Macroeconomic Forecasting: Introduction
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Introduction – Local Level Model

Program :

- Introduction
- Local level model
- Statistical dynamic properties
- Signal extraction, filtering and prediction
- Likelihood function and parameter estimation.
- Literature : J. Durbin and S.J. Koopman (2012), "Time Series Analysis by State Space Methods", Second Edition, Oxford: Oxford University Press. Chapter 2.

Time Series

A time series is a set of observations y_t, each one recorded at a specific time t.

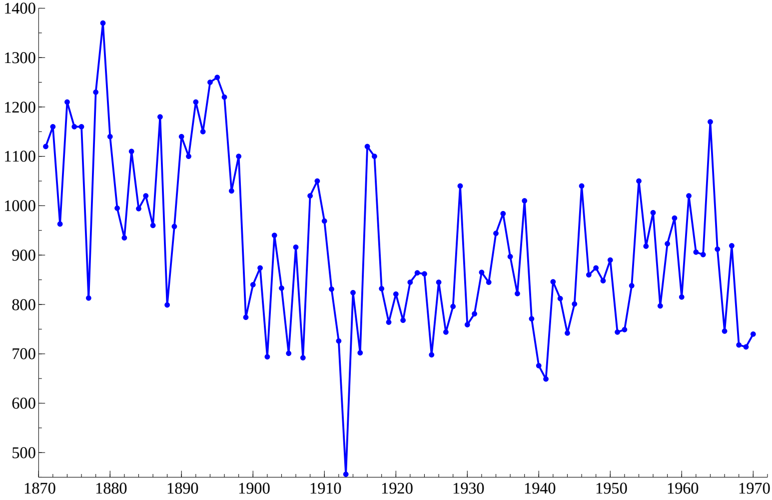
The observations are ordered over time.
We assume to have n observations, $t = 1, \ldots, n$.

Examples of time series are:

- Number of cars sold, every year
- Gross Domestic Product, of a country, every quarter
- Stock price changes, tick-by-tick, within one trading day
- CO_2 emissions, of a country, every month

Time series modeling is relevant for a wide variety of tasks and fields, including economic policy, financial decision making, climate change monitoring, and forecasting

Nile Data



1400
1300
1200
1100
1000
900
800
700
600
500
1870
1880
1890
1900
1910
1920
1930
1940
1950
1960
1970

Time Series

A time series for a single entity is typically denoted by

$$y_1, \ldots, y_n \quad \Leftrightarrow \quad y_t, \qquad t = 1, \ldots, n,$$

where t is the time index and n is time series length.
The current value is y_t.
The first lagged value, or first lag, is y_{t-1}.
The τth lagged value, or τ-th lag, is $y_{t-\tau}$ for $\tau = 1, 2, 3, \ldots$.

The change between period $t-1$ and period t is $y_t - y_{t-1}$.
This is called the first difference denoted by $\Delta y_t = y_t - y_{t-1}$.

In economic time series, we often take the first difference of the logarithm, or the log-difference, that is

$$\Delta \log y_t = \log y_t - \log y_{t-1} = \log(y_t / y_{t-1}),$$

is a proxy of proportional change, see Appendix.
Percentage change is then $100\Delta \log y_t$.

Autoregressive model: AR(1)

The AR(1) model is given by

$$y_t = \mu + \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2),$$

with three parameter coefficients μ, ϕ and σ_ε^2 with $0 < \sigma_\varepsilon < \infty$.

Stationarity condition: $|\phi| < 1$.

Statistical dynamic properties:

- Mean $\mathbb{E}(y_t) = \mu \,/\, (1 - \phi)$; in case $\mu = 0$, $\mathbb{E}(y_t) = 0$;
- Variance $\mathbb{V}\mathrm{ar}(y_t) = \sigma^2 \,/\, (1 - \phi^2)$;
- Autocovariance lag 1 is $\mathbb{C}\mathrm{ov}(y_t, y_{t-1}) = \phi\, \sigma^2 \,/\, (1 - \phi^2)$;
- and for lag $\tau = 2, 3, 4, \ldots$ is $\mathbb{C}\mathrm{ov}(y_t, y_{t-\tau}) = \phi^\tau \sigma^2 \,/\, (1 - \phi^2)$;
- Autocorrelation lag $\tau = 1, 2, 3, \ldots$ is $\mathbb{C}\mathrm{orr}(y_t, y_{t-\tau}) = \phi^\tau$.

Autoregressive model: AR(1)

The AR(1) model is given by

$$y_t = \mu + \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2),$$

with three parameter coefficients μ, ϕ and σ_ε^2 with $0 < \sigma_\varepsilon < \infty$.

- Unconditional distribution:

$$\mathbb{E}(y_t) = \mu \,/\, (1 - \phi), \qquad \mathbb{V}\mathrm{ar}(y_t) = \sigma^2 \,/\, (1 - \phi^2)$$

- Conditional distribution:

$$\mathbb{E}(y_t | Y_{t-1}) = \mu + \phi y_{t-1}, \qquad \mathbb{V}\mathrm{ar}(y_t | Y_{t-1}) = \sigma^2$$

where $Y_t = \{y_1, \ldots, y_t\}$.

Moving Average model: MA(1)

The MA(1) model is given by

$$y_t = \mu + \theta\varepsilon_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2),$$

with three parameter coefficients μ, θ and σ_ε^2 with $0 < \sigma_\varepsilon < \infty$.

Invertibility condition: $|\theta| < 1$.

Statistical dynamic properties:

- Mean $\mathbb{E}(y_t) = \mu$; in case $\mu = 0$, $\mathbb{E}(y_t) = 0$;
- Variance $\mathbb{V}\text{ar}(y_t) = \sigma^2\,(1 + \theta^2)$;
- Autocovariance lag 1 is $\mathbb{C}\text{ov}(y_t, y_{t-1}) = \theta\,\sigma^2$;
- ... for lag $\tau = 2, 3, 4, \ldots$ is $\mathbb{C}\text{ov}(y_t, y_{t-\tau}) = 0$;
- Autocorrelation lag 1 is $\mathbb{C}\text{orr}(y_t, y_{t-1}) = \theta\,/\,(1 + \theta^2)$.

Moving Average model: MA(1)

The MA(1) model is given by

$$y_t = \mu + \theta\varepsilon_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2),$$

with three parameter coefficients μ, θ and σ_ε^2 with $0 < \sigma_\varepsilon < \infty$.

- Unconditional distribution:

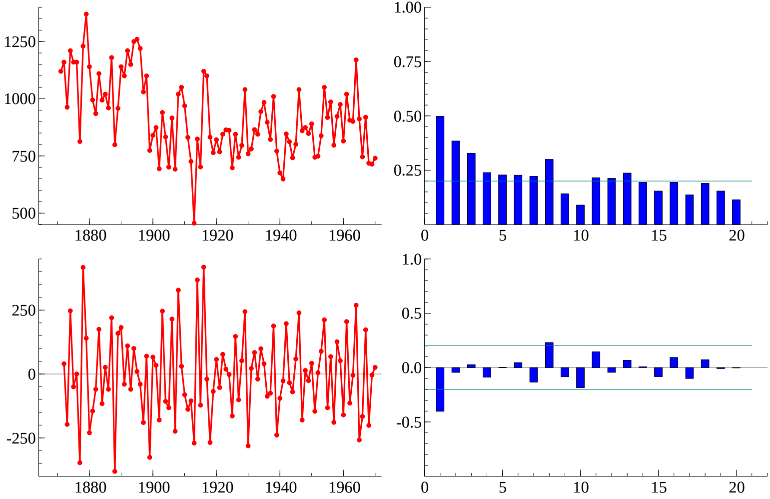
$$\mathbb{E}(y_t) = \mu, \qquad \mathbb{V}\text{ar}(y_t) = \sigma^2\,(1 + \theta^2)$$

- Conditional distribution:

$$\mathbb{E}(y_t | Y_{t-1}) = \mu + \theta\varepsilon_{t-1}, \qquad \mathbb{V}\text{ar}(y_t | Y_{t-1}) = \sigma^2$$

where ε_{t-1} can be reconstructed from Y_{t-1} and with $\varepsilon_0 = 0$.

Example: Nile in levels and Nile in differences



Local Level:

model formulation and statistical properties

Constant vs Time Varying Mean Model

- Constant mean:
  - fixed level μ :

$$y_t = \mu + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2)$$

- Time Varying mean: replace μ by μ_t with
  - deterministic function of time:

$$\mu_t = a + b\,t + c\,t^2 + \dots$$

  - stochastic function of time, for example:

$$\mu_t = \mu_{t-1} + \eta_t, \qquad \eta_t \sim \mathcal{NID}(0, \sigma_\eta^2)$$

Local Level Model

$$y_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2)$$
$$\mu_{t+1} = \mu_t + \eta_t, \qquad \eta_t \sim \mathcal{NID}(0, \sigma_\eta^2)$$

- Time-varying level is modelled as a random walk process;
- Notice the updating in terms of t or $t+1$;
- The disturbances ε_t, η_s are independent for all s, t;
- The model is incomplete without initial specification for μ_1;
- The process μ_t is nonstationary and y_t is nonstationary.

Local Level Model

The local level model or random walk plus noise model :

$$y_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2)$$
$$\mu_{t+1} = \mu_t + \eta_t, \qquad \eta_t \sim \mathcal{NID}(0, \sigma_\eta^2)$$

- The level μ_t and irregular ε_t are both unobserved;
- We still need to define μ_1;
- Parameters σ_ε^2 and σ_η^2 are unknown;
- Define q as the *signal-to-noise ratio* : $q = \sigma_\eta^2 \,/\, \sigma_\varepsilon^2$.

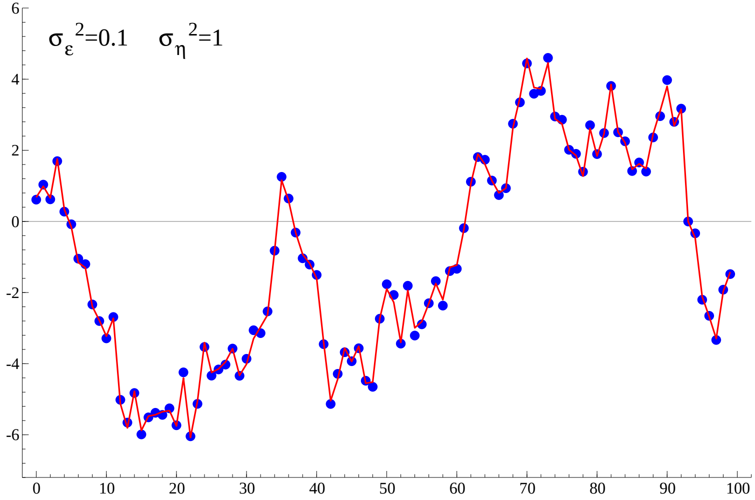
Local Level Model

The local level model or random walk plus noise model :

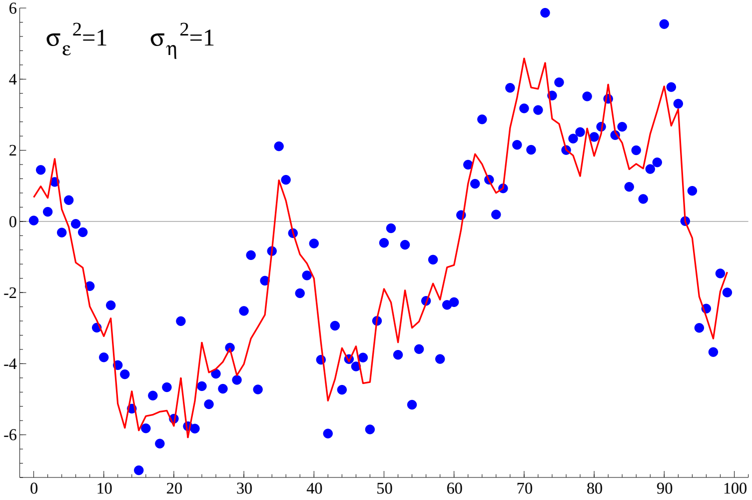
$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t, \qquad && \varepsilon_t \sim \mathcal{NID}(0, \sigma_\varepsilon^2) \\
\mu_{t+1} &= \mu_t + \eta_t, \qquad && \eta_t \sim \mathcal{NID}(0, \sigma_\eta^2)
\end{aligned}
$$

- Trivial special cases:
  - $\sigma_\eta^2 = 0 \implies y_t \sim \mathcal{NID}(\mu_1, \sigma_\varepsilon^2)$ (IID, global level);
  - $\sigma_\varepsilon^2 = 0 \implies y_{t+1} = y_t + \eta_t$ (random walk);
- Local Level model is basic illustration of state space model.
- It is very easy to simulate data from the local level model.

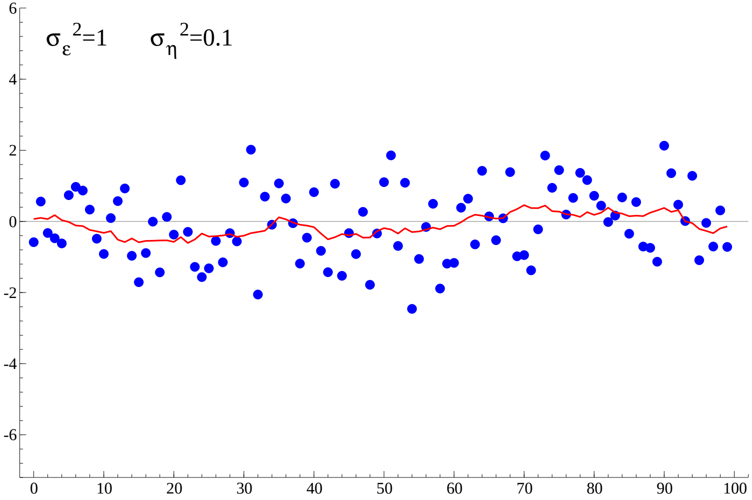
Simulated Local Level data, $q = 10$



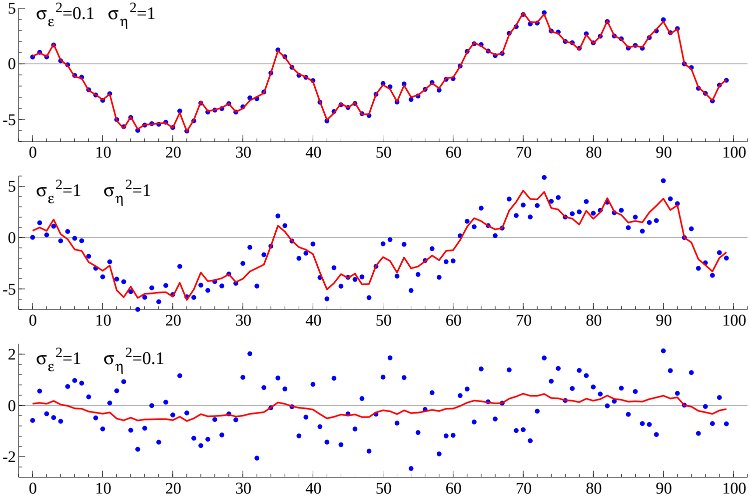
Simulated Local Level data, $q = 1$



Simulated Local Level data, $q = 0.1$



Simulated Local Level data



Statistical Properties

- Local Level Model, non-stationary y_t :

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, \qquad & \varepsilon_t &\sim \mathcal{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \eta_t, \qquad & \eta_t &\sim \mathcal{NID}(0, \sigma_\eta^2). \end{aligned}$$

- First difference of y_t, stationary Δy_t:

$$\Delta y_t = \Delta \mu_t + \Delta \varepsilon_t = \eta_{t-1} + \varepsilon_t - \varepsilon_{t-1}.$$

- Dynamic properties of Δy_t: $\mathbb{E}(\Delta y_t) = 0$ and

$$\begin{aligned} \gamma_0 &= \mathbb{E}(\Delta y_t \Delta y_t) &&= \sigma_\eta^2 + 2\sigma_\varepsilon^2, \\ \gamma_1 &= \mathbb{E}(\Delta y_t \Delta y_{t-1}) &&= -\sigma_\varepsilon^2, \\ \gamma_\tau &= \mathbb{E}(\Delta y_t \Delta y_{t-\tau}) &&= 0, \qquad \text{for } \tau \geq 2. \end{aligned}$$

Properties of Local Level model

- Define theoretical autocorrelation function (acf)

$$\rho_\tau = \gamma_\tau \,/\, \gamma_0, \qquad \tau = 1, 2, \ldots$$

- We have q as the *signal-to-noise ratio* : $q = \sigma_\eta^2 \,/\, \sigma_\varepsilon^2$.
- We have $\gamma_0 = \sigma_\eta^2 + 2\sigma_\varepsilon^2$, $\gamma_1 = -\sigma_\varepsilon^2$, $\gamma_\tau = 0$ for $\tau \geq 2$.
- The theoretical acf of Δy_t in local level model is

$$\rho_1 = \frac{-\sigma_\varepsilon^2}{\sigma_\eta^2 + 2\sigma_\varepsilon^2} = -\frac{1}{q+2},$$
$$\rho_\tau = 0, \qquad \tau \geq 2.$$

- This acf is the same as acf of AR(1) or MA(1) ?
- We notice that

$$-1/2 \leq \rho_1 \leq 0$$

Properties of Local Level model (ctd)

- The local level model implies that $\Delta y_t \sim$ MA(1) but with the acf function $\rho_1 = -1/(q+2)$, it is restricted;
- Hence $y_t \sim$ ARIMA$(0,1,1)$.
- An alternative representation for Δy_t is the MA(1) model

$$\Delta y_t = \xi_t + \theta \xi_{t-1}, \qquad \xi_t \sim \mathcal{NID}(0, \sigma^2).$$

- The acf of an MA(1) process is $\rho_1 = \theta \,/\, (1+\theta^2)$.
- When y_t comes from a local level model, we have a restricted parameter space for $\theta :\ -1 < \theta < 0$.
- To express θ as function of q, solve equality for the two ρ_1's:

$$\theta = \frac{1}{2}\left(\sqrt{q^2 + 4q} - 2 - q\right).$$

Local Level Model

The Local Level model is given by

$$y_t = \mu_t + \varepsilon_t, \qquad \mu_{t+1} = \mu_t + \eta_t, \qquad t = 1, \ldots, n.$$

- The parameters σ^2_ε and σ^2_η are unknown and need to be estimated, typically via maximum likelihood estimation;
- Parameter estimation via maximum likelihood will be discussed soon.
- When we treat parameters σ^2_ε and σ^2_η as known, how to "estimate" the unobserved series $\mu_1, \ldots, \mu_n$?
- This "estimation" is referred to as signal extraction.
- We base this "estimation" on conditional expectations.
- Signal extraction is the recursive evaluation of conditional means and variances of the unobserved μ_t for $t = 1, \ldots, n$.

What are the take-aways so far ?

- Statistical formulation of local level model
- Role of signal-to-noise ratio
- Statistical (dynamic) properties and relation to ARIMA model

Signal extraction and prediction

some basics on the bivariate normal distribution

Normal density

Consider a random variable x that is normally distributed

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2).$$

The density function for x is given by

$$f(x) = c \exp(-\frac{1}{2} Q_x),$$

where

$$c^{-1} = \sigma_x \sqrt{2\pi}, \qquad Q_x = (x - \mu_x)^2 \,/\, \sigma_x^2.$$

The logdensity function for x is given by

$$\log f(x) = -\frac{1}{2} \log 2\pi - \frac{1}{2} \log \sigma_x^2 - \frac{1}{2} Q_x.$$

Bivariate normal distribution

Consider two random variable x and y that are normally distributed

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2), \qquad y \sim \mathcal{N}(\mu_y, \sigma_y^2), \qquad \mathbb{C}\mathrm{ov}(x, y) = \sigma_{xy},$$

where we let $\sigma_{xy} = \sigma_x \sigma_y \rho$ with $-1 \leq \rho \leq 1$.

In case $\rho = 0$ and hence $\sigma_{xy} = 0$, the variables are *independent* or *uncorrelated*. The joint normal density function is simply

$$f(x, y) = f(x)\, f(y).$$

Also,

$$\mathbb{E}(x|y) = \mathbb{E}(x) = \mu_x, \qquad \mathbb{V}\mathrm{ar}(x|y) = \mathbb{V}\mathrm{ar}(x) = \sigma_x^2.$$

Bivariate normal distribution, ctd.

Consider two random variable x and y that are normally distributed

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2), \qquad y \sim \mathcal{N}(\mu_y, \sigma_y^2), \qquad \mathbb{C}\text{ov}(x, y) = \sigma_{xy},$$

where $\sigma_{xy} = \sigma_x \sigma_y \rho$ with $-1 \leq \rho \leq 1$.

In case $-1 \leq \rho \leq 1$ and hence $\sigma_{xy} \neq 0$, the variables are *dependent* and hence *correlated*. The joint density function is normal and (to avoid matrix algebra for now) it can be expressed by

$$f(x, y) = f(x|y)\, f(y).$$

What are the expressions for $\mathbb{E}(x|y)$ and $\mathbb{V}\text{ar}(x|y)$?

Conditional mean and variance

Consider two random variable x and y that are normally distributed

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2), \qquad y \sim \mathcal{N}(\mu_y, \sigma_y^2), \qquad \mathbb{C}\text{ov}(x, y) = \sigma_{xy} \neq 0,$$

where $\sigma_{xy} = \sigma_x \sigma_y \rho$. To obtain expressions for the conditional mean $\mathbb{E}(x|y)$ and variance $\mathbb{V}\text{ar}(x|y)$, we define

$$y = \mu_y + \sigma_y z_y, \qquad x = \mu_x + \sigma_x[\rho z_y + \sqrt{1-\rho^2} z_x],$$

where z_x , $z_y \sim \mathcal{N}(0, 1)$ are independently distributed such that

$$f(z_x, z_y) = \frac{1}{2\pi} \exp\left[\frac{1}{2}(z_x^2 + z_y^2)\right].$$

Please verify that statistical properties of x and y (mean, var, cov) are the same when based on the two expressions above.

Conditional mean, derivation for bivariate $\mathcal{N}$

Given that

$$y = \mu_y + \sigma_y z_y, \qquad x = \mu_x + \sigma_x[\rho z_y + \sqrt{1-\rho^2} z_x],$$

we have

$$\begin{aligned}
\mathbb{E}(x|y) &= \mathbb{E}\left[\mu_x + \sigma_x \left(\rho z_y + \sqrt{1-\rho^2} z_x\right) |y\right] \\
&= \mu_x + \sigma_x \, \mathbb{E}\left[\left(\rho \frac{y-\mu_y}{\sigma_y} + \sqrt{1-\rho^2} z_x\right) |y\right] \\
&= \mu_x + \sigma_x \sigma_y \rho \frac{y-\mu_y}{\sigma_y^2} + \sqrt{1-\rho^2} \, \mathbb{E}(z_x|z_y) \\
&= \mu_x + \sigma_{xy}(y-\mu_y) \, / \, \sigma_y^2.
\end{aligned}$$

Conditional variance, derivation for bivariate $\mathcal{N}$

Given that

$$y = \mu_y + \sigma_y z_y, \qquad x = \mu_x + \sigma_x[\rho z_y + \sqrt{1-\rho^2} z_x],$$

we have

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(x|y) &= \mathbb{V}\mathrm{ar}\left[\mu_x + \sigma_x\left(\rho z_y + \sqrt{1-\rho^2} z_x\right)|y\right] \\
&= \mathbb{V}\mathrm{ar}\left[\mu_x + \sigma_x\left(\rho \frac{y-\mu_y}{\sigma_y} + \sqrt{1-\rho^2} z_x\right)|y\right] \\
&= \mathbb{V}\mathrm{ar}\left[\sigma_x\sqrt{1-\rho^2} z_x | z_y\right] \\
&= \sigma_x^2(1-\rho^2) \\
&= \sigma_x^2 - \sigma_{xy}^2 \,/\, \sigma_y^2.
\end{aligned}$$

Conditional mean and variance of bivariate $\mathcal{N}$

Consider two random variable x and y that are normally distributed

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2), \qquad y \sim \mathcal{N}(\mu_y, \sigma_y^2), \qquad \mathbb{C}\mathrm{ov}(x, y) = \sigma_{xy} \neq 0.$$

The conditional mean and variance are given by

$$\mathbb{E}(x|y) = \mu_x + \sigma_{xy}(y - \mu_y) \,/\, \sigma_y^2, \qquad \mathbb{V}\mathrm{ar}(x|y) = \sigma_x^2 - \sigma_{xy}^2 \,/\, \sigma_y^2.$$

Verify these results and get familiar with these basic principles.

Notice that (1) $\mathbb{E}(x|y)$ is a function of y but $\mathbb{V}\mathrm{ar}(x|y)$ is not;
(2) when $\sigma_{xy} = 0$, $\mathbb{E}(x|y) = \mu_x$ and $\mathbb{V}\mathrm{ar}(x|y) = \sigma_x^2$.

Estimation Error / Prediction Error

Consider two random variable x and y that are normally distributed

$$x \sim \mathcal{N}(\mu_x, \sigma_x^2), \qquad y \sim \mathcal{N}(\mu_y, \sigma_y^2), \qquad \mathbb{C}\mathrm{ov}(x, y) = \sigma_{xy} \neq 0.$$

When considering the conditional mean $\mathbb{E}(x|y)$ as an estimate of x, the estimation error is

$$e = x - \mathbb{E}(x|y) = (x - \mu_x) - \sigma_{xy}(y - \mu_y) \,/\, \sigma_y^2,$$

and its properties are

$$\mathbb{E}(e) = 0, \quad \mathbb{V}\mathrm{ar}(e) = \sigma_x^2 - \sigma_{xy}^2 \,/\, \sigma_y^2 = \mathbb{V}\mathrm{ar}(x|y),$$

and

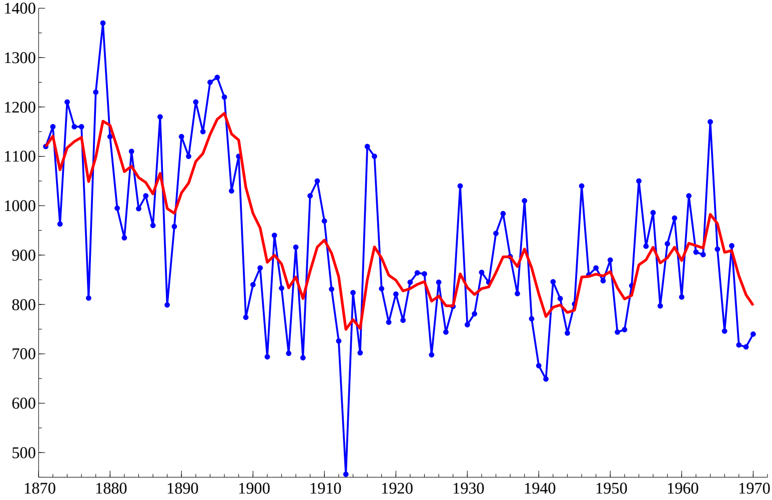
$$\mathbb{C}\mathrm{ov}(e, y) = \mathbb{E}[e\,(y - \mu_y)] = \sigma_{xy} - \sigma_{xy} = 0,$$

implying $\mathbb{E}(e|y) = \mathbb{E}(e) = 0$ and $\mathbb{V}\mathrm{ar}(e|y) = \mathbb{V}\mathrm{ar}(e) = \mathbb{V}\mathrm{ar}(x|y)$.

Signal extraction and prediction

back to local level model

Signal Extraction for Nile Data: filtered estimate of level



Local Level Model: signal extraction

Local Level model :

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad \mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2).$$

We are at time point t.
Assume we have collected observations for $y_1, \ldots, y_{t-1}$ and that the conditional density $f(\mu_t | y_1, \ldots, y_{t-1})$ is normal with known mean a_t and known variance p_t, we have

$$\mu_t | y_1, \ldots, y_{t-1} \sim f(\mu_t | y_1, \ldots, y_{t-1}) \equiv \mathcal{N}(a_t, p_t).$$

We collect observation for y_t, the conditional density of interest is

$$f(\mu_t | y_1, \ldots, y_t),$$

This conditional density turns out to be normal as well

$$f(\mu_t | y_1, \ldots, y_t) \equiv \mathcal{N}(a_{t|t}, p_{t|t}),$$

What are the expressions for $a_{t|t}$ and $p_{t|t}$? This is next !

Prediction error

Local Level model :

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad \mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2).$$

Given $f(\mu_t | y_1, \ldots, y_{t-1}) \equiv \mathcal{N}(a_t, p_t)$, we construct a forecast for y_t, that is

$$\widehat{y}_t = \mathbb{E}(y_t | Y_{t-1}) = \mathbb{E}(\mu_t + \varepsilon_t | Y_{t-1}) = a_t.$$

The corresponding prediction error is

$$v_t = y_t - \widehat{y}_t = y_t - a_t,$$

with $\mathbb{E}(v_t) = \mathbb{E}(y_t - a_t) = \mathbb{E}[(\mu_t - a_t) + \varepsilon_t] = 0$ and

$$\mathbb{V}\mathrm{ar}(v_t) = \mathbb{V}\mathrm{ar}(y_t - a_t) = \mathbb{V}\mathrm{ar}[(\mu_t - a_t) + \varepsilon_t] = p_t + \sigma_\varepsilon^2,$$

see results in Slide "Estimation Error / Prediction Error".

Prediction error

We have prediction error

$$v_t = y_t - a_t, \qquad \text{with} \quad \mathbb{E}(v_t) = 0, \qquad \mathbb{V}\text{ar}(v_t) = p_t + \sigma_\varepsilon^2.$$

It also follows from Slide "Estimation Error / Prediction Error" that

$$\mathbb{C}\text{ov}(v_t, y_{t-j}) = 0, \quad \text{for} \quad j = 1, \ldots, t-1.$$

Hence, we also have

$$\mathbb{E}(v_t | Y_{t-1}) = \mathbb{E}(v_t) = 0,$$
$$\mathbb{V}\text{ar}(v_t | Y_{t-1}) = \mathbb{V}\text{ar}(v_t) = p_t + \sigma_\varepsilon^2,$$

and

$$f(y_t | Y_{t-1}) = \mathcal{N}(a_t, p_t + \sigma_\varepsilon^2), \quad f(v_t) = f(v_t | Y_{t-1}) = \mathcal{N}(0, p_t + \sigma_\varepsilon^2).$$

Local Level Model: signal extraction

To obtain an expression for $a_{t|t}$ and $p_{t|t}$ in $f(\mu_t|Y_t) = \mathcal{N}(a_{t|t}, p_{t|t})$, we next make the point that we can "*re-shuffle*" linearly the conditional information set: we have $f(a|b,c) \equiv f(a|b,d)$, when, for example, $d = c - b$. Hence we can also have

$$f(\mu_t|Y_t) \equiv f(\mu_t|v_t, Y_{t-1})$$

since $v_t = y_t - \mathbb{E}(\mu_t|Y_{t-1})$ is a fixed linear function of Y_t, more on this below. Then we have

$$\begin{aligned} f(\mu_t|v_t, Y_{t-1}) &= f(\mu_t, v_t|Y_{t-1})/f(v_t|Y_{t-1}) \\ &= f(\mu_t|Y_{t-1})f(v_t|\mu_t, Y_{t-1})/f(v_t|Y_{t-1}), \end{aligned}$$

where all $f()$'s are normal densities.

Local Level Model: signal extraction

We have

$$f(\mu_t|v_t, Y_{t-1}) = f(\mu_t|Y_{t-1})f(v_t|\mu_t, Y_{t-1})/f(v_t|Y_{t-1}),$$

with

$$f(\mu_t|Y_{t-1}) = \mathcal{N}(a_t, p_t), \qquad f(v_t|Y_{t-1}) = \mathcal{N}(0, p_t + \sigma_\varepsilon^2),$$

but what about $f(v_t|\mu_t, Y_{t-1})$?

Given that $v_t = y_t - a_t = \mu_t + \varepsilon_t - a_t$, we have

$$\mathbb{E}(v_t|\mu_t, Y_{t-1}) = \mu_t - a_t, \qquad \mathbb{V}\mathrm{ar}(v_t|\mu_t, Y_{t-1}) = \sigma_\varepsilon^2.$$

and hence

$$f(v_t|\mu_t, Y_{t-1}) = \mathcal{N}(\mu_t - a_t, \sigma_\varepsilon^2).$$

Local Level Model: signal extraction

It follows that

$$\begin{aligned} f(\mu_t|v_t, Y_{t-1}) &= f(\mu_t|Y_{t-1}) \times f(v_t|\mu_t, Y_{t-1}) \,/\, f(v_t|Y_{t-1}) \\ &= \mathcal{N}(a_t, p_t) \times \mathcal{N}(\mu_t - a_t, \sigma_\varepsilon^2) \,/\, \mathcal{N}(0, p_t + \sigma_\varepsilon^2). \end{aligned}$$

Given the functional form of the normal density, we have $f(\mu_t|Y_t) = \text{const.} \times \exp\big(-\frac{1}{2}Q_t\big)$ with

$$Q_t = (\mu_t - a_t)^2/p_t + (v_t - \mu_t + a_t)^2/\sigma_\varepsilon^2 - v_t^2/(p_t + \sigma_\varepsilon^2).$$

After some algebra ("completing the square"), we have

$$Q_t = \frac{p_t + \sigma_\varepsilon^2}{p_t\,\sigma_\varepsilon^2}\Big(\mu_t - a_t - \frac{p_t\,v_t}{p_t + \sigma_\varepsilon^2}\Big)^2.$$

Local Level Model: filter density

The filter density function $f(\mu_t|Y_t)$ is normal and has functional form,

$$f(\mu_t|Y_t) = \text{const.} \times \exp\big(-\frac{1}{2}Q_t\big),$$

with

$$Q_t = \frac{p_t + \sigma_\varepsilon^2}{p_t\,\sigma_\varepsilon^2}\big(\mu_t - a_t - \frac{p_t\,v_t}{p_t + \sigma_\varepsilon^2}\big)^2.$$

It implies that

$$f(\mu_t|Y_t) \equiv \mathcal{N}(a_{t|t}, p_{t|t}),$$

with

$$a_{t|t} = a_t + k_t\,v_t, \qquad p_{t|t} = k_t\,\sigma_\varepsilon^2, \qquad k_t = \frac{p_t}{p_t + \sigma_\varepsilon^2}.$$

Presto ! We have expressions for $a_{t|t}$ and $p_{t|t}$, congratulations !!

Local Level Model: prediction

Local Level model :

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2_\varepsilon), \quad \mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma^2_\eta).$$

In addition, we are typically interested in the *predicted* signal density

$$f(\mu_{t+1}|Y_t) \equiv \mathcal{N}(a_{t+1}, p_{t+1}),$$

where

$$\begin{aligned} a_{t+1} &= \mathbb{E}(\mu_{t+1}|Y_t) = \mathbb{E}(\mu_t + \eta_t|Y_t) = a_{t|t}, \\ p_{t+1} &= \mathbb{V}\text{ar}(\mu_t + \eta_t|Y_t) = p_{t|t} + \sigma^2_\eta. \end{aligned}$$

We have obtained the updating equations

$$a_{t+1} = a_t + k_t\, v_t, \qquad p_{t+1} = k_t\, \sigma^2_\varepsilon + \sigma^2_\eta, \qquad k_t = \frac{p_t}{p_t + \sigma^2_\varepsilon}.$$

These are **recursive equations**.

Kalman filter for the Local Level Model

Local Level model :

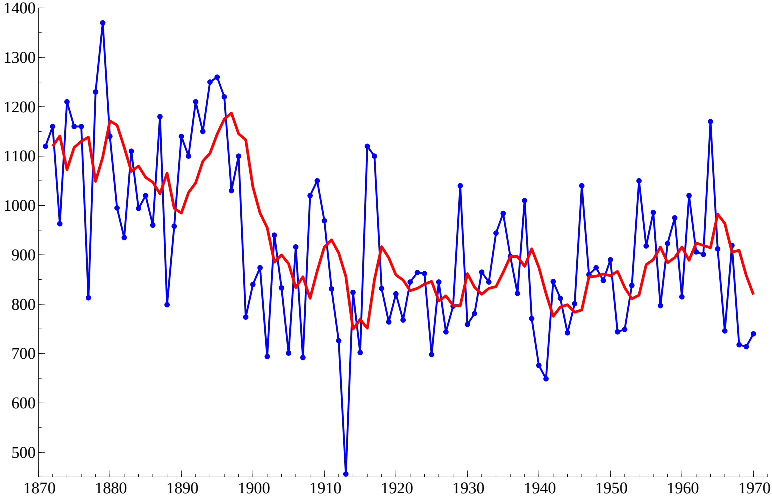
$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2_\varepsilon), \quad \mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma^2_\eta).$$

For $f(\mu_t | Y_{t-1}) = \mathcal{N}(a_t, p_t)$, with given values of a_t and p_t, the Kalman filter update equation are given by

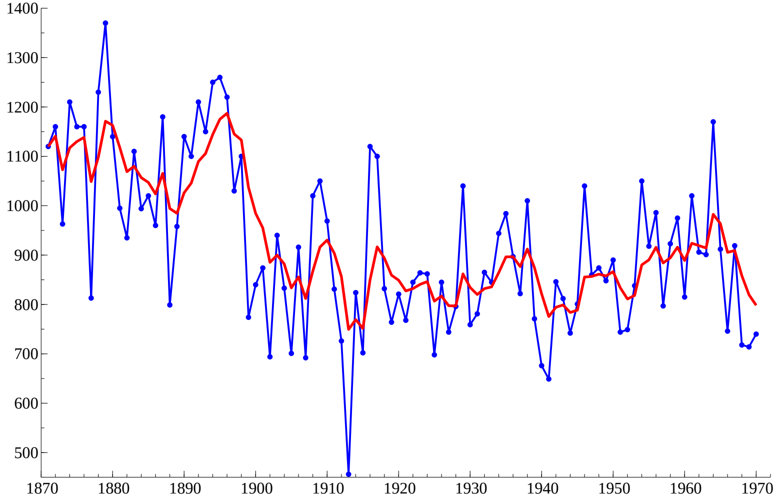
$$\begin{aligned} v_t &= y_t - a_t, & k_t &= p_t \,/\, (p_t + \sigma^2_\varepsilon), \\ a_{t|t} &= a_t + k_t v_t, & p_{t|t} &= k_t\, \sigma^2_\varepsilon, \\ a_{t+1} &= a_{t|t}, & p_{t+1} &= p_{t|t} + \sigma^2_\eta. \end{aligned}$$

We repeat this for each t, starting with $t = 1$: we let $t = 1, \ldots, n$.
What initial values for a_1 and p_1 should we consider ?
See Session 2 !!

Signal Extraction: predicted estimates of local level



Signal Extraction: filtered estimates of local level



What are the take-aways here ?

- We derived a way to estimate μ_t from the Local Level model.
- The estimate a_t is the mean of the predicted conditional density $f(\mu_t|Y_{t-1}) = \mathcal{N}(a_t, p_t)$.
- The estimate $a_{t|t}$ is the mean of the filtered conditional density $f(\mu_t|Y_t) = \mathcal{N}(a_{t|t}, p_{t|t})$.
- The estimates are computed using recursive equations.
- The derivations rely on basic principles from the bivariate normal distribution.

APPENDICES

Textbooks

- A.C.Harvey (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge University Press
- G.Kitagawa & W.Gersch (1996). *Smoothness Priors Analysis of Time Series*. Springer-Verlag
- J.Harrison & M.West (1997). *Bayesian Forecasting and Dynamic Models*. Springer-Verlag
- J.Durbin & S.J.Koopman (2012). *Time Series Analysis by State Space Methods, Second Edition*. Oxford University Press
- J.J.F.Commandeur & S.J.Koopman (2007). *An Introduction to State Space Time Series Analysis*. Oxford University Press

Appendix – Taylor series

The Taylor expansion for function $f(x)$ around some value x^* is

$$f(x) = f(x = x^*) + f'(x = x^*)[x - x^*] + \frac{1}{2} f''(x = x^*)[x - x^*]^2 + \ldots,$$

where

$$f'(x) = \frac{\partial f(x)}{\partial x}, \qquad f''(x) = \frac{\partial^2 f(x)}{\partial x \partial x},$$

and $g(x = x^*)$ means that we evaluate function $g(x)$ at $x = x^*$.

Example: consider $f(x) = \log(1 + x)$ with $f'(x) = (1 + x)^{-1}$ and $f''(x) = -(1 + x)^{-2}$; the expansion of $f(x)$ around $x^* = 0$ is

$$\log(1 + x) = 0 + 1 \cdot (x - 0) + \frac{1}{2}(-1) \cdot (x - 0)^2 + \ldots = x - \frac{1}{2}x^2 + \ldots$$

Notice that $f(x = 0) = 0$, $f'(x = 0) = 1$ and $f''(x = 0) = -1$. For small enough x (when x is close to $x^* = 0$), we have

$$\log(1 + x) \approx x.$$

Check: $\log(1.01) = .00995 \approx 0.01$ and $\log(1.1) = 0.0953 \approx 0.1$.

Appendix – Percentage growth

Observation at time t is y_t and observation at time $t-1$ is y_{t-1}.

We define rate r_t as the proportional change of y_t wrt y_{t-1}, that is

$$r_t = \frac{y_t - y_{t-1}}{y_{t-1}} \Rightarrow y_t - y_{t-1} = y_{t-1} \cdot r_t \Rightarrow y_t = y_{t-1} \cdot (1 + r_t).$$

We notice that r_t can be positive and negative !

When we take logs of $y_t = y_{t-1} \cdot (1 + r_t)$, we obtain

$$\log y_t = \log y_{t-1} + \log(1 + r_t) \Rightarrow \log y_t - \log y_{t-1} = \log(1 + r_t) \Rightarrow$$

$$\Delta \log y_t = \log(1 + r_t).$$

Since $\log(1 + r_t) \approx r_t$, see previous slide, when r_t is small, we have

$$r_t \approx \Delta \log y_t.$$

The percentage growth is defined as $100 \times r_t \approx 100 \cdot \Delta \log y_t$.

Appendix – Lag operators and polynomials

- Lag operator $Ly_t = y_{t-1}$, $L^\tau y_t = y_{t-\tau}$, for $\tau = 1, 2, 3, \ldots$
- Difference operator $\Delta y_t = (1 - L)y_t = y_t - y_{t-1}$
- Autoregressive polynomial $\phi(L)y_t = (1 - \phi L)y_t = y_t - \phi y_{t-1}$
- Other polynomial $\theta(L)\varepsilon_t = (1 + \theta L)\varepsilon_t = \varepsilon_t + \theta\varepsilon_{t-1}$
- Second difference
  $\Delta^2 y_t = \Delta(\Delta y_t) = \Delta(y_t - y_{t-1}) = y_t - 2y_{t-1} + y_{t-2}$
- Seasonal difference $\Delta_s y_t = y_t - y_{t-s}$ for typical $s = 2, 4, 7, 12, 52$
- Seasonal sum operator
  $S(L)y_t = (1 + L + L^2 + \ldots + L^{s-1})y_t = y_t + y_{t-1} + \ldots + y_{t-s+1}$
- Show that $\Delta\, S(L) = \Delta_s$.