and* $T^*\mathbb{S}^2 \subset T^*\mathbb{R}^3 \simeq \mathbb{R}^6$ *with coordinates* $(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3)$ *induced from* $\mathbb{R}^6$ *and the induced symplectic form. Let* $\Gamma := (0, 0, 1)^T \in \mathbb{R}^3$ *and consider* q, p *as 'vectors' in* $\mathbb{R}^3$ *and denote by* $\langle \cdot, \cdot \rangle_{eu}$ *the Euclidean scalar product and by* $\times$ *the vector product. Then* $h := (J, H) : T^2\mathbb{S}^2 \to \mathbb{R}^2$ *given by*

$$J(q, p) := \langle q \times p, \Gamma \rangle_{eu} \quad \textit{and} \quad H(q, p) := \frac{1}{2} \|p\|^2 - \langle \Gamma, q \rangle_{eu}$$

is a completely integrable system, usually referred to as **spherical pendulum**. *It describes a pendulum that moves in three dimensional space.*

Proof. Left as an exercise to the reader or see [**Cushman & Bates**, Part I, Chapter 4]. □

CHAPTER 2

Behaviour of completely integrable systems due to regular and singular points

When we want to analyse the flow of a smooth vector field, the theory of ordinary differential equations suggests to consider the vector field locally and patch our local observations together to a global picture. The reason is that the behaviour near regular points (= points where the vector field is nonzero) and singular points (= points where the vector field vanishes) differs significantly: given a regular point, then Theorem A.16 (Flow box) states that the flow nearby can be 'straightened' by a smooth change of coordinates to a flow parallel to one of the coordinate axes. For singular points, the situation is more complicated: If the singular point is hyperbolic then, by Theorem A.18 (Hartman-Grobman), the flow is C^0-conjugated to the flow of the linearization at this point. If the singular point is not hyperbolic one either has to require stronger properties (like linear systems conjugated by linear changes of coordinates) or involve higher derivatives to analyse the local dynamics.

Since a completely integrable system $(M, \omega, h = (h_1, \dots, h_n))$ consists of n vector fields $X^{h_1}, \dots, X^{h_n}$ we certainly may apply the above techniques to study each of the n flows $\Phi^{h_1}, \dots, \Phi^{h_n}$ separately. The natural question is if the Poisson commutativity of the flows allows us to combine the results for each of the flows to a result of the integrable system h and its flow Φ^h. The rough answer is *yes* as we will see in this chapter. In fact, we are even able to obtain some 'semilocal' statements, i.e., statements that hold for (connected compontens of) the whole preimage (= fiber) of a regular or singular value.

2.1. The fibration of a completely integrable system

In this section, we will see how much knowledge or the system can already be gained from its fibres.

DEFINITION 2.1. *Let* $(M, \omega, h = (h_1, \dots, h_n))$ *be a completely integrable system. The decomposition* $M = \bigcup_{r \in h(M)} h^{-1}(r)$ *is called* **Liouville foliation** *of* (M, ω, h). *A connected component of a fiber* $h^{-1}(r)$ *is said to be a* **leaf**. *A fiber or leaf are called* **regular** *if all of its points are regular. Otherwise the fiber or leaf are called* **singular**.

The fibers and leaves of the Liouville foliation are invariant under the flow of the momentum map. By definition of an completely integrable system, almost all values in the image of the momentum map are regular, i.e., the set of singular values in the image is a zero set. Thus in particular, the set of points lying in singular leaves is a zero set in the underlying symplectic manifold.

EXAMPLE 2.2 (**Uncoupled harmonic oscillator**). *Let* $\nu_1, \nu_2 \in \mathbb{R}^{>0}$. *The fibers of the Liouville foliation of the uncoupled harmonic oscillator* $h := (h_1, h_2) : \mathbb{R}^4 \to \mathbb{R}^2$ *with*

$$h_1(q,p) = \frac{\nu_1}{2}(q_1^2 + p_1^2) \quad \text{and} \quad h_2(q,p) = \frac{\nu_2}{2}(q_2^2 + p_2^2)$$

(see Example 1.54 and Figure 2.1) are given by

$$h^{-1}(r_1, r_2) \simeq \left\{ (q_1, p_1) \,\middle|\, q_1^2 + p_1^2 = \frac{r_1 \nu_1}{2} \right\} \times \left\{ (q_2, p_2) \,\middle|\, q_2^2 + p_2^2 = \frac{r_2 \nu_2}{2} \right\}.$$

The image of the momentum map h *is given by the (unbounded) polygon* $h(\mathbb{R}^4) = \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$. *Depending on the value of* $r = (r_1, r_2)$, *we find*

- *If* $r_1, r_2 > 0$, *then the fibers are regular and given by* 2*-tori.*
- *If* $r_1 = 0$ *and* $r_2 > 0$ *or* $r_1 > 0$ *and* $r_2 = 0$, *then the fibers are singular and given by* 1*-tori.*
- *If* $r_1 = 0 = r_2$, *then the fiber is singular and is given by a single point.*
- *If* $r_1 < 0$ *or* $r_2 < 0$, *then the fiber is the empty set.*

In particular, all fibers are connected and compact.

PROOF. Since $\nu_1, \nu_2 > 0$, we have $h_j(q,p) \geq 0$ with $h_j(q,p) = 0$ precisely on the plane $\{q_j = 0 = p_j\} \subset \mathbb{R}^4$ for $j \in \{1, 2\}$. Thus we obtain as image of the momentum map the quadrant

$$h(\mathbb{R}^4) = \{(h_1(q,p), h_2(q,p)) \mid (q,p) \in \mathbb{R}^4\} = \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}.$$

If $r = (r_1, r_2) \notin h(\mathbb{R}^4)$ then $h^{-1}(r) = \emptyset$. If $r \in h(\mathbb{R}^4) = \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ then $h^{-1}(r) = h_1^{-1}(r_1) \cap h_1^{-1}(r_2)$ where

$$h_1^{-1}(r_1) = \left\{ (q_1, q_2, p_1, p_2) \in \mathbb{R}^4 \;\middle|\; q_1^2 + p_1^2 = \frac{r_1 \nu_1}{2} \right\} \tag{2.3}$$

$$= \text{circle of radius } \sqrt{\frac{r_1 \nu_1}{2}} \text{ in } (q_1, p_1)\text{-plane} \;\times\; \mathbb{R}^2. \tag{2.4}$$

For $r_1 = 0$ this yields

$$h_1^{-1}(0) = \{(0, q_2, 0, p_2) \mid q_1, p_2 \in \mathbb{R}\} = \text{origin in } (q_1, p_1)\text{-plane} \;\times\; \mathbb{R}^2.$$

Analogous statements hold true for $h_2^{-1}(r_2)$. Thus we get

$$\begin{aligned} h^{-1}(r) &= h_1^{-1}(r_1) \cap h_1^{-1}(r_2) \\ &= \left\{ (q_1, q_2, p_1, p_2) \;\middle|\; q_1^2 + p_1^2 = \frac{r_1 \nu_1}{2},\ q_2^2 + p_2^2 = \frac{r_2 \nu_2}{2} \right\} \\ &\simeq \left\{ (q_1, p_1) \;\middle|\; q_1^2 + p_1^2 = \frac{r_1 \nu_1}{2} \right\} \times \left\{ (q_2, p_2) \;\middle|\; q_2^2 + p_2^2 = \frac{r_2 \nu_2}{2} \right\}. \end{aligned}$$

Thus the fibers are of the form claimed in the statement and sketched in Figure 2.1. Moreover, we calculate

$$Dh|_{(q,p)} = \begin{pmatrix} \nu_1 q_1 & 0 & \nu_1 p_1 & 0 \\ 0 & \nu_2 q_2 & 0 & \nu_2 p_2 \end{pmatrix}$$

and find $\mathrm{rk}(Dh|_{(q,p)}) = 0$ precisely for $(q, p) = (0, 0, 0, 0)$. Moreover, $\mathrm{rk}(Dh|_{(q,p)}) = 1$ if and only if $(q_1, p_1) = 0$ or $(q_2, p_2) = 0$ but not $(q, p) = (0, 0, 0, 0)$. In all other cases, $\mathrm{rk}(Dh|_{(q,p)}) = 2$. For fibers, this implies that $h^{-1}(0, 0) = \{(0, 0, 0, 0)\}$ is a singular point of rank zero, i.e., a fixed point. Moreover, $h^{-1}(0, r_2)$ and $h^{-1}(r_1, 0)$ with $r_1, r_2 > 0$ are singular fibers or rank one. $h^{-1}(r_1, r_2)$ with $r_1, r_2 > 0$ are fibers of rank two, thus regular ones. □

Thus the fiber over the (zero dimensional) vertex of the image of the momentum map of the uncoupled harmonic oscillator consists of a (zero dimensional) fixed point, the fibers over the (one dimensional) edges of the image of the momentum map are (one dimensional) circles, and the fibers over the (2-dimensional) interior are 2-tori. We will see in Theorem 2.41 (Local normal form) and Theorem **??** (Delzant) that this is no coincidence.

If the fibers of an integrable system (M, ω, h) are not connected, then the *leaf space* (given by collapsing each connected component to a point), differs from the image of the momentum map $h(M)$ which causes many difficulties. Therefore one often imposes conditions to insure that the fibers are connected. For more details on the leaf space, see for example [**Hohloch & Sabatini & Sepe & Symington**] and the references therein.

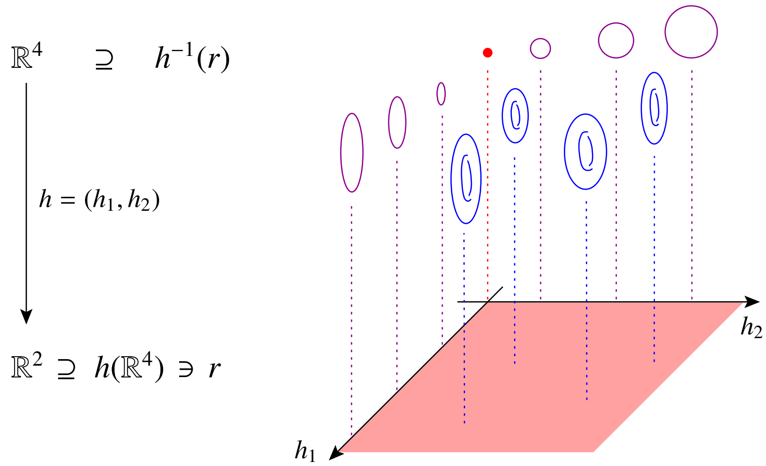


Figure 2.1. The Liouville fibration of the uncoupled harmonic oscillator: The image of the momentum map is the positive quadrant. The fiber over the origin is a fixed point. The fibers over the edges minus the origin are circles, and the fibers over the interior are 2-tori.

Let us now investigate if the Liouville foliation also carries some symplectic properties.

Definition 2.5. *Let (M, ω) be a symplectic manifold and $N \subseteq M$ a submanifold.*

1) N is **isotropic** *if $\omega_p(u, v) = 0$ for all $p \in N$ and all $u, v \in T_pN$, i.e., ω vanishes along N.*
2) N is **Lagrangian** *if N is isotropic and* $\dim N = \frac{1}{2} \dim M$.

Recall that Theorem A.26 (Implicite functions) implies that a regular fiber or a regular leaf is a submanifold whose dimension equals half of the dimension of the underlying symplectic manifold.

Lemma 2.6. *Let (M, ω, h) be a completely integrable system. Then all fibers and leafs of the Liouville foliation are isotropic where ever a resonable tangent space can be defined. In addition, all regular fibers are even Lagrangian submanifolds.*

Proof. Let $h = (h_1, \dots, h_n)$ and $x \in h^{-1}(r)$. The tangent space $T_x(h^{-1}(r))$ coincides with $\mathrm{Span}_{\mathbb{R}}\{X^{h_1}(x), \dots, X^{h_n}(x)\}$. We compute

$$\omega_x(X^{h_i}(x), X^{h_j}(x)) = -\{h_i, h_j\} = 0 \quad \forall\, 1 \leq i, j \leq n.$$

Thus ω vanishes on $h^{-1}(r)$. If r is a regular value then, by Theorem A.26 (Implicite functions), $h^{-1}(r)$ is a submanifold of dimension $\frac{1}{2}\dim M$ and thus not just isotropic but even Lagrangian. □

For this reason, Liouville foliations can be seen as 'singular Lagrangian fibrations'. Theorem 2.7 (Arnold-Liouville) will refine this to 'Lagrangian 2-torus fibrations' for connected compact regular fibers.

2.2. The Arnold-Liouville theorem

Recall from ODE theory that a **regular point of a vector field** is a point where the vector field does not vanish. According to Theorem A.16 (Flow box), the flow of an autonomous vector field can be 'straightened out' in the neighbourhood of a regular point as sketched in Figure 2.2.

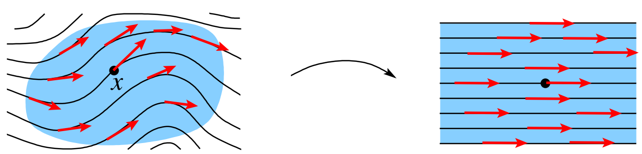


Figure 2.2. The 'straightening out' of the flow of a vector field near a regular point by means of a coordinate transformation as described in Theorem A.16 (Flow box).

Given a completely integrable system $(M, \omega, h = (h_1, \dots, h_n))$, one is naturally interested in a coordinate change like in Theorem A.16 (Flow box) that 'straightens out' the flows of $h_1, \dots h_n$ simultaneously as sketched in Figure 2.3 while respecting the integrability condition $\{h_i, h_j\} = 0$ for all $1 \leq i, j \leq n$, i.e., the straightening procedure should be (locally) fiber preserving. This will be achieved in Lemma 2.22 (Liouville coordinates) and its corollary Corollary 2.23 (Integrable flow box theorem).

In fact, we will see that we can even do better by *not just locally* straightening out the combined flows of a completely integrable system, but by actually straightening out a *whole neighbourhood of a fiber in a fiber preserving way* as sketched in Figure 2.4. This is achieved by the so-called *Arnold-Liouville theorem* (see Theorem 2.7 (Arnold-Liouville)) that in fact also comprises the work of several other mathematicians, in particular [**Liouville**] and [**Arnold & Avez**] important contributions by [**Mineur 1936**], [**Mineur 1937**], [**Jost**], and [**Markus & Meyer**].

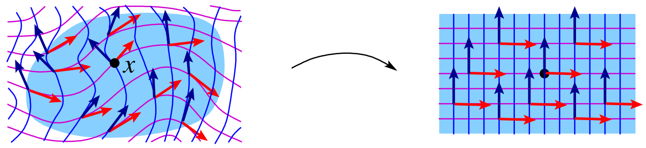


FIGURE 2.3. Simultaneously 'straightening out' of the flows of a completely integrable system locally near a regular point of the integrable system is achieved by means of Liouville coordinates in Corollary 2.23 (Integrable flow box theorem).

THEOREM 2.7 (**Arnold-Liouville**). *Let $(M, \omega, h = (h_1, \ldots, h_n))$ be a completely integrable system and $r \in \mathbb{R}^n$ a regular value. If $h^{-1}(r)$ is compact and connected, then*

1) $h^{-1}(r)$ is an embedded n-torus.

2) There exists
- *open sets $D, E \subseteq \mathbb{R}^n$ with $0 \in D$ and $r \in E$,*
- *an open neighbourhood $U := \bigcup_{\rho \in E} h^{-1}(\rho) \subseteq M$ of $h^{-1}(r)$,*
- *a diffeomorphism $\varphi : \mathbb{T}^n \times D \to U$,*
- *a diffeomorphism $\mu : E \to D$ with $\mu(r) = 0$*

such that
- *$\varphi^*\omega = -\sum_{i=1}^n dq_i \wedge dp_i$, i.e., φ is a symplectomorphism to the standard model,*
- *$\mu \circ h \circ \varphi : \mathbb{T}^n \times D \to D$ satisfies $(\mu \circ h \circ \varphi)(q, p) = p$.*

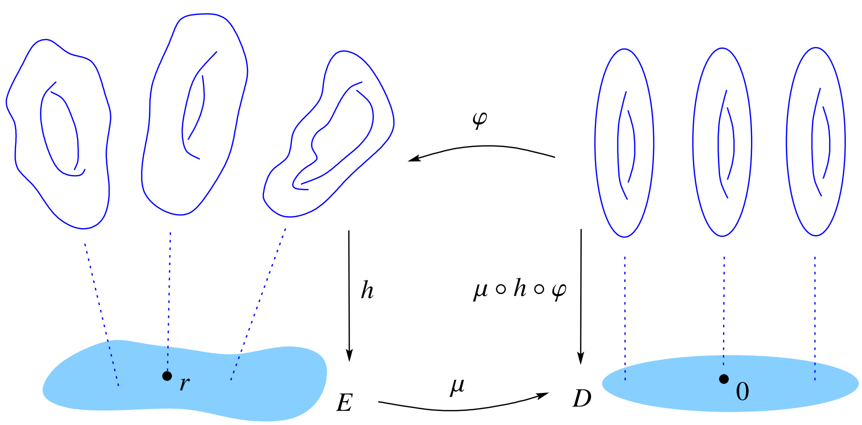


FIGURE 2.4. The fiber preserving 'straightening transformation' of the flow of a completely integrable system as described in Theorem 2.7 (Arnold-Liouville).

We require $h^{-1}(r)$ *compact and connected* in Theorem 2.7 (Arnold-Liouville) to reduce the complexity of the statement:

- Dropping the compactness requirement will admit the fiber to be diffeomorphic to a cylinder $T^k \times \mathbb{R}^{n-k}$ for some $0 \le k \le n$ instead of the torus $\mathbb{T}^n$. For a proof, see [**Cushman & Bates**].
- Dropping the connectedness requirement will allow the fiber to have more than one connected component.

Without copmpactness and connectivity of the fiber, we would have to deal e.g. with fibers consisting of up to infinitely many connected components and where each component could be of the form $T^k \times \mathbb{R}^{n-k}$ for some (maybe varying) $0 \le k \le n$.

Example 2.8. *The mathematical pendulum* $(\mathbb{R}^2, dp \wedge dq, h)$ *given by* $h : \mathbb{R}^2 \to \mathbb{R}$, $h(q, p) = \frac{1}{2}p^2 + \lambda \cos(q)$ *yields the Hamiltonian system* $q' = p$ *and* $p' = -\lambda \sin(q)$. *It contains fibers that consist of infinitely many circles as well as fibers that consists of two copies of the real line.*

The intuition for turning a completeley integrable system (M, ω, h) into the 'semilocal normal form'

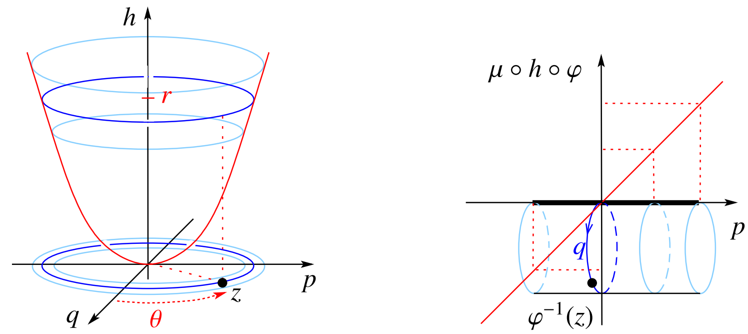
$$(\mathbb{T}^n \times D,\ -\sum_{k=1}^{n} dq_k \wedge dp_k,\ \mu \circ h \circ \varphi)$$

given by Theorem 2.7 (Arnold-Liouville) is sketched and explained in Figure 2.5.

In order to simplyfy a completely integrable system considerably, it is actually enough to consider $h \circ \varphi$ instead of the full semilocal normal form $\mu \circ h \circ \varphi$ given by Theorem 2.7 (Arnold-Liouville):

Corollary 2.9 (**Action-angle coordinates**). *Let* (M, ω, h) *satisfy the hypotheses of Theorem 2.7 (Arnold-Liouville) and let* φ *and* μ *be as stated in Theorem 2.7 (Arnold-Liouville).*

1) $\mathcal{H} := (\mathcal{H}_1, \ldots, \mathcal{H}_n) := h \circ \varphi : \mathbb{T}^n \times D \to \mathbb{R}^n$ *depends solely on the* **action variables** $p = (p_1, \ldots, p_n)$ *and is totally independent of the* **angle variables** $q = (q_1, \ldots, q_n)$, *i.e., we have* $\mathcal{H}(p, q) = \mathcal{H}(p)$. *Thus the concatenation with* μ *only adjusts the values of* h *to yield the linear function* $\mu \circ h \circ \varphi(q, p) = p$ *but has no influence on the dependency on* p *or* q.



(a) (b)

FIGURE 2.5. Let (M, ω, h) be a completely integrable system that satisfies the hypothesis of Theorem 2.7 (Arnold-Liouville). **(a)** Intuition for Theorem 2.7 (Arnold-Liouville): $z \in M$ is completely determined by its energy level r and the angle θ. **(b)** The local model given in Theorem 2.7 (Arnold-Liouville) near the fiber containing z: The role of θ from (a) is taken over by $q \in \mathbb{T}^n$ and h is transformed to $\mu \circ h \circ \varphi : \mathbb{T}^n \times D \to D$ given by $\mu \circ h \circ \varphi(q, p) = p$. Thus the role of r is taken over by p as value of the new Hamiltonian $\mu \circ h \circ \varphi$.

2) *The Hamiltonian vector field $X^{\mathcal{H}_i}(q, p)$ has $\partial_{p_j}\mathcal{H}_i(q, p) = \partial_{p_j}\mathcal{H}_i(p)$ as jth entry for $1 \leq j \leq n$ while all other entries vanish, briefly $X^{\mathcal{H}_i} = (\partial_p \mathcal{H}_i(p), 0)^T$. Thus $\mathcal{H}_i$ induces the Hamiltonian system*

$$q'_j = \partial_{p_j}\mathcal{H}_i(p), \qquad p'_j = 0 \qquad \text{for } 1 \leq j \leq n.$$

The components of a Hamiltonian solution $(q(t_i), p(t_i))$ of $\mathcal{H}_i$ are

$$t_i \mapsto p_j(t_i) = p_j(0),$$
$$t_i \mapsto q_j(t_i) = q_j(0) + t_i \partial_{p_j}\mathcal{H}_i(p_j(t_i))$$

for $1 \leq j \leq n$. Hence the Hamiltonian flow of $\mathcal{H}_i$ is given by

$$\Phi^{\mathcal{H}_i}_{t_i}(q, p) = (q + t_i \partial_p \mathcal{H}_i(p), p)$$

Thus the flow is linear and leaves each torus $\mathbb{T}^n \times \{p\} \subset \mathbb{T}^n \times D$ invariant. The coefficient $\partial_p \mathcal{H}_i(p) =: \nu_i(p)$ is called **frequency** *of the flow $\Phi^{\mathcal{H}_i}$ and depends only on p.*

3) In the semilocal normal form $\tilde{\mathcal{H}}(q,p) := \mu \circ h \circ \varphi(q,p) = p$, the Hamiltonian vector field and flow simplify even more: In $X^{\tilde{\mathcal{H}}_i}$, the ith entry equals 1 whereas all others vanish. Thus the components of a Hamiltonian solution $(q(t_i), p(t_i))$ of $\tilde{\mathcal{H}}_i$ are

$$\begin{aligned} t_i &\mapsto q_i(t_i) = q_i(0) + t_i, & t_i &\mapsto p_i(t_i) = p_i(0), \\ t_i &\mapsto q_j(t_i) = q_j(0), & t_i &\mapsto p_j(t_i) = p_j(0) \qquad \forall\, j \neq i \end{aligned}$$

with $\Phi_{t_i}^{\tilde{\mathcal{H}}_i}(q,p) = (q + t_i e_i, p)$ where e_i is the ith standard basis vector of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Proof. Assume that the notations and hypotheses of Theorem 2.7 (Arnold-Liouville). The Poisson commutativity $\{h_i, h_j\} = 0$ for all $1 \leq i, j \leq n$ together with Lemma 1.49 implies

$$0 = \{h_i, h_j\} \circ \varphi = \{h_i \circ \varphi, h_j \circ \varphi\} \overset{1.49}{=} \{\mathcal{H}_i, \mathcal{H}_j\} \qquad \forall\, 1 \leq i, j \leq n.$$

Moreover, concatenating $(\mathcal{H}_1, \dots, \mathcal{H}_n) = \mathcal{H} = h \circ \varphi$ with $\mu = (\mu_1, \dots, \mu_n)$ or, more precisely, the component functions μ_i yields together with Lemma 1.49

$$\{p_i, \mathcal{H}_j\} = \{\mu_i \circ \mathcal{H}, \mathcal{H}_j\} \overset{1.49}{=} 0 \qquad \forall\, 1 \leq i, j \leq n.$$

Thus, for $1 \leq j \leq n$, the flow of $\mathcal{H}_j$ preserves $\mathbb{T}^n \times \{p\}$. Moreover, we calculate

$$0 = \{p_i, \mathcal{H}_j\} = -\omega(X^{p_i}, X^{\mathcal{H}_j}) = dp_i(X^{\mathcal{H}_j}) = \partial_{q_i}\mathcal{H}_j \qquad \forall\, 1 \leq i, j \leq n$$

i.e., for all $1 \leq j \leq n$, the functions $\mathcal{H}_j$ do not depend on $q = (q_1, \dots, q_n)$. The formulas of $X^{\mathcal{H}_j}$ and of $\Phi^{\mathcal{H}_j}$ follow immediately. □

Notation 2.10. *In the situation of Corollary 2.9, we write by abuse of notation $X^{\mathcal{H}}(q,p) = \begin{pmatrix} \partial_p \mathcal{H}(p) \\ 0 \end{pmatrix}$ with*

$$q' = \partial_p \mathcal{H}(p) \quad \textit{and} \quad p' = 0$$

and

$$\Phi_{t_1}^{\mathcal{H}_1} \circ \dots \circ \Phi_{t_n}^{\mathcal{H}_n}(q,p) =: \Phi_t^{\mathcal{H}}(q,p) =: (q + t\partial_p \mathcal{H}(p), p)$$

where $\partial_p \mathcal{H}(p) = (\nu_1(p), \dots, \nu_n(p)) =: \nu(p)$ is the **frequency** *of the flow $\Phi^{\mathcal{H}}$.*

Now we look into the uniqueness of action-angle coordinates:

REMARK 2.11. *The action-angle coordinates in Corollary 2.9 are not unique. We may change the basis of the torus or apply certain translations and still have action-angle coordinates. More precisely, let* $A \in GL(n, \mathbb{Z})$ *with* $\det A = \pm 1$, $c \in \mathbb{R}$ *and* $w : D \to \mathbb{R}$ *a smooth function. Then*

$$\begin{cases} \tilde{q} := A(q + \partial_p w(p)), \\ \tilde{p} := (A^T)^{-1} p + c \end{cases}$$

are also action-angle coordinates.

We will encounter obstructions to the global existence of action-angle variables in Section **??**. For a detailed analysis 'how globally' action-angle coordinates can be defined, we refer the interested reader to [**Duistermaat**].

Now we will see where the name *action-angle* coordinates actually originates from.

REMARK 2.12. *Assume the situation of Theorem 2.7 (Arnold-Liouville), recall* $U = \varphi(\mathbb{T}^n \times D)$ *and* $\omega|_U = \left(\varphi^{-1}\right)^* \omega_{st}$ *with* $\omega_{st} = -\sum_{k=1}^n q_k \wedge dp_k$ *and* $\sigma_{st} = \sum_{k=1}^n p_k dq_k$ *on* $\mathbb{T}^n \times D$.

1) *Since* $\omega_{st} = d\sigma_{st}$ *is exact its pullback*
$$\omega|_U = (\varphi^{-1})^* \omega_{st} = (\varphi^{-1})^* d\sigma_{st} = d((\varphi^{-1})^* \sigma_{st}) =: d\sigma$$
is exact, too.
2) *Assume that* $\gamma_1, \ldots, \gamma_n$ *are loops that form a basis of the first homology class* $H_1(h^{-1}(r) \cap U, \mathbb{Z})$ *with* $\mu(r) = p$. *Then the 'action variables'* $(p_1, \ldots, p_n) = p = \mu \circ h \circ \varphi(q, p)$ *are given by the 'action integral'*
$$p_i = \int_{\gamma_i} \sigma.$$
Since the variable $q = (q_1, \ldots, q_n)$ *describes 'angles' in the torus* $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$, *they are referred to as 'angle variables'.*

2.3. Generating functions and the maximal number of independent integrals

This section is quite technical since it deals with the construction of certain local symplectic coordinates that do not just turn the symplectic form into the standard form like Theorem 1.34 (Darboux) does but also 'straighten' locally the flow of an integrable system. These coordinates are essential for the proof of Theorem 2.7 (Arnold-Liouville). We follow hereby the ideas in [**Hofer & Zehnder**, Appendix A.1].

We start with a 'scrambled' symplectic change of coordinates: given a symplectic change of coordinates $(\xi, \eta) \mapsto (x, y)$, we will see that we can recover from ξ and y the coordinates η and x and that the recovered coordinates locally can be expressed as partial derivatives of a certain function that depends only on the given coordinates. More precisely:

Lemma 2.13 (**Generating functions I**). *Consider a symplectomorphism* $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ *and think of it as change of coordinates from* $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ *to*

$$\psi(\xi, \eta) =: (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) =: (x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

1) *If* $\det(\partial_\xi \mathfrak{a}) \neq 0$ *in* $(\hat{\xi}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *then there exists an open neighbourhood* U *of* $(\mathfrak{a}(\hat{\xi}, \hat{\eta}), \hat{\eta}) =: (\hat{x}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and a smooth function* $W : U \to \mathbb{R}$ *such that*

$$\xi = \partial_\eta W(x, \eta) \quad \textit{and} \quad y = \partial_x W(x, \eta) \quad \forall\, (x, \eta) \in U$$

with $\det(\partial_x \partial_\eta W) \neq 0$ *in* U.

2) *For all* $(\hat{x}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and all open neighbourhoods* U *of* $(\hat{x}, \hat{\eta})$ *and all smooth functions* $W : U \to \mathbb{R}$ *with* $\det(\partial_x \partial_\eta W) \neq 0$ *in* U, *the map defined on* U *given by*

$$(x, \eta) \mapsto (\xi, y) := (\partial_\eta W(x, \eta), \partial_x W(x, \eta))$$

is a symplectomorphism.

Such a function W *is usually referred to as* **generating function**.

Before we prove this statement, let us discuss it a bit: From the point of view of *geometry* the recovered coordinates are the components of the 1-form $dW = \partial_\eta W d\eta + \partial_x W dx$ which is by construction exact and therefore also closed. From the point of view of *analysis*, the expression $(\xi, y) = (\partial_\eta W(x, \eta), \partial_\eta W(x, \eta))$ shows that the $n+n = 2n$ components of a symplectic map are not 'functionally independent' of each other in the sense that they both satisfy a functional equation originating from the same function W.

Generating functions seems to be very abstract concept, but, in certain situations, it is not impossible to write a generating function down explicitly:

Example 2.14. *The Euclidean scalar product is a generating function for the identity* $\mathrm{Id} : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$.

Proof. Consider the identity as map $\mathrm{Id} : (\mathbb{R}^n \times \mathbb{R}^n, \omega_{st}) \to (\mathbb{R}^n \times \mathbb{R}^n, \omega_{st})$ with

$$(\xi, \eta) \mapsto \mathrm{Id}(\xi, \eta) =: (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) =: (x, y)$$

where in fact $(\xi, \eta) = \mathrm{Id}(\xi, \eta) = (x, y)$. Note that $\partial_\xi \mathfrak{a}(\xi, \eta) = 1 \neq 0$ and define the map

$$W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad W(x, \eta) := \langle x, \eta \rangle_{eu} = \sum_{k=1}^{n} x_k \eta_k$$

where $x = (x_1, \ldots, x_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$. Now compute the partial derivatives $\partial_{x_k} W(x, \eta) = \eta_k$ and $\partial_{\eta_k} W(x, \eta) = x_k$ which yield

$$\xi = x = \partial_\eta W(x, \eta) \qquad \text{and} \qquad y = \eta = \partial_x W(x, \eta).$$

Note that $\partial_{x_i} \partial_{\eta_j} W(x, \eta) = \delta_{ij}$ and thus $\det(\partial_x \partial_\eta W) = \det(\mathrm{Id}) = 1 \neq 0$. □

More generally, we have

Example 2.15. *Function of the form $W(x, \eta) = \langle x, \eta \rangle_{eu} + w(x, \eta)$, where w and its first and second derivatives Dw and $D^2 w$ have sufficiently small norm, are generating functions for symplectic mappings that are sufficiently close to the identity. Conversely, all symplectic mappings close to the identity can locally be described by means of generating functions of the form $W(x, \eta) = \langle x, \eta \rangle_{eu} + w(x, \eta)$.*

Proof. Left as an exercise to the reader. □

The analytic reason behind the proof of Lemma 2.13 (Generating functions I) is that $\det(\partial_\xi \mathfrak{a}) \neq 0$ allows us to use Theorem A.23 (Implicite functions) to solve the equation $\mathfrak{a}(\xi, \eta) = x$ for ξ, i.e., it allows us to write $\xi = \alpha(x, \eta)$ for a suitable function α.

Proof of Lemma 2.13 (Generating functions I). Consider $(\mathbb{R}^{2n}, \omega_{st})$ with coordinates $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$. Let $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ be a symplectomorphism and consider ψ as (symplectic) change of coordinates

$$(\xi, \eta) \mapsto \psi(\xi, \eta) =: (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) =: (x, y).$$

Step 1 (finding suitable local coordinates): By assumption, we have $\det(\partial_\xi \mathfrak{a}(\hat{\xi}, \hat{\eta})) \neq 0$ in $(\hat{\xi}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$ and denote $\mathfrak{a}(\hat{\xi}, \hat{\eta}) =: \hat{x}$. Thus, according to Theorem A.23 (Implicite functions) applied to $\mathfrak{a} : \mathbb{R}^{2n} \to \mathbb{R}^n$, there exists a neighbourhood $L \subset \mathbb{R}^n$ of $\hat{\eta}$ and a neighbourhood $K \subseteq \mathbb{R}^n$ of $\hat{\xi}$ and a smooth $\alpha : L \to K$ such that $\alpha(\eta) = \xi$ if and only if $(\xi, \eta) \in K \times L$ and $\mathfrak{a}(\xi, \eta) = \hat{x}$. Since $\det(\partial_\xi \mathfrak{a}(\hat{\xi}, \hat{\eta})) \neq 0$ is an open condition we may vary $(\hat{\xi}, \hat{\eta})$ (and thus $\hat{x}$) slightly and obtain locally a smooth α that depends now additionally on x solving $x = \mathfrak{a}(\xi, \eta)$ locally for ξ as $\xi = \alpha(x, \eta)$. We now get a local diffeomorphism φ and a smooth function β via

$$\varphi(x, \eta) = (\alpha(x, \eta), \eta) \quad \text{and} \quad \beta(x, \eta) := \mathfrak{b}(\alpha(x, \eta), \eta).$$

Step 2 (finding the generating function): Consider $\mathbb{R}^{4n}$ with coordinates (ξ, η, x, y) and define embeddings i and j on suitable subsets U_i, $U_j \subseteq \mathbb{R}^{2n}$ via

$$\begin{aligned} &i : U_i \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{4n}, \quad j : U_j \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{4n} \\ &\ i(x, \eta) := (\alpha(x, \eta), \eta, x, \beta(x, \eta)), \\ &\ j(\xi, \eta) := (\xi, \eta, \mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) = (\xi, \eta, \psi(\xi, \eta)). \end{aligned}$$

We have $i = j \circ \varphi$ since $\beta(x, \eta) = \mathfrak{b}(\alpha(x, \eta), \eta)$. Now we want to show that the pullback of 1-form $\sigma := ydx + \xi d\eta := \sum_{k=1}^n y_k dx_k + \xi_k d\eta_k$ under i is closed. Herefore we calculate

$$\begin{aligned} d\sigma &= dy \wedge dx + d\xi \wedge d\eta = dy \wedge dx - d\eta \wedge d\xi, \\ d(i^*\sigma) &= i^* d\sigma = \varphi^*(j^*(d\sigma)) = \varphi^*(\psi^*\omega_{st} - \omega_{st}). \end{aligned}$$

Since ψ is a symplectomorphism, we have $\psi^*\omega_{st} - \omega_{st} = 0$ and, since φ is a diffeomorphism, we conclude $d(i^*\sigma) = 0$, i.e., $i^*\sigma$ is closed. Therefore Lemma A.12 (Poincaré) implies the local existence of a 0-form (= function) W such that $i^*\sigma = dW$. Writing this in coordinates (x, η), we get

$$\begin{aligned} dW &= \sum_{k=1}^n \partial_{x_k} W(x, \eta) dx_k + \partial_{\eta_k} W(x, \eta) d\eta_k, \\ i^*\sigma &= \sum_{k=1}^n \beta_k(x, \eta) dx_k + \alpha_k(x, \eta) d\eta_k \end{aligned}$$

so that $dW = i^*\sigma$ implies

$$\xi = \alpha(x, \eta) = \partial_\eta W(x, \eta) \quad \text{and} \quad y = \beta(x, \eta) = \partial_x W(x, \eta).$$

□

The identity is a symplectomorphism and satisfies $\det(\partial_\xi \mathfrak{a}) \neq 0$ in the notation of Lemma 2.13 (Generating functions I) as shown in the proof of Example 2.14. The rotation $(\xi, \eta) \mapsto (\eta, -\xi) =: (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta))$ is also a symplectomorphism but has $\det(\partial_\xi \mathfrak{a}) = 0$. Since the rotation is bijective $\det(\partial_\eta \mathfrak{a})$ cannot also vanish, it makes sense to adapt Lemma 2.13 (Generating functions I) to the case $\det(\partial_\eta \mathfrak{a}) \neq 0$. Roughly, this means in the notation of the proof of Lemma 2.13 (Generating functions I):

- If $\det(\partial_\xi \mathfrak{a}(\xi, \eta)) \neq 0$ then solve $\mathfrak{a}(\xi, \eta) = x$ for ξ via $\xi = \alpha(\eta, x)$.
- If $\det(\partial_\eta \mathfrak{a}(\xi, \eta)) \neq 0$ then solve $\mathfrak{a}(\xi, \eta) = x$ for η via $\eta = \alpha(\xi, x)$.

Lemma 2.16 (**Generating functions II**). *Consider a symplectomorphism $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ and think of it as change of coordinates from $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ to*

$$\psi(\xi, \eta) =: \big(\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)\big) =: (x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

1) *If* $\det(\partial_\eta \mathfrak{a}) \neq 0$ *in* $(\hat{\xi}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *then there exists an open neighbourhood* U *of* $(\hat{\xi}, \mathfrak{a}(\hat{\xi}, \hat{\eta})) =: (\hat{\xi}, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and a smooth function* $V : U \to \mathbb{R}$ *such that*
$$\eta = -\partial_\xi V(\xi, x) \quad \text{and} \quad y = \partial_x V(\xi, x) \quad \forall\, (\xi, x) \in U$$
with $\det(\partial_x \partial_\xi V) \neq 0$ *in* U.
2) *For all* $(\hat{\xi}, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and all open neighbourhoods* U *of* $(\hat{\xi}, \hat{x})$ *and all smooth functions* $V : U \to \mathbb{R}$ *with* $\det(\partial_x \partial_\xi V) \neq 0$ *in* U, *the map defined on* U *given by*
$$(\xi, x) \mapsto (\eta, y) := \big(-\partial_\xi V(\xi, x), \partial_x V(\xi, x)\big)$$
is a symplectomorphism.

Such a function V *is usually also referred to as* **generating function**.

Proof. Consider the 1-form $\sigma = \sum_{k=1}^n y_k dx_k - \eta_k d\xi_k = y dx - \eta d\xi$ and proceed as in the proof of Lemma 2.13 (Generating functions I). □

Note that, for the identity, we only can apply Lemma 2.13 (Generating functions I) but not Lemma 2.16 (Generating functions II). For the rotation $(\xi, \eta) \mapsto (\eta, -\xi)$, it is precisely the other way around.

Example 2.17. *Let* $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ *be symplectic and write* $\psi(\xi, \eta) = \big(\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)\big) =: (x, y)$. *Let* $\det(\partial_\xi \mathfrak{a}) \neq 0$ *and assume that* $\mathfrak{a}(\xi, \eta) = \mathfrak{a}(\xi)$ *does not depend on* η. *Then* ψ *is in fact of the form*

$$\begin{aligned} x &= \mathfrak{a}(\xi), \\ y &= \big((\partial_\xi \mathfrak{a}(\xi))^T\big)^{-1} (\eta + \partial_\xi u(\xi)) \end{aligned}$$

for a smooth function $u : \mathbb{R}^n \to \mathbb{R}$. *If* $\mathfrak{a} = \mathrm{Id}$ *then* $y = \eta + \partial_\xi u(\xi)$.

Proof. Write ψ by means of a generating function U as

$$\mathfrak{a}(\xi) = x = \partial_y U(\xi, y) \quad \text{and} \quad \mathfrak{b}(\xi, \eta) = \eta = \partial_\xi U(\xi, y).$$

Now determine U as precise as possible in terms of $\mathfrak{a}$: In coordinates $y = (y_1, \ldots, y_n)$, the first equation reads $\mathfrak{a}_k(\xi) = \partial_{y_k} U(\xi, y)$ for $1 \leq k \leq n$.

Integrating w.r.t. y_k now yields

$$U(\xi,\eta) = \int \mathfrak{a}_k(\xi)\, dy_k = \mathfrak{a}_k(\xi) y_k + u_k(\xi) \quad \text{for } 1 \leq k \leq n$$

with $u_k : \mathbb{R}^n \to \mathbb{R}$. Thus, given any smooth function $u : \mathbb{R}^n \to \mathbb{R}$, setting $U(\xi,\eta) := \langle \mathfrak{a}(\xi), y\rangle_{eu} - u(\xi)$ is the most general choice that still satisfies $\mathfrak{a}(\xi) = \partial_y U(\xi, y)$. This yields $\eta = \partial_\xi U(\xi, y) = (\partial_\xi \mathfrak{a}(\xi))^T y - \partial_\xi u(\xi)$. Solving for y yields the desired expression. □

Recall from Lemma 1.49 that concatenation with a symplectomorphism leaves the Poisson bracket invariant.

LEMMA 2.18. *Consider* $(\mathbb{R}^{2n}, \omega_{st})$ *with standard coordinates* $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ *and Poisson bracket* $\{\cdot,\cdot\}$ *induced by* ω_{st}*.*

1) For the coordinate functions $q_1, \ldots, q_n, p_1, \ldots, p_n$ *given by* $(q, p) \mapsto q_i$ *and* $(q, p) \mapsto p_i$*, we calculate*

$$\begin{aligned} \{q_i, q_j\} &= 0 = \{p_i, p_j\} && \forall\, 1 \leq i, j \leq n, \\ \{q_i, p_i\} &= 1 && \forall\, 1 \leq i \leq n, \\ \{q_i, p_j\} &= 0 && \forall\, i \neq j,\ 1 \leq i, j \leq n. \end{aligned}$$

2) Let $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ *be a symplectomorphism and write* $\psi = (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}_1, \ldots, \mathfrak{b}_n)$*. Then*

$$\begin{aligned} \{\mathfrak{a}_i, \mathfrak{a}_j\} &= 0 = \{\mathfrak{b}_i, \mathfrak{b}_j\} && \forall\, 1 \leq i, j \leq n, \\ \{\mathfrak{a}_i, \mathfrak{b}_i\} &= 1 && \forall\, 1 \leq i \leq n, \\ \{\mathfrak{a}_i, \mathfrak{b}_j\} &= 0 && \forall\, i \neq j,\ 1 \leq i, j \leq n. \end{aligned}$$

PROOF. **1)** Simple calculation.
2) Write $\mathfrak{a}_i = q_i \circ \psi$ and $\mathfrak{b}_i = p_i \circ \psi$ and use Lemma 1.49 and **1)**. □

This leads to the following important conclusion:

COROLLARY 2.19. *In symplectic standard coordinates, the first n components* $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \mathfrak{a}$ *of a symplectomorphism* $\psi = (\mathfrak{a}, \mathfrak{b})$ *mutually Poisson commute. Since they also satisfy* $\mathrm{rk}(D\mathfrak{a}) = n$*, they form a completely integrable system. The same is true for the last n components* $\mathfrak{b} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$*.*

Now we want to investigate the converse, i.e., if one can always extend n Poisson commuting functions $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ with $\mathrm{rk}(D\mathfrak{a}) = n$ to a symplectomorphism.

THEOREM 2.20 (**Liouville**). *Consider $(\mathbb{R}^{2n}, \omega_{st})$ with standard coordinates $(\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$ and let $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}^n$ be smooth with $\mathrm{rk}(D\mathfrak{a}) = n$ and $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ for all $1 \le i, j \le n$. Then we can extend $\mathfrak{a}$ locally to a symplectomorphism ψ whose first n components coincide with $\mathfrak{a}$.*

PROOF. *Step 1:* W.l.o.g. assume that $\det(\partial_\xi \mathfrak{a}) \neq 0$ and solve $x = \mathfrak{a}(\xi, \eta)$ as in the proof of Lemma 2.13 (Generating functions I) locally for ξ via $\xi = \alpha(x, \eta)$ with a suitable function α. We want to show that α is locally of the form $\alpha(x, \eta) = \partial_\eta W(x, \eta)$ for a smooth function W and that this W has the properties of a generating function. If this is the case, then part 2) in Lemma 2.13 (Generating functions I) describes how to extend α locally to a symplectomorphism. Taking the inverse delivers a symplectomorphism whose first n coordinate functions are given by $x = \mathfrak{a}(\xi, \eta)$, i.e., we have extended $\mathfrak{a}$ locally to a symplectomorphism.
Step 2: Since the second partial derivatives of a smooth function commute, the matrix $\partial_\eta \alpha$ has to be symmetric to allow for the existence of any smooth W with $\alpha(x, \eta) = \partial_\eta W(x, \eta)$. On the other hand, writing Lemma A.12 (Poincaré) in local coordinates shows that symmetry of $\partial_\eta \alpha$ is not only necessary but also sufficient for the existence of such a W.
Step 3: It suffices to show that the $(n \times n)$-matrix $\partial_\eta \alpha$ is symmetric. We calculate

$$\begin{aligned}
& 0 = \{\mathfrak{a}_i, \mathfrak{a}_j\} = \partial_\xi \mathfrak{a}_i \, (\partial_\eta \mathfrak{a}_j)^T - \partial_\eta \mathfrak{a}_i \, (\partial_\xi \mathfrak{a}_j)^T \\
\Leftrightarrow \quad & \partial_\xi \mathfrak{a}_i \, (\partial_\eta \mathfrak{a}_j)^T = \partial_\eta \mathfrak{a}_i \, (\partial_\xi \mathfrak{a}_j)^T \\
\Leftrightarrow \quad & (\partial_\eta \mathfrak{a}_j)^T \, ((\partial_\xi \mathfrak{a}_j)^T)^{-1} = (\partial_\xi \mathfrak{a}_i)^{-1} \, \partial_\eta \mathfrak{a}_i \\
\Leftrightarrow \quad & ((\partial_\xi \mathfrak{a}_j)^{-1} \, \partial_\eta \mathfrak{a}_j)^T = (\partial_\xi \mathfrak{a}_i)^{-1} \, \partial_\eta \mathfrak{a}_i
\end{aligned}$$

Thus the matrix $(\partial_\xi \mathfrak{a})^{-1} \, \partial_\eta \mathfrak{a}$ is symmetric. Moreover, differentiating the equation

$$\xi = \alpha(x, \eta) = \alpha(\mathfrak{a}(\xi, \eta), \eta)$$

w.r.t. ξ yields $\mathrm{Id} = \partial_\xi \xi = \partial_x \alpha \, \partial_\xi \mathfrak{a}$, implying $\partial_x \alpha = (\partial_\xi \mathfrak{a})^{-1}$. Differentiating w.r.t. η yields $0 = \partial_\eta \xi = \partial_\eta \alpha(x, \eta) = \partial_x \alpha \, \partial_\eta \mathfrak{a} + \partial_\eta \alpha$. Altogether, we get $\partial_\eta \alpha = -\partial_x \alpha \, \partial_\eta \mathfrak{a} = -(\partial_\xi \mathfrak{a})^{-1} \, \partial_\eta \mathfrak{a}$. We already showed the last term to be symmetric, thus so is the matrix $\partial_\eta \alpha$. □

Now we give an answer to the question how many functionally independent Poisson commuting functions we may find at most on $\mathbb{R}^{2n}$.

Corollary 2.21. *Let* $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n) : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}^n$ *and* $\mathfrak{a}_0 : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}$ *be smooth functions with* $\mathrm{rk}(D\mathfrak{a}) = n$ *and* $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ *for* $0 \leq i, j \leq n$. *Then* $\mathfrak{a}_0$ *can be expressed as a function of* $\mathfrak{a}_1, \dots, \mathfrak{a}_n$. *In particular, there are maximally n functionally independent, Poisson commuting functions on* $\mathbb{R}^{2n}$.

Proof. Consider $(\mathbb{R}^{2n}, \omega_{st})$ with standard coordinates (ξ, η) and $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n) : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}^n$ with $\mathrm{rk}(D\mathfrak{a}) = n$ and Poisson commuting components $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ for $0 \leq i, j \leq n$. Theorem 2.20 (Liouville) implies the existence of a local symplectomorphism ψ whose first n components are given by $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n)$, i.e., $\psi(\xi, \eta) = (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) = (x, y)$. We rewrite this as $(\mathfrak{a} \circ \psi^{-1})(x, y) = \mathfrak{a}(\xi, \eta) = x$ and set $f := \mathfrak{a}_0 \circ \psi^{-1}$. Then we get

$$0 = \{\mathfrak{a}_0, \mathfrak{a}_i\} \circ \psi^{-1} \overset{1.49}{=} \{\mathfrak{a}_0 \circ \psi^{-1}, \mathfrak{a}_i \circ \psi^{-1}\} = \{f, x_i\} \overset{??}{=} -\partial_{y_i} f$$

for all $1 \leq i \leq n$. Therefore f does not depend on $(y_1, \dots, y_n)$ but is a function solely depending on $(x_1, \dots, x_n) = x = (\mathfrak{a}_1, \dots, \mathfrak{a}_n)$. □

Now we strengthen Theorem 1.34 (Darboux).

Lemma 2.22 (**Liouville coordinates**). *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold and* $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n) : M \to \mathbb{R}^n$ *be smooth with* $\mathrm{rk}(D\mathfrak{a}) = n$ *and* $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ *for all* $1 \leq i, j \leq n$. *Then, for all* $z \in M$, *there exists an open neighbourhood* $U \subseteq M$ *of* z *and an open neighbourhood* $V \subseteq \mathbb{R}^{2n}$ *of* $0 \in \mathbb{R}^{2n}$ *and a diffeomorphism* $\psi : V \to U$ *such that*

- $\psi(0) = z$,
- $\psi^*\omega = \omega_{st}$,
- $(\mathfrak{a} \circ \psi)(q, p) = p$ *for all* $(q, p) \in V$.

Proof. Start with local coordinates as given by Theorem 1.34 (Darboux) and use Theorem 2.20 (Liouville) and Corollary 2.21 to tweak the change of coordinates into the desired form. □

In Liouville coordinates, the flow of $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ becomes linear, thus generalizing Theorem A.16 (Flow box) to n Poisson commuting flows (cf. Figure 2.3):

Corollary 2.23 (**Integrable flow box theorem**). *In the setting of Lemma 2.22 (Liouville coordinates), the flow* $\Phi_t^{\mathfrak{a}} := \Phi_{t_1}^{\mathfrak{a}_1} \circ \dots \circ \Phi_{t_n}^{\mathfrak{a}_n}$ *is* *given by $\Phi^{\mathfrak{a}}_t(\psi(q,p)) = \psi(q+t,p)$ for all $t = (t_1,\ldots,t_n)$ as long as (q,p) and $(q+t,p)$ lie in V.*

Proof. Let $t = (t_1,\ldots,t_n)$. The flow of $X^{\mathfrak{a}_i\circ\psi}$ is given by $\psi^{-1}\circ\Phi^{\mathfrak{a}_i}_{t_i}\circ\psi$ for all $1 \leq i \leq n$. Since $(\mathfrak{a}_i\circ\psi)(q,p) = p_i$, we compute $X^{\mathfrak{a}_i\circ\psi}(q,p) = e_i \in \mathbb{R}^{2n}$ where e_i is the ith unit vector for $1 \leq i \leq n$. Hence $\Phi^{\mathfrak{a}_i\circ\psi}_{t_i}(q,p) = (q,p) + t_ie_i$ and thus

$$\Phi^{\mathfrak{a}}_t(q,p) := \Phi^{\mathfrak{a}_1}_{t_1}\circ\cdots\circ\Phi^{\mathfrak{a}_n}_{t_n}(q,p) = (q+t,p)$$

with $t = (t_1,\ldots,t_n)$. □

2.4. The proof of Theorem 2.7 (Arnold-Liouville)

Theorem 2.7 (Arnold-Liouville) is a central theorem within the theory of integrable systems and therefore appears in almost every text book on integrable systems, see for example [**Arnold 1974**], [**Bolsinov & Fomenko**], [**Cushman & Bates**], [**Duistermaat**], [**Fassò**], [**Hofer & Zehnder**], [**Sepe & Vũ Ngọc**] etc. But since the details of the proof are somewhat lengthy and tedious, many authors opt for skipping them in favor of presenting a more concise (idea of) proof.

To our knowledge, [**Hofer & Zehnder**, Appendix A.1, A.2] gives a very detailed proof and we will use it as base for the proof of Theorem 2.7 (Arnold-Liouville) in the present section.

We split the proof of Theorem 2.7 (Arnold-Liouville) into two main parts:

I) We prove the first item of Theorem 2.7 (Arnold-Liouville) in Proposition 2.24.
II) We prove the second item of Theorem 2.7 (Arnold-Liouville) in a succession of statements that extend the existence of the desired map from a very local statement to a 'semiglobal' statement for a neighbourhood of the fiber $h^{-1}(r)$:
    1) Lemma 2.27 extends Lemma 2.22 (Liouville coordinates) to a fundamental domain of the lattice Γ defined in (2.25).
    2) Lemma 2.28 incorporates the periods of the lattice Γ into a coordinate change.
    3) Lemma 2.29 extends the lattice Γ to a lattice $\Gamma(y)$ with $\Gamma(0) = \Gamma$ for all values y in a neighbourhood of the regular value $r = 0 \in \mathbb{R}^n$ and shows the corresponding fibres to be n-tori.

4) Lemma 2.30 extends the local coordinate change from Lemma 2.27 in a way compatible with the lattices $\Gamma(y)$ on the nearby fibers.
5) The second item of Theorem 2.7 (Arnold-Liouville) is restated as Lemma 2.32 and proven.

We start with

Proposition 2.24. *Let $(M, \omega, h = (h_1, \dots, h_n))$ be a completely integrable system and $r \in \mathbb{R}^n$ a regular value. If $h^{-1}(r)$ is compact and connected, then $h^{-1}(r)$ is an embedded n-torus.*

Proof. Let $r \in \mathbb{R}^n$ be a regular value of h and $h^{-1}(r)$ compact and connected. Since all components $h_1, \dots, h_n$ of h mutually Poisson commute, their flows preserve the fibers of h. Thus the compactness of $h^{-1}(r)$ implies that the flow lines $t_j \mapsto \Phi^{h_j}_{t_j}(z)$ are defined on whole $\mathbb{R}$ for all $z \in h^{-1}(r)$ and all $1 \leq j \leq n$. This yields a well-defined $\mathbb{R}^n$-action

$$\mathbb{R}^n \times h^{-1}(r) \to h^{-1}(r), \quad (t = (t_1, \dots, t_n), z) \mapsto \Phi^h_t(z) := \Phi^{h_1}_{t_1} \circ \dots \circ \Phi^{h_n}_{t_n}(z).$$

Since solutions of smooth ODEs depend smoothly on their initial value conditions, the map

$$F^z : \mathbb{R}^n \to h^{-1}(r), \qquad F^z(t) := \Phi^h_t(z)$$

is smooth for all $z \in h^{-1}(r)$. Since $\operatorname{rk} Dh = n$ on $h^{-1}(r)$, Corollary 2.23 (Integrable flow box theorem) is at our disposal which implies that F^z is a local diffeomorphism for all $z \in h^{-1}(r)$. Thus its image is open and closed at the same time. Since the fiber $h^{-1}(r)$ is connected, the only subsets satisfying this are the empty set or $h^{-1}(r)$ itself, implying $F^z(\mathbb{R}^n) = h^{-1}(r)$. Thus F^z is surjective. But F^z cannot be injective: If F^z were injective then it would in fact be a diffeomorphism and since diffeomorphisms map compact sets to compact sets, mapping the noncompact $\mathbb{R}^n$ to the compact fiber $h^{-1}(r)$ would yield a contradiction. Thus F^z is an immersion (since $\operatorname{rk} DF^z|_t = n$ for all $t \in \mathbb{R}^n$) but no global diffeomorphism. Since F^z is not injective the isotropy group

$$\Gamma_z := \{t \in \mathbb{R}^n \mid \Phi^h_t(z) = z\}$$

of the $\mathbb{R}^n$-action on the fiber is nontrivial for all $z \in h^{-1}(r)$. The transitivity of the $\mathbb{R}^n$-action implies for all $z, \tilde{z} \in h^{-1}(r)$ that there is $s \in \mathbb{R}^n$ such that $\tilde{z} = \Phi^h_s(z)$. This yields

$$\begin{aligned} \Gamma_{\tilde{z}} &= \{\tilde{t} \in \mathbb{R}^n \mid \Phi^h_{\tilde{t}}(\tilde{z}) = \tilde{z}\} = \{\tilde{t} \in \mathbb{R}^n \mid \Phi^h_{\tilde{t}}(\Phi^h_s(z)) = \Phi^h_s(z)\} \\ &= \{\tilde{t} \in \mathbb{R}^n \mid \Phi^h_{\tilde{t}}(z) = z\} = \Gamma_z \\ &=: \Gamma \qquad (2.25) \end{aligned}$$

independent of the chosen foot point. Since DF^z has full rank for all $z \in h^{-1}(r)$ and all $t \in \mathbb{R}^n$, the points $t \in \mathbb{R}^n$ with $\Phi_t^h(z) = z$ are isolated in $\mathbb{R}^n$. Therefore $\Gamma \subset \mathbb{R}^n$ is a discrete subgroup. Hence there exists $k \in \{0, \dots, n\}$ and linearly independent vectors $\gamma_1, \dots, \gamma_k \in \mathbb{R}^n$ such that $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \dots, \gamma_k\}$. By definition, we have for all $\gamma \in \Gamma$

$$F^z(t + \gamma) = \Phi_{t+\gamma}^h(z) = \Phi_t^h(z) = F^z(t) \quad \forall\, t \in \mathbb{R}^n$$

and thus in particular $h^{-1}(r) \simeq \mathbb{R}^n/\Gamma \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}$. Since $h^{-1}(r)$ is compact but $\mathbb{T}^k \times \mathbb{R}^{n-k}$ not if $k < n$, we conclude $k = n$. Therefore $h^{-1}(r) \simeq \mathbb{R}^n/\Gamma \simeq \mathbb{T}^n$ is an n-torus. □

We remark

Definition 2.26. *Discrete subgroups of $\mathbb{R}^n$ are often called* **lattices**. *A set of linearly independent vectors spanning a lattice is called a* **basis of the lattice**. *The number of basis vectors is called the* **rank of the lattice**. *If $v_1, \dots, v_k$ is a basis of a lattice, then*

$$\left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in [0,1] \text{ for } 1 \le i \le k \right\}$$

is called a **fundamental domain** *of the lattice.*

Now let us start with the proof of the second claim in the statement of Theorem 2.7 (Arnold-Liouville).

W.l.o.g. we assume for the remainder of this section concerning the completely integrable system $(M, \omega, h = (h_1, \dots, h_n))$ with regular value $r \in \mathbb{R}^n$ and compact and connected fiber $h^{-1}(r)$ from the statement of Theorem 2.7 (Arnold-Liouville) that

$$r = 0 \in \mathbb{R}^n$$

In the rest of this section, we will be busy extending Liouville coordinates to larger and larger neighbourhoods until they fulfill the claim of Theorem 2.7 (Arnold-Liouville). The idea is similar to the proof of Theorem A.16 (Flow box) of ordinary differential equations, i.e., we will turn the first n coordinates of the given $2n$ 'space coordinates' into n time coordinates of the flow and replace the missing space coordinates by zero, thus effectively 'decoupling' space coordinates into half time and half space coordinates.

Lemma 2.27. *Let $(M, \omega, h = (h_1, \ldots, h_n)$ be a completely integrable system with regular value $0 \in \mathbb{R}^n$ and compact, connected fiber $h^{-1}(0)$. Let $\psi : V \to U$ be Liouville coordinates associated to (M, ω, h) with $z = \psi(0) \in h^{-1}(0)$. Then*

1) $$\theta : V \to M, \quad \theta(x, y) := \Phi^h_x(\psi(0, y))$$ *satisfies $\theta(x, y) = \Phi^h_x(\psi(0, y)) = \psi(x, y)$.*

2) $\theta(\cdot, 0)$ is defined on all of $\mathbb{R}^n$ with $\theta(\mathbb{R}^n \times \{0\}) = h^{-1}(0)$.

3) $\theta(x, y) \in h^{-1}(y)$ whenever $\theta(x, y)$ is defined.

4) There exists a small open neighbourhood $E \subseteq \mathbb{R}^n$ of $0 \in \mathbb{R}^n$ and a compact $K \subset \mathbb{R}^n$ containing a fundamental domain of the rank n lattice Γ from (2.25) such that θ is welldefined as map

$$\theta : K \times E \to M, \quad \theta(x, y) = \Phi^h_x(\psi(0, y)).$$

Proof. **1)** This follows from Corollary 2.23 (Integrable flow box theorem).
2) For $y = 0$, we find $\theta(x, 0) = \Phi^h_x(\psi(0, 0)) = \Phi^h_x(z)$ which, according to (the proof of) Proposition 2.24, is defined for all $x \in \mathbb{R}^n$ and satisfies $h^{-1}(0) = \Phi^h_{\mathbb{R}^n}(z) = \theta(\mathbb{R}^n \times \{0\})$.
3) Since h is invariant under its flow, we compute

$$h(\theta(x, y)) = h(\Phi^h_x(\psi(0, y))) = h(\psi(0, y)) \overset{2.22}{=} y$$

implying $\theta(x, y) \in h^{-1}(y)$ for all (x, y) where this expression is defined.
4) According to Proposition 2.24, $x \mapsto \Phi^h_x(z) = \theta(x, 0)$ covers $h^{-1}(0)$ surjectively and descends to the quotient $\mathbb{R}^n/\Gamma \simeq h^{-1}(0)$ as a diffeomorphism. Smooth dependence of the flow on initial conditions and 0 being a regular value for h imply that θ must at least be defined on a domain of the form $K \times E$ where $E \subseteq \mathbb{R}^n$ is a small open neighbourhood of the origin and $K \subset \mathbb{R}^n$ is a compact set containing a fundamental domain of Γ. □

Now we want to extend the lattice Γ on $h^{-1}(0)$ to a lattice $\Gamma(y)$ associated with the $\mathbb{R}^n$-action on $h^{-1}(y)$ for all $y \in E$ satisfying $\Gamma(0) = \Gamma$. Let us first compute the 'period shift' of the transition from $\Gamma = \Gamma(0)$ to $\Gamma(y)$ with $y \neq 0$.

Lemma 2.28. *Let $(M, \omega, h = (h_1, \ldots, h_n)$ be a completely integrable system with regular value $0 \in \mathbb{R}^n$ and compact, connected fiber $h^{-1}(0)$. Let $\psi : V \to U$ be Liouville coordinates associated to (M, ω, h) with $z = \psi(0) \in h^{-1}(0)$. Let $\Gamma =: \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \ldots, \gamma_n\}$ be the lattice associated with the $\mathbb{R}^n$-action on the regular fiber $h^{-1}(0)$. Then for all $1 \leq i \leq n$, the concatenation $\psi^{-1} \circ \Phi^h_{\gamma_i} \circ \psi$ defined on V is a local symplectic change of coordinates $(x, y) \mapsto (\xi(x, y), \eta(x, y))$ of* *the form*

$$\begin{cases}\xi(x,y) = x - \partial_y u_i(y),\\ \eta(x,y) = y\end{cases}$$

where $u_i : \{y \in \mathbb{R}^n \mid (x,y) \in V\} \to \mathbb{R}$ *is smooth and* $\partial_y u_i(0) = 0$. *To simplify notation, we abbreviate*

$$\partial_y u_i(y) =: v_i(y),$$

getting in particular $v_i(0) = 0$.

Proof. Since ψ^{-1}, $\Phi^h_{\gamma_i}$, ψ each are symplectic, so is their concatenation. By definition, we have $(\Phi^h_{\gamma_i} \circ \psi)(x,y) = \psi(\xi(x,y), \eta(x,y))$ and the invariance of h under its flow implies

$$y \overset{2.22}{=} (h \circ \psi)(x,y) = (h \circ \Phi^h_{\gamma_i} \circ \psi)(x,y) = (h \circ \psi)(\xi(x,y), \eta(x,y)) = \eta(x,y).$$

Now we extend the relation $y = \eta(x,y)$ by means of the generating function in Example 2.17 locally to a symplectomorphism, i.e., Example 2.17 implies the existence of smooth functions u_i such that

$$\xi(x,y) = x - \partial_y u_i(y) =: x - v_i(y).$$

We compute

$$(\xi,\eta)(0,0) = (\psi^{-1} \circ \Phi^h_{\gamma_i} \circ \psi)(0) = \psi^{-1}(\Phi^h_{\gamma_i}(z)) = \psi^{-1}(z) = (0,0)$$

implying $v_i(0) = 0$. □

Now we can give an explicit formula for the lattice $\Gamma(y)$ that extends Γ.

Lemma 2.29. *Let* $(M, \omega, h = (h_1, \dots, h_n)$ *be a completely integrable system with regular value* $0 \in \mathbb{R}^n$ *and compact, connected fiber* $h^{-1}(0)$. *Then the rank n lattice* $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \dots, \gamma_n\}$ *extends for all* $y \in \mathbb{R}^n$ *sufficiently close to* $0 \in \mathbb{R}^n$ *to a rank n lattice*

$$\Gamma(y) = \{\gamma_1 + v_1(y), \dots, \gamma_n + v_n(y)\}$$

on $h^{-1}(y)$ *with* v_i *defined as in Lemma 2.28. The lattice satisfies* $\Gamma(0) = \Gamma$ *and induces a diffeomorphism* $h^{-1}(y) \simeq \mathbb{R}^n/\Gamma(y)$.

Proof. Let ψ be Liouville coordinates associated with (M, ω, h). Close to $(0,0) \in \mathbb{R}^{2n}$, we compute

$$\Phi^h_{\gamma_i}(\psi(x,y)) = \psi(\xi,\eta) \overset{2.28}{=} \psi(x - v_i(y), y) \overset{2.23}{=} \Phi^h_{-v_i(y)}(\psi(x,y))$$

where the smooth functions v_i stem from Lemma 2.28. This identity is equivalent to $\Phi^h_{\gamma_i+v_i(y)}(\psi(x,y)) = \psi(x,y)$ for all (x,y) sufficiently close to $(0,0) \in \mathbb{R}^{2n}$. Therefore we get as isotropy group

$$\{t \in \mathbb{R}^n \mid \Phi^h_t(\psi(x,y)) = \psi(x,y)\} = \mathrm{Span}_{\mathbb{Z}}\{\gamma_i + v_i(y) \mid 1 \le i \le n\} =: \Gamma(y).$$

Because of $v_i(0) = 0$ we recover $\Gamma(0) = \Gamma$. Since the $\gamma_1, \dots, \gamma_n$ are linearly independent, the vectors $\gamma_i(y) := \gamma_i + v_i(y)$ with $1 \le i \le n$ are also linearly independent for (x,y) sufficiently close to $(0,0)$. Thus $\Gamma(y)$ is a rank n lattice. Arguing as in the proof of Proposition 2.24, we obtain $h^{-1}(y) \simeq \mathbb{R}^n/\Gamma(y)$. □

Now we are ready to extend θ.

Lemma 2.30. *Let $(M,\omega,h = (h_1,\dots,h_n)$ be a completely integrable system with regular value $0 \in \mathbb{R}^n$ and compact, connected fiber $h^{-1}(0)$. Let $\psi : V \to U$ be Liouville coordinates associated to (M,ω,h) with $z = \psi(0) \in h^{-1}(0)$. Then there exists a neighbourhood $E \subseteq \mathbb{R}^n$ of the origin such that*

$$\theta : (\mathbb{R}^n \times E, \omega_{st}) \to (M,\omega), \quad \theta(x,y) = \Phi^h_x(\psi(0,y))$$

is welldefined and satisfies

- $\theta^*\omega = \omega_{st}$,
- $\theta(x + \gamma(y), y) = \theta(x,y)$ *for all* $\gamma(y) \in \Gamma(y)$.

Moreover, for all $y \in E$, the map θ induces diffeomorphisms

$$\bar\theta_y : \mathbb{R}^n/\Gamma(y) \to h^{-1}(y), \quad \bar\theta_y(x + \Gamma(y)) := \theta(x,y).$$

Proof. *Step 1:* Lemma 2.27 states the existence of an open neighbourhood $E \subseteq \mathbb{R}^n$ of the origin and of a compact $K \subseteq \mathbb{R}^n$ containing a fundamental domain of $\Gamma(0)$ such that θ is welldefined on $K \times E$. Using the definition of of θ and $\gamma_i(y) \in \Gamma(y)$, we find that θ is in fact welldefined for points of the form $(x + \gamma_i(y), y)$ with $(x,y) \in K \times E$ since

$$\begin{aligned}\theta(x+\gamma_i(y), y) &= \Phi^h_{x+\gamma_i(y)}(\psi(0,y)) = (\Phi^h_x \circ \Phi^h_{\gamma_i(y)} \circ \psi)(0,y) \\ &= (\Phi^h_x \circ \psi)(0,y) = \theta(x,y).\end{aligned} \tag{2.31}$$

Since K is compact the flow lines are defined up to $\partial K \times E$ of $K \times E$ where they overlap with flow lines defined on 'shifted' domains $(K + \gamma(y)) \times E$ for suitable $\gamma(y) \in \Gamma(y)$ due to (2.31). Since h is smooth an initial value problem of the associated Hamiltonian system has a unique maximal solution. Thus using all of Γ to create overlaps the resulting maximal flow lines are defined on whole $\mathbb{R}$, i.e., θ is in fact defined on $\mathbb{R}^n \times E$.

Step 2: In order to show $\theta^*\omega = \omega_{st}$, we will write θ as concatenation of suitable symplectic maps. Let $(x, y) \in \mathbb{R}^n \times E$ and consider Liouville coordinates $\psi : V \to U$ of (M, ω, h). Choose $\tilde{x} \in \mathbb{R}^n$ sufficiently close to x such that $(x - \tilde{x}, y) \in V$. The translation

$$f : \mathbb{R}^n \times E \to \mathbb{R}^n \times E, \quad f(x, y) := (x - \tilde{x}, y),$$

certainly satisfies $f^*\omega_{st} = \omega_{st}$. Using the compatibility of Liouville coordinates with the flow we find

$$\begin{aligned}\theta(x, y) &= \Phi^h_x(\psi(0, y)) = \Phi^h_{\tilde{x}+(x-\tilde{x})}(\psi(0, y)) = (\Phi^h_{\tilde{x}} \circ \Phi^h_{x-\tilde{x}} \circ \psi)(0, y)\\ &= (\Phi^h_{\tilde{x}} \circ \psi)(x - \tilde{x}, y) = (\Phi^h_{\tilde{x}} \circ \psi \circ f)(x, y)\end{aligned}$$

and, since $\Phi^h_{\tilde{x}}$ and ψ are symplectic, we eventually get

$$\theta^*\omega = f^* \circ \psi^* \circ (\Phi^h_{\tilde{x}})^*\omega = f^* \circ \psi^*\omega = f^*\omega_{st} = \omega_{st}.$$

Step 3: Since $\theta(\cdot, y) : \mathbb{R}^n \to h^{-1}(y)$ is surjective for all $y \in E$, so is the quotient map

$$\bar{\theta}_y : \mathbb{R}^n/\Gamma(y) \to h^{-1}(y), \quad \bar{\theta}_y(x + \Gamma(y)) := \theta(x, y).$$

Moreover, the map $\bar{\theta}_y$ is injective for all $y \in E$ since $\bar{\theta}_y(u) = \bar{\theta}_y(\tilde{u})$ implies $\theta(u, y) = \theta(\tilde{u}, y)$ which in turn implies $(u - \tilde{u}) \in \Gamma(y)$ for all $y \in E$. □

To conclude the proof of Theorem 2.7 (Arnold-Liouville), we now normalize the lattices $\Gamma(y)$ to the standard lattice $\mathbb{Z}^n$.

Lemma 2.32. *Let $(M, \omega, h = (h_1, \ldots, h_n)$ be a completely integrable system with regular value $0 \in \mathbb{R}^n$ and compact, connected fiber $h^{-1}(0)$. Then there exist*

- *open neighbourhoods $D, E \subseteq \mathbb{R}^n$ of the origin,*
- *an open neighbourhood $U \subseteq M$,*
- *a symplectomorphism $\overline{\psi} : ((\mathbb{R}^n/\mathbb{Z}^n) \times D, \omega_{st}) \to U \subseteq (M, \omega)$,*
- *a diffeomorphism $\mu : E \to D$*

such that

- *$\mu \circ h \circ \overline{\psi}(x, y) = y$,*
- *$\overline{\psi}^*\omega = \omega_{st}$.*

Thus the map $\overline{\psi}$ in Lemma 2.32 has the desired properties of the map φ in Theorem 2.7 (Arnold-Liouville) which finishes the proof of Theorem 2.7 (Arnold-Liouville).

Proof of Lemma 2.32. The lattice $\Gamma(y) = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1(y), \ldots, \gamma_n(y)\}$ was defined in Lemma 2.28 by means of generating functions $u_1, \ldots, u_n$ satisfying $\gamma_j(y) = \gamma_j(0) + \partial_y u_j(y)$ and $\partial_y u_j(0) = 0$ for all $1 \leq j \leq n$. Now define for $1 \le j \le n$ the functions $\mu_j(y) := \langle \gamma_j(0), y\rangle_{eu} + u_j(y)$. They satisfy $\partial_{y_k}\mu_j(y) = \gamma_j^k(0) + \partial_{y_k} u_j(y)$ where $\gamma_j^k(0)$ is the kth component of $\gamma_j(0)$. Thus $\partial_y \mu_j(y) = \gamma_j(0) + \partial_y u_j(y) = \gamma_j(y)$. Since $\Gamma(y)$ has rank n for y near the origin so has $\partial_y \mu(y)$. Thus $y \mapsto \mu(y)$ is a local diffeomorphism near the origin. Now pick suitable neighbourhoods $D, E \subseteq \mathbb{R}^n$ such that $\mu : E \to D$ is a diffeomorphism and consider it as change of coordinates $y \mapsto \mu(y) =: \eta$. Set $V(\xi, y) := \sum_{k=1}^n \xi_k \mu_k(y) = \langle \xi, \mu\rangle_{eu}$ and consider

$$\begin{cases} \eta_j = \partial_{\xi_j} V(\xi, y) = \mu_j(y), \\ x_j := \partial_{y_j} V(\xi, y) = \displaystyle\sum_{k=1}^n \xi_k \partial_{y_j} \mu_k(y). \end{cases}$$

By construction, $\partial_\xi V(\xi, y) = \mu = (\mu_1, \dots, \mu_n)$ has rank n near the origin and so does $\partial^2_{y\xi} V(\xi, y) = \partial_y \mu = (\gamma_1(y), \dots, \gamma_n(y))$. By means of Lemma 2.13 (Generating functions I) [with $x \leftrightsquigarrow \xi$ and $y \leftrightsquigarrow \eta$ exchanged in the formulation], we may reverse the construction and obtain a symplectomorphism

$$\zeta : \mathbb{R}^n \times D \to \mathbb{R}^n \times E, \qquad (\xi, \eta) \mapsto (x, y).$$

It satisfies $\zeta(e_j, \eta) = (\gamma_j(y), y)$ on the standard unit vectors $e_1, \dots, e_n \in \mathbb{R}^n$. Moreover, we find $\zeta : \mathbb{Z}^n \times \{\eta\} \to \Gamma(y) \times \{y\}$ for $\eta = \mu(y)$. Now recall θ from Lemma 2.30 and set

$$\tilde{\psi} := \theta \circ \zeta : (\mathbb{R}^n \times D, \omega_{st}) \to U \subseteq (M, \omega)$$

which is symplectic and a diffeomorphism if U is a suitably chosen open set. $\tilde{\psi}$ satisfies

$$\begin{aligned} \tilde{\psi}(\xi + e_j, \eta) &= \theta\left(\sum_{k=1}^n (\xi + e_j)_k \partial_{y_1} \mu_k(y), \dots, \sum_{k=1}^n (\xi + e_j)_k \partial_{y_n} \mu_k(y), y\right) \\ &= \theta\left(\sum_{k=1}^n \xi_k \partial_{y_1} \mu_k(y) + \partial_{y_1} \mu_j(y), \dots, \sum_{k=1}^n \xi_k \partial_{y_n} \mu_k(y) + \partial_{y_n} \mu_j(y), y\right) \\ &= \theta(x + \partial_y \mu_j(y), y) = \theta(x + \gamma_j(y), y) = \theta(x, y) \\ &= \tilde{\psi}(\xi, \eta) \end{aligned}$$

for all $1 \le j \le n$. Therefore we may pass to the quotient as

$$\overline{\psi} : (\mathbb{R}^n / \mathbb{Z}^n) \times D \to U, \quad \overline{\psi}(\xi + \mathbb{Z}^n, \eta) := \tilde{\psi}(\xi, \eta)$$

and we compute

$$(\mu \circ h \circ \overline{\psi})(\xi, \eta) = (\mu \circ h \circ \theta \circ \zeta)(\xi, \eta) = (\mu \circ h \circ \theta)(x, y) = \mu(y) = \eta$$

which finished the proof. $\square$

2.5. Local normal form for nondegenerate singular points

Before we launch into the local study of singular points let us first fix some necessary notions and conventions.

Definition en Lemma 2.33. *Let $(M, \omega, h = (h_1, \dots, h_n))$ be a completely integrable system. Then all points in the orbit of a point of rank k also have rank k. Thus the notion of rank generalizes to orbits by saying that an* **orbit has rank** *k if the points of the orbit are of rank k. Fixed points are precisely the rank zero points.*

Proof. Let $z \in M$ be a point and recall that $h = h \circ \Phi_t^h$ for all $t \in \mathbb{R}$. Thus $Dh|_z = Dh|_{\Phi_t^h(z)}.D\Phi_t^h|_z$. Since $\mathrm{rk}(Dh|_{\Phi_t^h(z)}) = 2n$, we get $\mathrm{rk}(Dh|_z) = \mathrm{rk}(Dh|_{\Phi_t^h(z)})$. Moreover, we have $\mathrm{rk}(Dh|_z) = \mathrm{rk}(X^{h_1}|_z, \dots, X^{h_n}|_z)$. Since z is a fixed point if and only if $(X^{h_1}|_z, \dots, X^{h_n}|_z) = (0, \dots, 0)$, fixed points are precisely the rank zero points. □

Corollary 2.34. *Apart from fixed points, singular points always come in a family whose dimension equals the rank of these points.*

Usually, the notions of *regular* and *singular values* are only used for *values of which the fibers are not empty*. But requiring nonemty fibers becomes cumbersome when we want to discuss 'typical' behaviour, i.e., so-called *generic* behaviour (see Definition A.27), of C^k-functions $f : M \to N$ between smooth manifolds M and N. Here notations become significantly easier if we work with the following conventions:

$$N_{sing} = \{y \in N \mid f^{-1}(y) \neq \emptyset,\ \exists\, x \in f^{-1}(y) : \mathrm{rk}(Df|_x) < \dim N\},$$
$$N_{reg,\emptyset} = N \setminus N_{sing} = N_{reg} \cup (N \setminus f(M)).$$

With this notation, Theorem A.28 (Sard, finite dimensional version) implies that N_{sing} has vanishing Lebesgue measure and $N_{reg,\emptyset}$ has full Lebesgue measure for any C^k-sfunction $f : M \to N$, even for constant functions where the range consists precisely of one point whereas the pre-image of that point is the whole domein of definition. If we worked with N_{reg} instead of $N_{reg,\emptyset}$ then one would have to adapt the formulation of Theorem A.28 (Sard, finite dimensional version) to something like 'for all *generic* functions, the set of singular values in the range of the function is of measure zero.' But this is more complicated than working with $N_{reg,\emptyset}$.

Now apply this knowledge to the smooth function $h : M \to \mathbb{R}^n$ of an $2n$-dimensional completely integrable system $(M, \omega, h = (h_1, \dots, h_n))$: The set $\mathbb{R}^n_{reg,\emptyset}$ has full Lebesgue measure due to Theorem A.28 (Sard, finite dimensional version) and $\mathbb{R}^n_{sing}$ has zero Lebesgue measure. Nevertheless, from a dynamical point of view, $\mathbb{R}^n_{reg,\emptyset}$ is 'boring' since locally the dynamics of the flow of h near regular fibers all look the same due to Theorem 2.7 (Arnold-Liouville). Thus it is in fact the measure zero set $\mathbb{R}^n_{sing}$ and the dynamics near its singular fibers that needs to be studied to find 'significant' distinctions between integrable systems.

The distinction between *regular* and *singular* points of the momentum map $h : M \to \mathbb{R}^n$ of a completely integrable system is due to the behaviour of the *first derivative*, namely the question if Dh has full rank or not. Having full rank is generic whereas having lower rank is not. The same way as the behaviour of the first derivative distinguishes generic (= regular) from nongeneric (= singular) points among the set of all points, the second derivative distinguishes 'typical' (= generic) behaviour from 'atypical' (= nongeneric) behaviour *among the set of singular points*. In the situation of singular points, generic behaviour is usually referred to as *nondegenerate* and nongeneric hehaviour as *degenerate*. This procedure can be repeated on the set of degenerate points by means of the third derivative etc.

The distinction into generic and nongeneric behaviour and its iteration on the nongeneric set is necessary if one aims at classifications. For there is no hope to classify dynamical systems locally by one and the same local normal form. The best one can hope for is a finite number of types of local behaviour for each generic behaviour within a given set, the same way as Theorem 2.7 (Arnold-Liouville) provides a normal form for regular fibers but nor for singular ones.

In this section, we are looking for a local normal form for 'generic' singularities. Hereby we will note that, under certain assumptions, the image of the momentum map of a completely integrable system carries a lot of information about the system. We will see that not all types of critical values can appear everywhere in the image of the momentum map.

Since our focus is on 4-dimensional examples and to keep notation at a minimum, we will define nondegeneracy of singular points only in four dimensions. Explicit applications of the definition in explicit examples can be found in [**Hohloch & Palmer**] and and [**Le Floch & Palmer**] and [**De Meulenaere & Hohloch**]. For the $2n$-dimensional situation, we refer the reader to [**Bolsinov & Fomenko**, Section 1.8.3].

Nondegeneracy is often defined via Lie theory using so-called *Cartan subalgebras*, but the reformulation in terms of linear algebra is much more useful for explicit calculations of examples. So we opted for presenting here the linear algebra approach.

First we define nondegeneracy for rank zero singular points, i.e., fixed points, see also [**Bolsinov & Fomenko**, Sections 1.8.1 and 1.8.2].

Definition 2.35. *Let $(M, \omega, h = (h_1, h_2))$ be a four dimensional completely integrable system and $z \in M$ a singular point of rank zero, i.e., a fixed point. Choose a basis of T_zM and let Ω_z be the matrix of ω_z and $d^2h_1|_z$ and $d^2h_2|_z$ the matrices representing the Hessians of h_1 and h_2 w.r.t. this basis. The fixed point z is* **nondegenerate** *if*

1) *$d^2h_1|_z$ and $d^2h_2|_z$ are linearly independent.*
2) *There exist $b_1, b_2 \in \mathbb{R}$ such that the following matrix has four distinct eigenvalues:*

$$b_1\Omega_z^{-1}d^2h_1|_z + b_2\Omega_z^{-1}d^2h_2|_z$$

Nondegeneracy for rank 1 singular points in dimension four needs some preparations. The idea is to 'get rid of' the remaining rank by 'dividing by the nonsingular direction' to end up with a fixed point for which we already have a notion of nondegeneracy.

Let $(M, \omega, h = (h_1, h_2))$ be a 4-dimensional completely integrable system, and $z \in M$ a singular point of rank one, i.e.

$$1 = \mathrm{rk}(Dh|_z) = \mathrm{rk}(Dh_1|_z, Dh_2|_z) = rk(X^{h_1}(z), X^{h_2}(z))$$

Thus $Dh_1|_z$ and $Dh_2|_z$ as well as $X^{h_1}(z)$ and $X^{h_2}(z)$ are linearly dependent, i.e., there exist some $c_1 := c_1^z$, $c_2 := c_2^z \in \mathbb{R}$ with $(c_1, c_2) \neq (0, 0)$ such that

$$c_1Dh_1|_z + c_2Dh_2|_z = 0 \tag{2.36}$$

and similar for the Hamiltonian vector fields. In particular, the orbit $\mathcal{O}_z := \{\Phi_t^h(z) \mid t \in \mathbb{R}^2\}$ of z is 1-dimensional and so is its tangent space in z

$$L_z := T_z\mathcal{O}_z = \mathrm{Span}_{\mathbb{R}}\{X^{h_1}(z), X^{h_2}(z)\} \subseteq T_zM.$$

Definition en Lemma 2.37. *Let (M, ω) be a symplectic manifold and $z \in M$. Consider the symplectic vector space (T_zM, ω_z) and let $V_z \subseteq T_zM$ be a subspace. Then the vector space*

$$V_z^{\omega} := \{u \in T_zM \mid \omega_z(u, v) = 0 \ \forall \, v \in V_z\}$$

is called **symplectic complement** *of V_z in T_zM. We have*

$$\dim V_z + \dim V_z^{\omega} = \dim M$$

but, different from the orthogonal complement, the intersection of a vector space and its symplectic complement need not be trivial.

Back to the situation of the rank one singular point z of the completely integrable system (M, ω, h), recall that $0 = \{h_1, h_2\} = \omega(X^{h_1}, X^{h_2})$ and that $L_z = \mathrm{Span}_{\mathbb{R}}\{X^{h_1}(z), X^{h_2}(z)\}$. This implies $L_z \subseteq L_z^{\omega}$. Moreover, we have $\dim L_z = 1$ and thus $\dim L_z^{\omega} = \dim M - \dim L_z = 3$.

Recall that the Hessian of a function $f : M \to \mathbb{R}$ can be expressed on vector fields A, B as

$$d^2 f(A, B) := A(B(f)) - df(\nabla_A B) \tag{2.38}$$

where ∇ is the Levi-Civita connection on the Riemannian manifold (M, g) with $g := \omega(\cdot, J\cdot)$ for a *suitable* almost complex structure J (which is a map $J_z : T_zM \to T_zM$ with $J \circ J = -\mathrm{Id}$). The Levi-Civita connection satisfies in particular $\nabla_A B - \nabla_B A = [A, B]$. Now recall c_1 and c_2 from (2.36) and set

$$H_{c_1,c_2}|_z := d^2(c_1h_1 + c_2h_2)|_z = c_1 d^2 h_1|_z + c_2 dh_2|_z.$$

Lemma 2.39. *The symmetric 2-form $H_{c_1,c_2}|_z$ can be expressed as*

$$H_{c_1,c_2}|_z(A, B) = A(B(c_1h_1 + c_2h_2))|_z$$

on vector fields A, B on M and L_z lies in its kernel. Therefore $H_{c_1,c_2}|_z$ descends to the 2-dimensional quotient L_z^{ω}/L_z.

Proof. Using the fact that $c_1dh_1 + c_2dh_2 = d(c_1h_1 + c_2h_2)$ vanishes in z together with formula (2.38) we obtain

$$H_{c_1,c_2}|_z(A, B) = A(B(c_1h_1 + c_2h_2)) - d(c_1h_1 + c_2h_2)(\nabla_A B) = A(B(c_1h_1 + c_2h_2)).$$

on two vector fields A, B on M. Since X^{h_1} and X^{h_2} generate L_z and are linearly dependent we may write $C \in L_z$ w.l.o.g. as $C = cX^{h_1}$ for some $c \in \mathbb{R}$. Then we compute

$$\begin{aligned} H_{c_1,c_2}|_z(A, C) &= A(C(c_1h_1 + c_2h_2)) = A(cX^{h_1}(c_1h_1 + c_2h_2)) \\ &= A(d(c_1h_1 + c_2h_2)(cX^{h_1})) = A(\{c_1h_1 + c_2h_2, ch_1\}) \\ &= 0 \end{aligned}$$

since $\{c_1h_1 + c_2h_2, ch_1\} = \{c_1h_1, ch_1\} + \{c_2h_2, ch_1\} = 0$. Thus L_z lies in the kernel of $H_{c_1,c_2}|_z$ so that $H_{c_1,c_2}|_z$ descends to the quotient L_z^{ω}/L_z. Moreover, we compute $\dim(L_z^{\omega}/L_z) = \dim L_z^{\omega} - \dim L_z = 3 - 1 = 2$. □

Now we are ready to define

Definition 2.40. *Let $(M, \omega, h = (h_1, h_2))$ be a 4-dimensional completely integrable system. Let $z \in M$ be a singular point of rank one and $c_1, c_2 \in \mathbb{R}$ with $(c_1, c_2) \neq (0, 0)$ such that $c_1 Dh_1|_z + c_2 Dh_2|_z = 0$. The point z is said to be* **nondegenerate** *if $c_1 d^2 h_1|_z + c_2 d^2 h_2|_z$ is invertible on L^ω/L. Otherwise z is called* **degenerate**.

Explicit examples in dimension four with proofs for the non-degeneracy of fixed points and singular points of rank one can be found in [**Hohloch & Palmer**], [**Le Floch & Palmer**], and [**De Meulenaere & Hohloch**].

The definition of nondegeneracy of higher rank singular points in dimension greater than four can be found in [**Bolsinov & Fomenko**, Section 1.8.3] and is needed for the formulation of the following theorem which provides a local normal form for nondegenerate singular points of $2n$-dimensional completely integrable systems.

Theorem 2.41 (**Local normal form**). *Let $(M, \omega, h = (h_1, \dots, h_n))$ be a $2n$-dimensional completely integrable system and $z \in M$ a nondegenerate singular point. Then there exist local symplectic Darboux coordinates $(x, y) := (x_1, \dots, x_n, y_1, \dots, y_n)$ in a neighbourhood $U \subset M$ of z such that there exists a function $f := (f_1, \dots, f_n) : U \to \mathbb{R}^n$ with $\{h_k, f_j\} = 0$ for all $1 \leq k, j \leq n$ whose component functions f_j stem from the following list:*

1) **elliptic component:**
   $f_j(x, y) = \frac{1}{2}(x_j^2 + y_j^2)$,
2) **hyperbolic component:**
   $f_j(x, y) = x_j y_j$,
3) **focus-focus component**, *comes always as a pair (f_j, f_{j+1}):*
   $$\begin{cases} f_j(x, y) & = x_j y_{j+1} - x_{j+1} y_j, \\ f_{j+1}(x, y) & = x_j y_j + x_{j+1} y_{j+1}, \end{cases}$$
4) **nonsingular component** *(also called* **regular component***):*
   $f_j(x, y) = y_j$.

Proof. The proof of this theorem is spread throughout the literature and there is, up to our knowledge, no comprehensive presentation of it. The local normal form was announced by [**Eliasson 1984**], but the proof appeared to be somewhat incomplete. Altogether, there are at least the following contributions:

- The C^∞ case in two dimensions is described by the *Lemme de Morse isochore* in [**Colin de Verdière & Vey**].
- The two dimensional analytic case appears in [**Rüssmann**].
- The analytic case in dimension $2n$ was done by [**Vey**].
- C^∞ for the elliptic case in dimension $2n$ was done by [**Eliasson 1990**].
- Another proof for C^∞ in the elliptic case in dimension $2n$ was provided by [**Dufour & Molino**].
- Low dimensional hyperbolic cases have been dealt with in [**Miranda**].
- The focus-focus case in dimension four has been dealt with by [**Vũ Ngọc & Wacheux**] and [**Chaperon**].
- The infinitesimal case was proven by [**Miranda & Vũ Ngọc**].
- A completely different approach was presented by [**Wang**].
- The equivariant case with an action of a compact group was treated in [**Miranda & Zung**].

□

Following [**Bolsinov & Fomenko**] or [**Vũ Ngọc 2006**], there is an interpretation of the components of Theorem 2.41 (Local normal form) in terms of eigenvalues. On 4-dimensional manifolds, this boils down to

Corollary 2.42. *Let $(M, \omega, h = (h_1, h_2))$ be a 4-dimensional completely integrable system with nondegenerate fixed point $z \in M$. Let Ω_z be a matrix representing ω_z in some chosen basis of T_zM. Pick $b_1, b_2 \in \mathbb{R}$ such that $K_{b_1,b_2}|_z := b_1\Omega_z^{-1}d^2h_1|_z + b_2\Omega_z^{-1}d^2h_2|_z$ has four distinct eigenvalues. Then*

1) *An elliptic component corresponds to a pair of imaginary eigenvalues $\pm i\beta$ of $K_{b_1,b_2}|_z$ where $\beta \in \mathbb{R}^{\neq 0}$.*
2) *A hyperbolic component corresponds to a pair of real eigenvalues $\pm\alpha \in \mathbb{R}^{\neq 0}$ of $K_{b_1,b_2}|_z$.*
3) *A focus-focus component corresponds to a quadruple of complex eigenvalues $\pm\alpha \pm i\beta$ of $K_{b_1,b_2}|_z$ where $\alpha, \beta \in \mathbb{R}^{\neq 0}$.*

The type of eigenvalue does not depend on the chosen $b_1, b_2 \in \mathbb{R}$.

Corollary 2.43. *Let $(M, \omega, h = (h_1, h_2))$ be a 4-dimensional completely integrable system and $z \in M$ a regular or a nondegenerate singular point. Then*

1) $\mathrm{rk}(z) = 2 \quad \Leftrightarrow \quad$ *z regular.*
2) $\mathrm{rk}(z) = 1 \quad \Leftrightarrow \quad$ *z elliptic-regular or hyperbolic-regular.*

3) $\mathrm{rk}(z) = 0 \quad \Leftrightarrow \quad$ *z elliptic-elliptic or elliptic-hyperbolic or hyperbolic-hyperbolic or focus-focus.*

Let us now get some geometric intuition for some of these types of singular points.

Corollary 2.44. *Let (M, ω, h) be a 4-dimensional completely integrable system with a elliptic-elliptic fixed point $z \in M$. Then the system looks around z like the uncoupled oscillator (see Example 1.54 and Figure 2.1) around the origin.*

Proof. Use Example 1.54 and Theorem 2.41 (Local normal form). □

Now let us get an intuition for focus-focus points.

Lemma 2.45. *Let (M, ω, h) be a 4-dimensional completely integrable system with a focus-focus point $z \in M$. The, near z, the fibers of the system can be locally seen as hyperboloids and the focus-focus fiber as transverse intersection of complex planes where the intersection point represents the focus-focus point.*

Proof. By setting $\zeta_1 := x_1 + ix_2$ and $\zeta_2 := y_1 + iy_2$, identify $\mathbb{R}^2 \simeq \mathbb{C}$ and consider the map

$$g : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad g(\zeta_1, \zeta_2) := \overline{\zeta}_1 \zeta_2 = (x_1 y_1 + x_2 y_2) + i(x_1 y_2 - x_2 y_1).$$

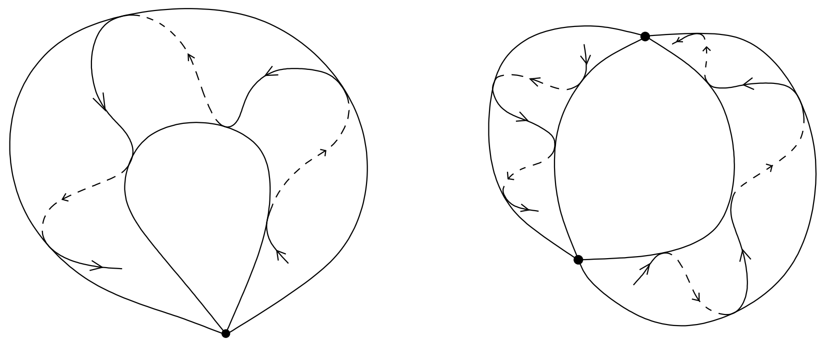
With the notations of Theorem 2.41 (Local normal form), we obtain

$$g(\zeta_1, \zeta_2) = f_2(x, y) + i f_1(x, y).$$

Then $g^{-1}(0) = (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$ is the fiber above the singular value $0 \in \mathbb{R}^2$, consisting of two transversely intersecting complex planes. The nearby fibers $g^{-1}(c)$ with $c \in \mathbb{C}^{\neq 0}$ are regular and can be seen as cylinders or hyperboloids, see Example **??**. □

A focus-focus point can also be seen as isolated fixed point with hyperbolic expansion and contraction behaviour while admitting a local $\mathbb{S}^1$-action, see for instance [**Chaperon**] and [**Vũ Ngọc & Wacheux**]. The flow on a focus-focus fiber behaves as in Figure 2.6.

Intuitively, a fiber over a focus-focus singular value can be seen as a (maybe multiply) 'pinched torus', i.e., a torus where (at least) one circle has been contracted to a point (which is precisely the focus-focus point).



(a) (b)

FIGURE 2.6. (a) A fiber with one focus-focus fixed point. The flow spirals away from the focus-focus points and is again attracted to it. (b) A fiber with two focus-focus points and flow lines on the regular parts in between.

REMARK 2.46. *A focus-focus fiber $\mathcal{F}$ that contains precisely one focus-focus point z consists of two distinct orbits, namely the singular focus-focus point $\{z\}$ and $\mathcal{F} \setminus \{z\}$ which consists of regular points. A focus-focus fiber $\mathcal{F}$ that contains precisely k focus-focus points $z_1, \dots, z_k$ consists of $2k$ distinct orbits, namely the singular focus-focus points $\{z_1\}, \dots, \{z_k\}$ and $\mathcal{F} \setminus \left(\bigcup_{j=1}^{k}\{z_j\}\right)$ which has k connected components that all consist of regular points.*

This can be strengthened to

THEOREM 2.47 ([**Zung 1996**, **Zung 1997**]). *A focus-focus fiber that contains exactly n focus-focus points consists of a 'chain' of n spheres where each of the spheres intersects transversally two other spheres. The intersection points are given by the n focus-focus points, cf. Figure 2.6.*

There are explicit examples of completely integrable systems with fibers that contain more than one focus-focus point: [**De Meulenaere & Hohloch**] study a system which has at first no focus-focus points, then four focus-focus points in four different fibers and then two focus-focus points each in two different fibers. The fibers containing two focus-focus points can be parametrized explicitly and display readily the intuition as 'double pinched tori', see [**De Meulenaere & Hohloch**, Proposition 1.3].

APPENDIX A

Appendix

This appendix recalls various necessary or helpful notions from ODE theory and differential geometry.

A.1. Manifolds and submanifolds

If one wants to work on, say, the sphere $\mathbb{S}^2$ one faces the problem that the sphere is described *implicitly* by the equation

$$\mathbb{S}^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \sum_{i=1}^{k} x_i^2 = 1 \right\}.$$

More precisely, this description of the sphere needs three coordinates although the sphere itself is only 2-dimensional. Therefore two coordinates should suffice to describe the sphere. Unfortunately, it is only possible to parametrise – by means of two coordinates – subsets of the sphere but never the whole sphere if one works with (open subsets of) $\mathbb{R}^2$ as domains of definition for the parametrization. Working with (partially) closed domains is not very practical since one then needs to define differentiability on the boundary of these sets (which is possible but cumbersome).

Let us now find 2-dimensional 'patches' on the sphere that can easily be parametrized by open sets of $\mathbb{R}^2$. Denote by $\mathbb{D}^2$ the open unit disk in $\mathbb{R}^2$. Consider the upper and lower halfspheres

$$S_u := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 > 0\} \quad \text{and} \quad S_\ell := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 < 0\},$$

with the maps $\psi_u : S_u \to \mathbb{D}^2$ and $\psi_\ell : S_\ell \to \mathbb{D}^2$ given by

$$\psi_u(x_1, x_2, x_3) = (x_1, x_2), \qquad \psi_u^{-1}(y_1, y_2) = \left(y_1, y_2, \sqrt{1 - y_1^2 - y_2^2}\right),$$
$$\psi_\ell(x_1, x_2, x_3) = (x_1, x_2), \qquad \psi_\ell^{-1}(y_1, y_2) = \left(y_1, y_2, -\sqrt{1 - y_1^2 - y_2^2}\right).$$

Moreover, there are the right and left halfspheres

$$S_r := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_2 > 0\} \quad \text{and} \quad S_l := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_2 < 0\},$$

with the maps $\psi_r : S_r \to \mathbb{D}^2$ and $\psi_l : S_l \to \mathbb{D}^2$ given by

$$\psi_r(x_1, x_2, x_3) = (x_1, x_3), \qquad \psi_r^{-1}(y_1, y_3) = \left(y_1, \sqrt{1 - y_1^2 - y_3^2}, y_3\right),$$

$$\psi_l(x_1, x_2, x_3) = (x_1, x_3), \qquad \psi_l^{-1}(y_1, y_3) = \left(y_1, -\sqrt{1 - y_1^2 - y_3^2}, y_3\right).$$

Analogously, we get the front and back halfspheres

$$S_f := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_1 > 0\} \quad \text{and} \quad S_b := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_1 < 0\}.$$

with the maps $\psi_f : S_f \to \mathbb{D}^2$ and $\psi_b : S_b \to \mathbb{D}^2$ given by

$$\psi_f(x_1, x_2, x_3) = (x_2, x_3), \qquad \psi_f^{-1}(y_2, y_3) = \left(\sqrt{1 - y_2^2 - y_3^2}, y_2, y_3\right),$$

$$\psi_b(x_1, x_2, x_3) = (x_2, x_3), \qquad \psi_b^{-1}(y_2, y_3) = \left(-\sqrt{1 - y_2^2 - y_3^2}, y_2, y_3\right).$$

The union

$$S_u \cup S_\ell \cup S_r \cup S_l \cup S_f \cup S_b = \mathbb{S}^2$$

covers the whole sphere and, on each 'patch' S_i, the sphere is described by the '2-dimensional coordinates' $\psi_i^{-1} : \mathbb{D}^2 \to S_i$ for all $i \in \{u, \ell, r, l, f, b\}$.

Let us now generalize this concept. Recall that a **homeomorphism** is a continuous, bijective map of which the inverse is also continuous. A **C^k-diffeomorphism** is a homeomorphism that is C^k-differentiable and whose inverse is also C^k-differentiable.

Definition A.1.

1) *An m-**dimensional** **C^k-differentiable manifold** is a topological space M together with open subsets $U_i \subseteq M$ and homeomorphisms $\psi_i : U_i \to \psi_i(U_i) \subseteq \mathbb{R}^m$ such that their composition*

$$\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \subseteq \mathbb{R}^m \to \psi_j(U_i \cap U_j) \subseteq \mathbb{R}^m$$

is a C^k-diffeomorphism for all i, j. In case of $k = 0$, we speak of **topological manifolds**, *in case of $k = \infty$ of* **smooth manifolds**.

2) *The pair (U_i, ψ_i) is called a* **(coordinate) chart** *of M and $\psi_j \circ \psi_i^{-1}$* **change of charts** *or* **change of coordinates**. *The union of all charts (U_i, ψ_i) is called an* **atlas** *of M.*

The suitable notion for 'subset of manifolds' is the following.

Definition A.2. *Let M be an m-dimensional C^k-manifold and $k \in \mathbb{N}_0$ with $k \leq m$. A subset $N \subseteq M$ is an* **(embedded) k-dimensional**

submanifold of M *if, for all* $x \in N$*, there exists a chart* (U, ψ) *of* M *with* $p \in U$ *such that* $\psi(U \cap M) = \psi(U) \cap \ (\mathbb{R}^k \times \{0\}^{m-k}) \subseteq \mathbb{R}^m$.

If $(U_i, \psi_i)_{i \in I}$ is an atlas of a manifold M and if N is a submanifold of M, then the restrictions $(N \cap U_i, \psi_i|_{N \cap U_i})$ form an atlas of N.

A.2. (Co)tangent bundle and differential forms in $\mathbb{R}^m$

If we want to measure the volume of an m-dimensional subset in $\mathbb{R}^m$, we may use the m-dimensional Lebesgue measure. But if the subset has dimension $\tilde{m} < m$, the m-dimensional Lebesgue mesure of this set is zero. It is a priori not clear how to use the $\tilde{m}$-dimensional Lebesgue measure in $\mathbb{R}^m$ to measure the volume of $\tilde{m}$-dimensional subsets since the sets can lie in a very complicated way in $\mathbb{R}^m$.

The idea is to come up with a notion that can handle ‘intermediate’ volumes. Let us see what kind of properties this notion must have. Given a parallelogram $P_{u,v}$ spanned by two vectors $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ in $\mathbb{R}^2$, its volume is given by

$$\mathrm{vol}(P_{u,v}) = \det(u, v) = u_1 v_2 - u_2 v_1.$$

Shearing the parallelogram by means of the map $(u, v) \mapsto (u + \lambda v, u)$ with $\lambda \in \mathbb{R}$ does not change its volume, algebraically expressed by

$$\det(u + \lambda v, v) = \det(u, v) + \lambda \det(v, v) = \det(u, v) + \lambda \cdot 0 = \det(u, v).$$

Scaling a vectors of the parallelogram by a scalars $\lambda \in \mathbb{R}$ corresponds to the volume transformation

$$\mathrm{vol}(P_{\lambda u,v}) = \det(\lambda u, v) = \lambda \det(u, v) = \lambda \,\mathrm{vol}(P_{u,v}).$$

This suggests that whatever notion we introduce should satisfy

- **Multilinearity**, i.e., linearity in each variable w.r.t. addition and scalar multiplication of vectors.
- **Skewsymmetry**, i.e., the property corresponding to $\det(u, u) = 0$ or, equivalently, $\det(u, v) = -\det(v, u)$.

Moreover, recall that the infinitesimal change of volume in the transformation formula of integrals

$$\int_V f(y)\, dy = \int_{\psi^{-1}(V)} (f \circ \psi)(x) \left|\det(D\psi|_x)\right| \, dx.$$

is given by the determinant of the Jacobian of the transformation, i.e., watching the change of volume of the *linearization* is enough to describe the change of volume under the (nonlinear) transformation. This suggests that our new notion should work on the level of functions and derivatives (and therefore tangent spaces).

A.2.1. Tangent bundle. We recall the definition of the tangent space for subsets of $\mathbb{R}^m$. For $V \subseteq \mathbb{R}^m$, we define the **tangent space of** V **in** $p \in V$ by

$$T_pV := \{p\} \times \left\{ v \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \exists\, \varepsilon > 0,\ \exists\, \gamma :]-\varepsilon, \varepsilon[\to V \text{ differentiable,} \\ \gamma(0) = p,\ \gamma'(0) = v \end{array} \right\}$$

which is, by neglecting the **foot point** p, often seen as

$$T_pV \simeq \{v \in \mathbb{R}^n \mid \exists\, \varepsilon > 0,\ \exists\, \gamma :]-\varepsilon, \varepsilon[\to V \text{ diff.}, \gamma(0) = p,\ \gamma'(0) = v\}.$$

Remark A.3. *1) Let $V \subseteq \mathbb{R}^m$ be open. Then $T_pV \simeq \{p\} \times \mathbb{R}^m$ for all $p \in V$.*
2) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and let $r \in \mathbb{R}^n$ be a regular value of f. Then the level set $f^{-1}(r)$ has dimension $m - n$ and $T_p(f^{-1}(r)) = \ker Df|_p$ for all $p \in f^{-1}(r)$.

The **tangent space** of V is given by

$$TV := \bigcup_{p \in V} T_pV$$

and comes with a natural projection

$$\pi : TV \to V,$$

namely sending an element to its foot point. The tangent space TV together with its projection π to the **base space** V is usually called **tangent bundle**. A map $\sigma : V \to TV$ satisfying $\pi(\sigma(p)) = p$ for all $p \in V$ is called a **section** of TV. It is a map that assigns to each $p \in V$ precisely one vector in T_pV, i.e., a map of the form

$$v : V \to TV, \quad p \mapsto v_p \in T_pV.$$

Remark A.4. *The sections of the tangent bundle $TV \to V$ are preciesely the vector fields on V.*

A.2.2. Cotangent bundle. Within this subsection, we assume that $V \subseteq \mathbb{R}^m$ is open. Let us use the coordinates $(x_1, \ldots, x_m)$ on V and consider a point $p \in V$. The curves

$$t \mapsto p + t\,(0, \ldots, 0, 1, 0, \ldots, 0),$$

with the 1 at the ith position, all equal p at time $t = 0$ and have linearly independent tangent vectors

$$\partial_{x_i}|_p := (0, \ldots, 0, 1, 0, \ldots, 0)^T \in T_pV.$$

Here we keep track of the foot point by writing $|_p$ ('at the foot point p'). Since $\dim(T_pV) = m$, the vectors $\partial_{x_1}|_p, \ldots, \partial_{x_m}|_p$ form a basis of T_pV. An arbitrary vector $v_p \in T_pV$ can thus be written as $v_p = \sum_{i=1}^m v_i|_p\, \partial_{x_i}|_p$. The dual vector space

$$(T_pV)^* := \{\alpha_p : T_pV \to \mathbb{R} \mid \alpha_p \text{ linear}\}$$

also has dimension m and we endow it with the 'dual' basis $dx_1|_p, \ldots, dx_m|_p$ by requiring

$$dx_i|_p\,(\partial_{x_j}|_p) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

A functional $\alpha_p \in (T_pV)^*$ can thus be written as $\alpha_p = \sum_{i=1}^m \alpha_i|_p\, dx_i|_p$. Evaluating α_p on v_p using linearity and duality gives

$$\alpha_p(v_p) = \left(\sum_{i=1}^m \alpha_i|_p\, dx_i|_p\right)\left(\sum_{i=1}^m v_i|_p\, \partial_{x_i}|_p\right) = \sum_{i=1}^m \alpha_i|_p v_i|_p \quad \in \mathbb{R}.$$

The union

$$(TV)^* := \bigcup_{p \in V} (T_pV)^*$$

is the **cotangent space** and it also comes with a projection $\pi : (TV)^* \to V$ by sending all elements to their foot points. The cotangent space $(TV)^*$ together with its projection π to the **base space** V is usually called **cotangent bundle**. Maps $\sigma : V \to (TV)^*$ satisfying $\pi(\sigma(p)) = p$ are called **sections**. This are maps that assigns to each $p \in V$ precisely one functional in T_pV, i.e., maps of the form

$$\alpha : V \to (TV)^*, \quad p \mapsto \alpha_p \in (T_pV)^*,$$

meaning $\alpha_p : T_pV \to \mathbb{R}$ is linear for all $p \in V$.

A.2.3. Differential forms in $\mathbb{R}^m$. Now we construct maps from the $k$fold product of the tangent space to $\mathbb{R}$ that are multilinear and skewsymmetric (often called **alternating** instead of skewsymmetric).

Let $V \subseteq \mathbb{R}^m$ be open with coordinates $(x_1, \ldots, x_m)$ and, for all $p \in V$, endow T_pV with the basis $\partial_{x_1}|_p, \ldots, \partial_{x_m}|_p$ and $(T_pV)^*$ with the basis $dx_1|_p, \ldots, dx_m|_p$.

Notation A.5. *1) Functions $f : V \to \mathbb{R}$ are called* 0**-forms** *on V. Evaluated at a point $p \in V$, a 0-form is a scalar $f(p) \in \mathbb{R}$.*
2) Sections $\alpha : V \to (TV)^*$ are called 1**-forms** *on V. Evaluated at a point $p \in V$, a 1-form is a functional $\alpha_p \in (T_pV)^*$, meaning, a linear map $\alpha_p : T_pV \to \mathbb{R}$.*

Now we introduce an operation that will produce forms of higher order.

Definition A.6. *1) The* **exterior product** *or* **wedge product** *of a 0-form f and a 1-form α on V is given by the 1-form*

$$f \wedge \alpha := f\alpha$$

defined by $(f \wedge \alpha)_p := (f\alpha)_p := f(p)\alpha_p \in (T_pV)^*$ for all $p \in V$.
2) The **exterior product** *or* **wedge product** *of two 1-forms α, β on V is given by the* 2**-form** *$\alpha \wedge \beta$ on V that is defined via*

$$(\alpha \wedge \beta)_p(u_p, v_p) := \alpha_p(u_p)\beta_p(v_p) - \alpha_p(v_p)\beta_p(u_p)$$

for all $p \in V$ and all $u_p, v_p \in T_pV$.

The wedge product for 0- and 1-forms satisfies the following properties:

- $f \wedge \alpha = \alpha \wedge f$ for all 0-forms f and all 1-forms α.
- $\alpha \wedge (f\beta) = f(\alpha \wedge \beta)$ for all 0-forms f and all 1-forms α and β.
- $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ for all 1-forms α, β, γ.
- $\alpha \wedge \beta = -\beta \wedge \alpha$ for all 1-forms α, β.

These properties lead to the representation of a 2-form ω as

$$\omega = \sum_{1 \le i_1 < i_2 \le m} \omega_{i_1 i_2} dx_{i_1} \wedge dx_{i_2}$$

with $\omega_p = \sum_{1 \le i_1 < i_2 \le m} \omega_{i_1 i_2}(p)(dx_{i_1} \wedge dx_{i_2})|_p$ for all $p \in V$, i.e., $\omega_{i_1 i_2} : V \to \mathbb{R}$ is a function for all indices $1 \le i_1 < i_2 \le m$. In particular, we find (neglecting the foot point notation for the moment)

$$(dx_i \wedge dx_j)(u, v) = u_i v_j - u_j v_i$$

which recovers the determinant of the vectors $u = (u_1, \dots, u_m)^T$ and $v = (v_1, \dots, v_m)^T$ in case $i = 1$, $j = 2$, and $m = 2$.

Iterating the wedge product with 0-, 1-, and 2-forms leads to arbitrary k-forms. More precisely

Definition and Proposition A.7. *Let $k, \ell \in \mathbb{N}_0$. The* **exterior product** *or* **wedge product** *of a k*-**form** *α and a ℓ*-**form** *β on V is defined as the $(k+\ell)$-form $\alpha \wedge \beta$ given by*

$$(\alpha \wedge \beta)_p(\mathbf{u}_p, \mathbf{v}_p) := \alpha_p(\mathbf{u}_p)\beta_p(\mathbf{v}_p) - \alpha_p(\mathbf{v}_p)\beta_p(\mathbf{u}_p)$$

for all $p \in V$ and all $\mathbf{u}_p \in (T_pM)^k$ and all $\mathbf{v}_p \in (T_pV)^\ell$. We have

- *$f \wedge \alpha = \alpha \wedge f = f\alpha$ for all 0-forms f and all k-forms α.*
- *$\alpha \wedge (f\beta) = f(\alpha \wedge \beta)$ for all 0-forms f, all k-forms α, and all ℓ-forms β.*
- *$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ for all k-forms α and all ℓ-forms β, γ.*
- *$\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$ for all k-forms α and all ℓ-forms β.*

These properties lead to the representation of a k-form α as

$$\alpha = \sum_{1 \le i_1 < \dots < i_k \le m} \alpha_{i_1 \dots i_2} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

with $\alpha_p = \sum_{1 \le i_1 < \dots < i_k \le m} \alpha_{i_1 \dots i_k}(p)(dx_{i_1} \wedge \dots \wedge dx_{i_k})|_p$ for all $p \in V$, meaning, $\alpha_{i_1 \dots i_k} : V \to \mathbb{R}$ is a function for all indices $1 \le i_1 < \dots < i_k \le m$.

Definition A.8. *Let $V \subseteq \mathbb{R}^m$ be open, $p \in V$, let $k \in \mathbb{N}^{\ge 2}$, and let $\mu_p : (T_pV)^k \to \mathbb{R}$ be a map.*

1) μ_p is **multilinear** *if μ_p is linear in each variable.*

2) μ_p **alternates** *or is* **alternating** *if for all $u_p, v_p \in T_pV$*

$$\mu_p(\dots, u_p, v_p, \dots) = -\mu_p(\dots, v_p, u_p, \dots).$$

We set

$$\Lambda^k((T_pV)^*) := \left\{ \mu_p : (T_pV)^k \to \mathbb{R} \,\middle|\, \mu_p \text{ multilinear and alternating} \right\}$$

and

$$\Lambda^k(V) := \Lambda^k((TV)^*) := \bigcup_{p \in V} \Lambda^k((T_pV)^*).$$

This space also carries a projection $\pi : \Lambda^k(V) \to V$ by sending multilinear, alternating maps μ_p to their footpoint p. Sections of the bundle $\Lambda^k(V) \to V$ are maps $\sigma : V \to \Lambda^k(V)$ satisfying $\pi(\sigma(p)) = p$.

REMARK A.9. *1) k-forms can be seen as multilinear, alternating maps. More precisely, a k-form α is a section*

$$\alpha : V \to \Lambda^k(V), \quad p \mapsto \alpha_p \in \Lambda^k((T_pV)^*).$$

2) Alternating implies that, on spaces of dimension m, all k-forms with $k > m$ vanish.

The following map is of high importance in (co)homology theory since it represents the boundary operator of De Rham cohomology.

DEFINITION AND PROPOSITION A.10.
Let $k \in \mathbb{N}_0$. The operator $d : \Lambda^k(V) \to \Lambda^{k+1}(V)$ given on 0-forms f by

$$(df)_p := \sum_{i=1}^{m} \partial_{x_i} f(p) dx_i|_p$$

for all $p \in V$ and on k-forms $\alpha = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \alpha_{i_1 \ldots i_2} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ by

$$\sum_{1 \leq i_1 < \cdots < i_k \leq m} (d\alpha_{i_1 \ldots i_2}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

is called **exterior derivative** *and satisfies $d \circ d = 0$.*

The following types of forms are the ‘building blocks’ of so-called chain complexes in cohomology theory.

DEFINITION A.11. *A k-form α is* **closed** *if $d\alpha = 0$. A k-form is* **exact** *if there exists a $(k-1)$-form β with $d\beta = \alpha$.*

Note that $d \circ d = 0$ implies that exact forms are closed.

LEMMA A.12 (**Poincaré**). *Locally, all closed forms are exact.*

PROOF. See for example [**Warner**] or [**Petersen**] or [**Bott & Tu**]. □

Given a map, there is a way to construct new k-forms out of old ones. If the map is a diffeomorphism, we can invert the construction.

DEFINITION AND PROPOSITION A.13. *Let $U, V \subseteq \mathbb{R}^m$ be open and $\psi : U \to V$ surjective and differentiable. Let α be a k-form on V. The*

pullback *of α under ψ defines the k-form $\psi^*\alpha$ on U via*

$$(\psi^*\alpha)_p((u_1)_p, \dots, (u_k)_p) := \alpha_{\psi(p)}(D\psi|_p.(u_1)_p, \dots, D\psi|_p.(u_k)_p)$$

for all $p \in U$ and all $(u_1)_p, \dots, (u_k)_p \in T_pU$. The pullback satisfies the following properties:

- *$\psi^* f = f \circ \psi$ for all 0-forms $f : V \to \mathbb{R}$.*
- *$\psi^*(c\alpha) = c(\psi^*\alpha)$ for all constants $c \in \mathbb{R}$ and all k-forms α.*
- *$\psi^*(\alpha + \beta) = \psi^*\alpha + \psi^*\beta$ for all k-forms α, β.*
- *$d(\psi^*\alpha) = \psi^*(d\alpha)$ for all surjective, differentiable $\psi : U \to V$ and all k-forms α on V.*

A.3. Differential forms on manifolds

This section still needs to be written... Roughly we can already say that one tries to 'pull back' all definitions from $\mathbb{R}^m$ to the manifold M by means of the charts (U_i, ψ_i). Since there may be several charts covering one point in the manifold, one has to show that this is welldefined... ...which is a bit tedious...

A.4. Flows of autonomous ODEs

Let M be an m-dimensional smooth manifold, $\zeta \in M$ and X a vector field on M that is locally Lipschitz. Then the initial value problem

$$z' = X(z), \qquad z(0) = \zeta$$

has a unique maximal solution defined on an interval I_ζ containing 0. We denote this solution by $z_\zeta : I_\zeta \to M$. Instead of only tracking the 'time variable' $t \in I_\zeta$, we can also consider the dependence of z_ζ on the 'space variable' ζ, meaning, we may study the mapping $(t, \zeta) \mapsto z_\zeta(t)$. It satisfies $z_\zeta(0) = \zeta$ and $\partial_t z_\zeta(t) = X(z_\zeta(t))$. This motivates

Definition and Proposition A.14. *Let M be a smooth manifold and X a C^k-vector field on M with $k \geq 1$. Set $V := \bigcup_{\zeta \in M} I_\zeta \times \{\zeta\} \subseteq \mathbb{R} \times M$. Then V is open and the* **(local) flow of** *$z' = X(z)$ is given by the C^k-mapping*

$$\Phi : V \to M, \quad \Phi_t(\zeta) := \Phi(t, \zeta) := z_\zeta(t).$$

It satisfies

(i) $\Phi_0(\zeta) = \zeta \quad \forall\, \zeta \in M$,

(ii) $\Phi_{t+s}(\zeta) = \Phi_t(\Phi_s(\zeta)) \quad \forall\, \zeta \in M,\ \forall\, s, (t+s) \in I_\zeta.$

Switching from the $\Phi(t,\zeta)$ to the $\Phi_t(\zeta)$ notation is motivated by the otherwise rather cumbersome formulation of properties (i) and (ii). The property $\Phi_t \circ \Phi_s = \Phi_{t+s}$ is often called **flow property**. It means that flowing first for time s and then additionally for time t is the same as flowing directly for time $(t+s)$. The property $\Phi_0 = \mathrm{Id}$ says that *flowing for time* $t = 0$ simply means *stay where you are*.

$\Phi_t(\zeta) = z_\zeta(t)$ has the following geometric meaning: consider $\zeta \in M$ and follow the solution z_ζ from $\zeta = z_\zeta(0)$ for time t. The point reached by the solution z_ζ at time t is $\Phi_t(\zeta)$. We often use the following short notation for properties (i) and (ii):

$$\Phi_0 = \mathrm{Id} : M \to M \qquad \text{en} \qquad \Phi_t \circ \Phi_s = \Phi_{t+s}.$$

Moreover, it is often useful to fix t and consider the induced map

$$\Phi_t : M \to M$$

that shows the 'evolution' of the differential equation when 'jumping' directly from time 0 to time t. In this notation, compare the meaning of the (partial) derivatives:

$$D\Phi_t|_p = \partial_p \Phi|_{(t,p)} \qquad \text{en} \qquad \left.\frac{d}{dt}\right|_{t=0} \Phi_t(p) = \partial_t \Phi|_{(t,p)}.$$

The flow property implies

Corollary A.15. *$\Phi_t : M \to M$ is invertible and $(\Phi_t)^{-1} = \Phi_{-t}$.*

An autonomous differential equation or its flow is called **complete** if $I_\zeta = \mathbb{R}$ for all $\zeta \in M$. For example, flows on compact (sub)manifolds without boundary like spheres or tori are always complete.

Around regular points, the flow can be 'ironed' into a nice normal form:

Theorem A.16 (**Flow box theorem**). *Let $x' = f(x)$ be an autonomous ODE with flow Φ on a smooth manifold M and let $z \in M$ be a regular point, i.e., $f(z) \neq 0$. Then there exists a neighbourhood $U \subseteq M$ of z and a neighbourhood $\tilde{U} \subseteq \mathbb{R}^n$ of the origin and a coordinate transformation $\psi : U \to \tilde{U}$ such that $\psi(z) = 0$ and $x' = f(x)$ is transformed* *on $\tilde{U}$ into*

$$\begin{cases} y_1' &= 1, \\ y_2' &= 0, \\ \vdots \\ y_n' &= 0, \end{cases}$$

which has flow $\tilde{\Phi}_t(\tilde{z}) = \tilde{z} + te_1$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$, i.e., the transformed flow is parallel to the y_1-axis.

Proof. See for example [**Hohloch2**] or [**Teschl**]. □

This means in particular that the flow of an ODE near regular points always looks the same up to a coordinate transformation.

Definition A.17. *A fixed point z of an ODE $x' = f(x)$ is* **hyperbolic** *if $Df|_z$ has no eigenvalues of the form $i\sigma$ with $\sigma \in \mathbb{R}$.*

Near hyperbolic fixed points, the ODE is C^0-conjugate to its linearization:

Theorem A.18 (**Hartman-Grobman**). *Let $x' = f(x)$ be an autonomous ODE with flow Φ on a smooth, n-dimensional manifold M. Let $z \in M$ be a hyperbolic fixed point. Denote the flow of $y' = Df|_z.y$ on $T_zM \simeq \mathbb{R}^n$ by Ψ. Then there exist open neighbourhoods $U \subseteq M$ of z and $V \subseteq \mathbb{R}^n$ of the origin $\mathbf{0} \in \mathbb{R}^n$ and a homeomorphism $h : U \to V$ such that*

$$h(\Phi_t(x)) = \Psi_t(h(x))$$

for all $t \in \mathbb{R}$ and $x \in U$ with $\Phi_t(x) \in U$.

Proof. See for example [**Palis & de Melo**]. □

A.5. Some results from (functional) analysis

We recall some useful notions from global and functional analysis. For more details, see for example [**Hohloch1**] and the references therein.

Definition A.19. *Let* $(X, \|\ \|_X)$ *and* $(Y, \|\ \|_Y)$ *be normed vector spaces over a field* $\mathbb{F}$ *and* $T : X \to Y$ *a map.*

1) T *is a* **linear operator** *if*
$$T(\lambda x + \tilde{x}) = \lambda T(x) + T(\tilde{x}) \quad \forall\, x, \tilde{x} \in X,\ \forall\, \lambda \in \mathbb{F}.$$
2) *A linear operator* T *is* **bounded** *if*
$$\exists\, C > 0 : \quad \|T(x)\|_Y \leq C\,\|x\|_X \quad \forall\, x \in X.$$
3) *The set of linear bounded operators from* X *to* Y *is denoted by* $\mathscr{B}(X, Y)$.
4) *The* **operator norm** *of* $T \in \mathscr{B}(X, Y)$ *is given by*
$$\|T\| := \sup\{\|T(x)\|_Y \mid x \in X, \|x\|_X \leq 1\}.$$
5) *The space* $\mathscr{B}(X, \mathbb{R})$ *and* $\mathscr{B}(X, \mathbb{C})$ *are the real and complex* **dual space** *of* X. *Sometimes they are denoted by* X^* *or* X' *in the literature.*

Moreover

Definition A.20. *Let* $(X, \|\ \|_X)$ *and* $(Y, \|\ \|_Y)$ *be Banach spaces and* $U \subseteq X$ *open.* $f : U \to Y$ *is* **(Fréchet) differentiable in** $x \in U$ *if there exists* $T_x \in \mathscr{B}(X, Y)$ *such that*
$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - T_x(h)\|_Y}{\|h\|_X} = 0.$$
If such a T_x *exists it is usually denoted by* $Df|_x$ *or* $df(x)$ *and called the* **(Fréchet) derivative** *of* f *in* $x \in U$. *The map* f *is* **(Fréchet) differentiable** *if* f *is (Fréchet) differentiable in all* $x \in U$.

Higher regularity is defined as follows.

Definition A.21. *Let* X, Y *be Banach spaces,* $U \subseteq X$ *open and* $f : U \to Y$ *Fréchet differentiable.* f *is* C^k *for* $k \in \mathbb{N}^{\geq 1}$ *if the following map is* C^{k-1} :
$$U \to \mathscr{B}(X, Y), \qquad x \mapsto Df|_x$$

The *Inverse function theorem* from the finite dimensional setting generalizes verbatim to Banach spaces:

Theorem A.22 (**Inverse function theorem**). *Let* X, Y *be Banach spaces and* $U \subseteq X$ *open, and* $f : U \to Y$ *a* C^k*-map with* $k \in \mathbb{N}^{\geq 1}$. *Let* $x_0 \in U$ *with* $Df|_{x_0} \in \mathscr{B}(X, Y)$ *bijective. Then there exists an open neighbourhood* $U_0 \subseteq U$ *of* x_0 *such that the restriction* $f|_{U_0} : U_0 \to Y$ *is injective,* $V_0 := f(U_0)$ *is open in* Y *and*
$$(f|_{U_0})^{-1} : V_0 \to U_0 \text{ is } C^k \quad \text{and} \quad (Df^{-1})|_y = \left(Df|_{f^{-1}(y)}\right)^{-1} \quad \forall\, y \in V_0.$$

Proof. Cf. appendix of [**McDuff & Salamon**]. □

Let us recall the *Implicit function theorem* in *finite* dimensions before we address the infinite dimensional version:

Theorem A.23 (**Implicite functions: finite dimensional version**). *Let $n, m \in \mathbb{N}_0$ with $n \geq m$ and $U \subseteq \mathbb{R}^n$ open. Let $f \in C^q(U, \mathbb{R}^m)$ with $q \geq 1$ and write $\mathbb{R}^n \simeq \mathbb{R}^{(n-m)} \times \mathbb{R}^m$ with coordinates $(z_1, \ldots, z_n) = (x_1, \ldots, x_{n-m}, y_1, \ldots, y_m) = (x, y)$. Moreover, assume that $p = (\kappa, \lambda) \in U \subseteq \mathbb{R}^{n-m} \times \mathbb{R}^m$ with $f(p) =: r$ satisfies*

$$\det D_y f|_p := \det \left(D_{((n-m)+j)} f_i|_p \right)_{1 \leq i, j \leq m} \neq 0.$$

Then $\exists\, K \subseteq \mathbb{R}^{n-m}$ open, $\exists\, L \subseteq \mathbb{R}^m$ open with $p = (\kappa, \lambda) \in K \times L \subseteq U$ and $\exists\, \varphi \in C^q(K, L)$ such that

$$(x, y) \in K \times L \ \text{ and } \ f(x, y) = r \quad \Leftrightarrow \quad x \in K \ \text{ and } \ y = \varphi(x).$$

In particular, we have $\lambda = \varphi(\kappa)$.

When generalizing the *Implicite function theorem* from the finite dimensional setting, things get more interesting. First we define

Definition A.24. *Let X, Y be Banach spaces, $U \subseteq X$ open and path-connected, and $f : U \to Y$ a C^k-map.*

1) *f is a* **Fredholm map** *if $Df|_x$ is a Fredholm operator for all $x \in U$.*
2) *If f is a Fredholm map we define its* **Fredholm index** *via*

$$\operatorname{Ind}(f) := \operatorname{Ind}(Df|_x) \quad \forall\, x \in U.$$

Moreover, we need

Definition A.25. *Let X, Y be Banach spaces, $U \subseteq X$ open and $f : U \to Y$ a C^k-map. $y \in Y$ is a* **regular value** *of f if $Df|_x \in \mathscr{B}(X, Y)$ is surjective for all $x \in f^{-1}(y) \subseteq U$.*

The following theorem displays the geometric implications Fredholm maps have. Note that $\mathbb{R}^n$ with its usual norms is a Banach space and that linear maps between finite dimensional vector spaces are always Banach. So the finite dimensional version of the *implicite function theorem* is well included in the following infinite dimensional version.

Theorem A.26 (**Implicite function theorem**). *Let X, Y be Banach spaces, $U \subseteq X$ open, $f : U \to Y$ a C^k-map with $k \in \mathbb{N}^{\geq 1}$, and $y \in Y$ a regular value of f. Then $M := f^{-1}(y) \subseteq X$ is a C^k-Banach manifold whose tangent space satisfies*

$$T_x M = \ker Df|_x \quad \forall\, x \in M.$$

If f is Fredholm then M is a finite dimensional manifold where the dimension of the connected component of M containing x is given by

$$\dim(T_x M) = \dim(\ker(Df|_x)) < \infty.$$

Proof. Cf. appendix of [**McDuff & Salamon**]. □

It remains to inquire how 'typical' it is for a value $y \in Y$ to be nonsingular.

Definition A.27. *A set is said to be* **of second Baire category** *if can be written as a countable intersection of open and dense sets. A property is called* **generic** *if it holds true on a set of second Baire category. A property is* **nongeneric** *if it is not generic.*

Note that we consider in the following not only regular values as 'nonsingular' but also values outside the range of the function, i.e., values with empty fibers.

Theorem A.28 (**Sard's Theorem**). *Let M, N be smooth manifolds and $f : M \to N$ a C^k-map with $k \geq \max\{1, \dim(M) - \dim(N)\}$ and consider $N_{sing} = \{y \in M \mid y$ singular value of $f\}$. Then $N \setminus N_{sing} = N_{reg,\emptyset}$ is of second Baire category. In particular, N_{sing} has Lebesgue measure zero.*

Proof. Cf. [**Hirsch**]. □

There exists also an infinite dimensional version of Theorem A.28 (Sard, finite dimensional version):

Theorem A.29 (**Sard's Theorem**). *Let X, Y be separable Banach spaces, $U \subseteq X$ open, $f : U \to Y$ a C^k-map with*

$$k \geq \max\{1, \operatorname{Ind}(f) + 1\}$$

and consider $Y_{sing} = \{y \in Y \mid y$ singular value of $f\}$. Then $Y \setminus Y_{sing} = Y_{reg,\emptyset}$ is of second Baire category.

PROOF. Cf. the appendix of [**McDuff & Salamon**]. ◻

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