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Research Foundation – Flanders

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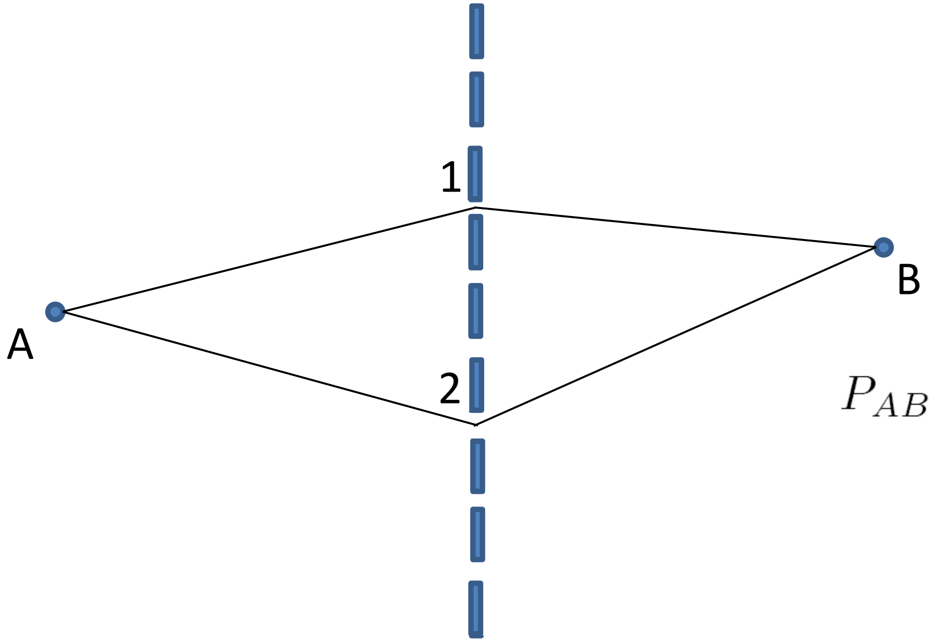
Theory of Quantum and Complex systems

Schrödinger's money: The strange marriage between quantum mechanics and finance

From left to right: **Dr. S. Klimin**, L. Liang, W. O'Kelly, **prof.dr. J.T.Devreese**, M. Baeten, **JT**, W. Casteels, K. Putteneers, prof.dr.em. F. Brosens, D.Lemmens, B.Anthonis, N. Van den Broeck, T. Verhulst, prof.dr. M. Wouters, J.Devreese, D.Sels
Not shown: S. Ceuppens, S. Koghee, G. Lombardi, E. Vermeyen

Part I: path integrals in quantum mechanics

Introduction: quantum mechanics with path integrals

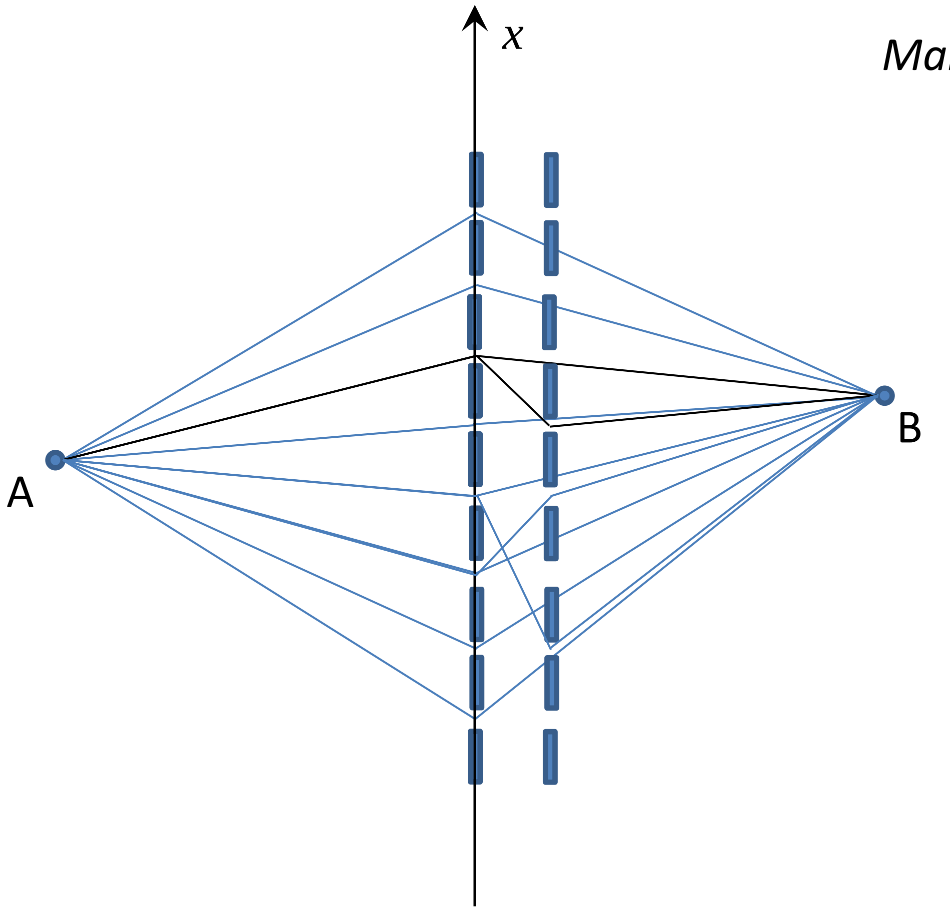


Two alternatives: add the amplitudes

$$\phi_{AB} = \phi_1 + \phi_2$$

$$P_{AB} = |\phi_{AB}|^2 = |\phi_1 + \phi_2|^2$$

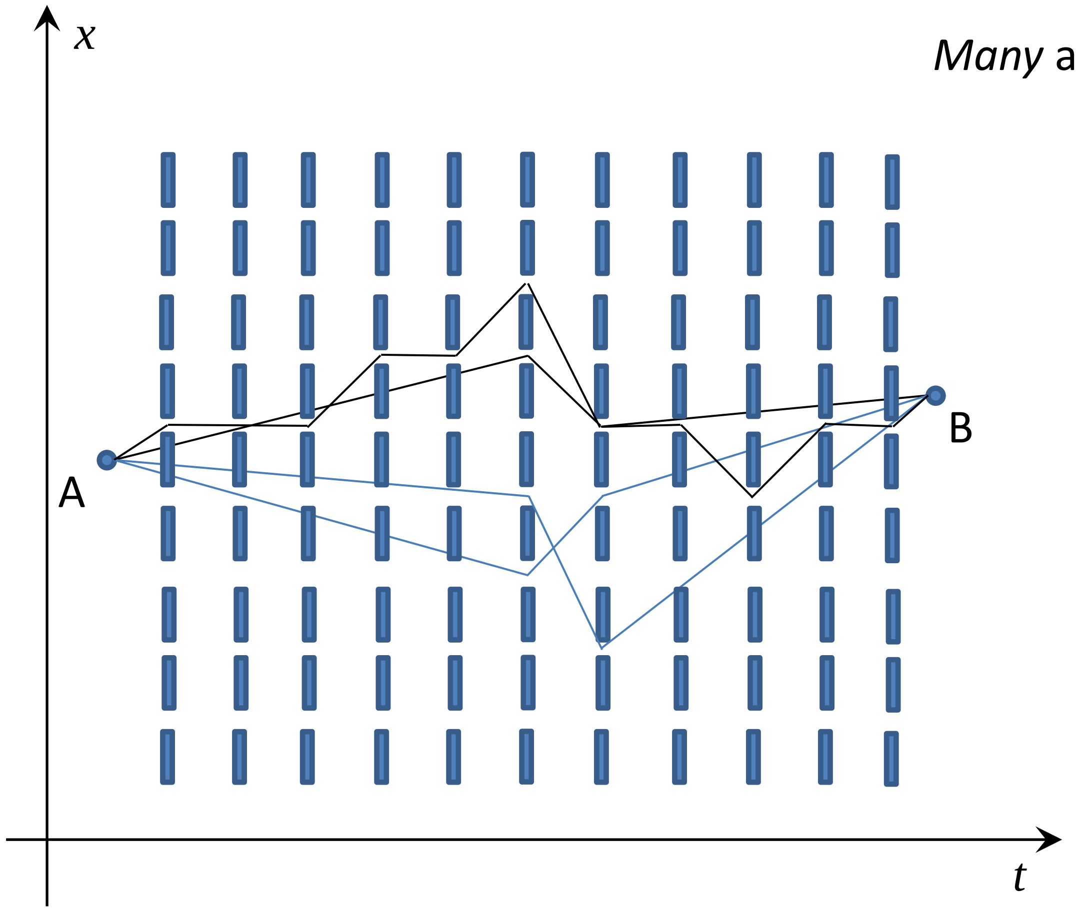
Introduction: quantum mechanics with path integrals



Many alternatives: add the amplitudes

$$\phi_{AB} = \sum_{x_1} \sum_{x_2} \phi_{x_1,x_2}$$

Introduction: quantum mechanics with path integrals



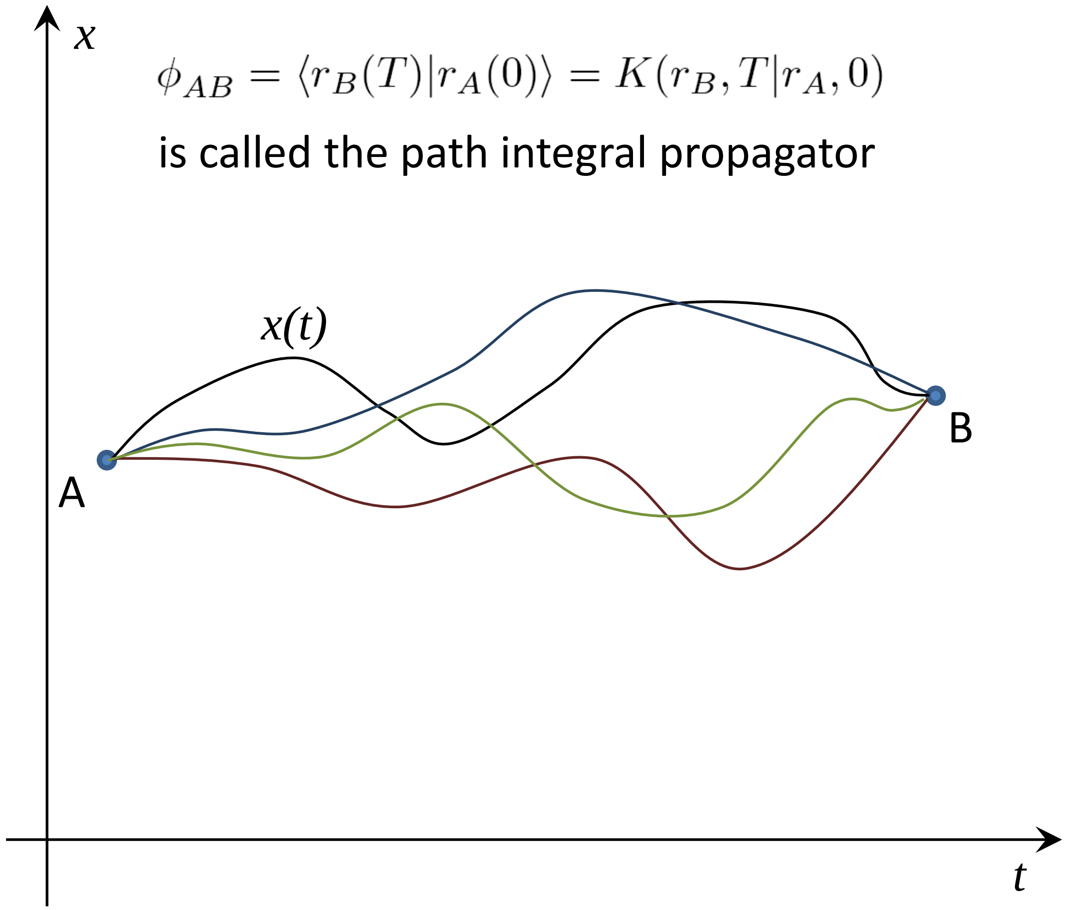
Many alternatives: add the amplitudes

$$\phi_{AB} = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_N} \phi_{x_1, x_2, \ldots, x_N}$$

Introduction: quantum mechanics with path integrals

$$\phi_{AB} = \langle r_B(T) | r_A(0) \rangle = K(r_B, T | r_A, 0)$$

is called the path integral propagator



Many alternatives: add the amplitudes

$$\phi_{AB} = \int \mathcal{D}x \; \phi\left[x(t)\right]$$

The amplitude corresponding to a given path $x(t)$ is

$$\phi\left[x(t)\right] = \exp\left\{\frac{i}{\hbar} S[x(t)]\right\}$$

Here, S is the action functional:

$$S[x(t)] = \int_0^T L(x, \dot{x}, t) dt$$

With L the Lagrangian, eg.

$$L(x, \dot{x}, t) = \frac{m}{2}\dot{x}^2 - V(x)$$

H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009)

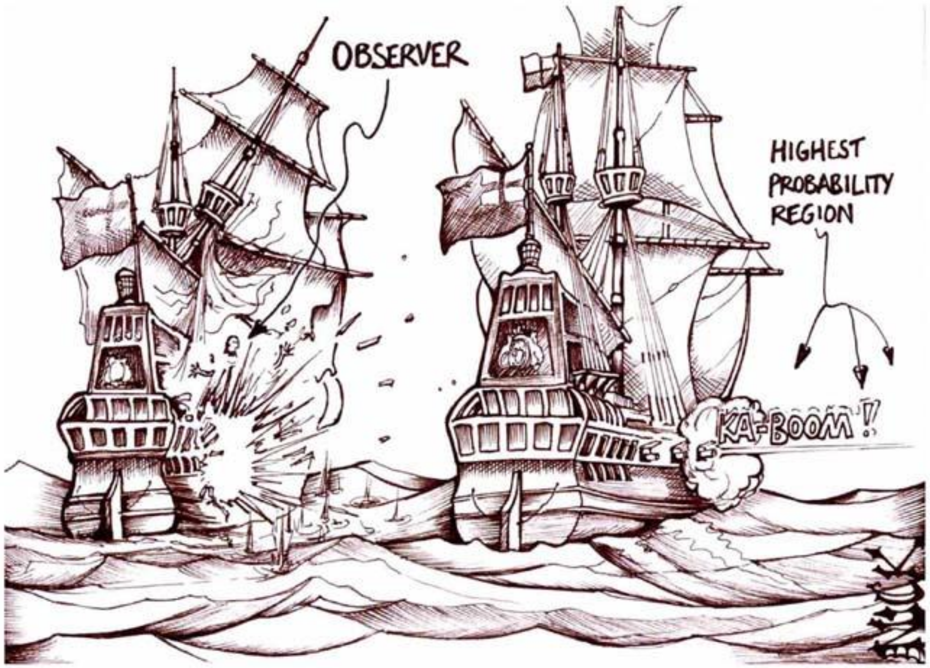
This prescription contains both

→ classical mechanics (as a limit) :

Only when neighbouring paths have the same action functional

$$\rightarrow \quad \delta\mathcal{S} = 0$$

will $\exp\{-iS[\mathbf{r}(t)]/\hbar\}$ interfere constructively.



Cannon balls: a quantum mechanical treatment.

→ and "Schrödinger-picture" quantum mechanics

$$K(\mathbf{r}_1, t|\mathbf{r}_0, 0) = \left\langle \mathbf{r}_1 \left| \exp\left\{i\hat{H}t/\hbar\right\}\right| \mathbf{r}_0 \right\rangle$$
$$= \sum_n \psi_n(\mathbf{r}_1)\psi_n(\mathbf{r}_0)\exp\{-iE_n t/\hbar\}$$

H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009)

Introduction: *Statistical* mechanics with path integrals

The particle propagator or Green's function

$$K(\mathbf{r}_1, t|\mathbf{r}_0, 0) = \left\langle \mathbf{r}_1 \left| \exp\left\{ i\hat{H}t/\hbar \right\} \right| \mathbf{r}_0 \right\rangle$$
$$= \sum_n \psi_n(\mathbf{r}_1)\psi_n(\mathbf{r}_0) \exp\{-iE_n t/\hbar\}$$

as a Feynman path integral:

$$K(\mathbf{r}_1, t|\mathbf{r}_0, 0)$$
$$= \int \mathcal{D}r \ \exp\left\{ \frac{i}{\hbar} \int_0^t \left(\frac{m\dot{r}^2}{2} - V(\mathbf{r}) \right) dt' \right\}$$

$$t = i\hbar\beta$$

$$\sum_n \psi_n(\mathbf{r}_1)\psi_n(\mathbf{r}_0) \exp\{-\beta E_n\}$$
$$= \rho(\mathbf{r}_1, \mathbf{r}_0; \beta)$$

The density matrix

$$\rho(\mathbf{r}_1, \mathbf{r}_0; \beta)$$
$$= \int \mathcal{D}r \ \exp\left\{ -\int_0^\beta \left(\frac{m\dot{r}^2}{2} + V(\mathbf{r}) \right) d\tau \right\}$$

as a 'Wiener' path integral

H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009)

Part II: Finance, and trading things in the future

Question #1

Choose between two rewards:

(a) 100 € , one year from now
(b) 100 € , right now

Conclusion: Money now is worth more than money in the future.

Question #2

Choose between two rewards:

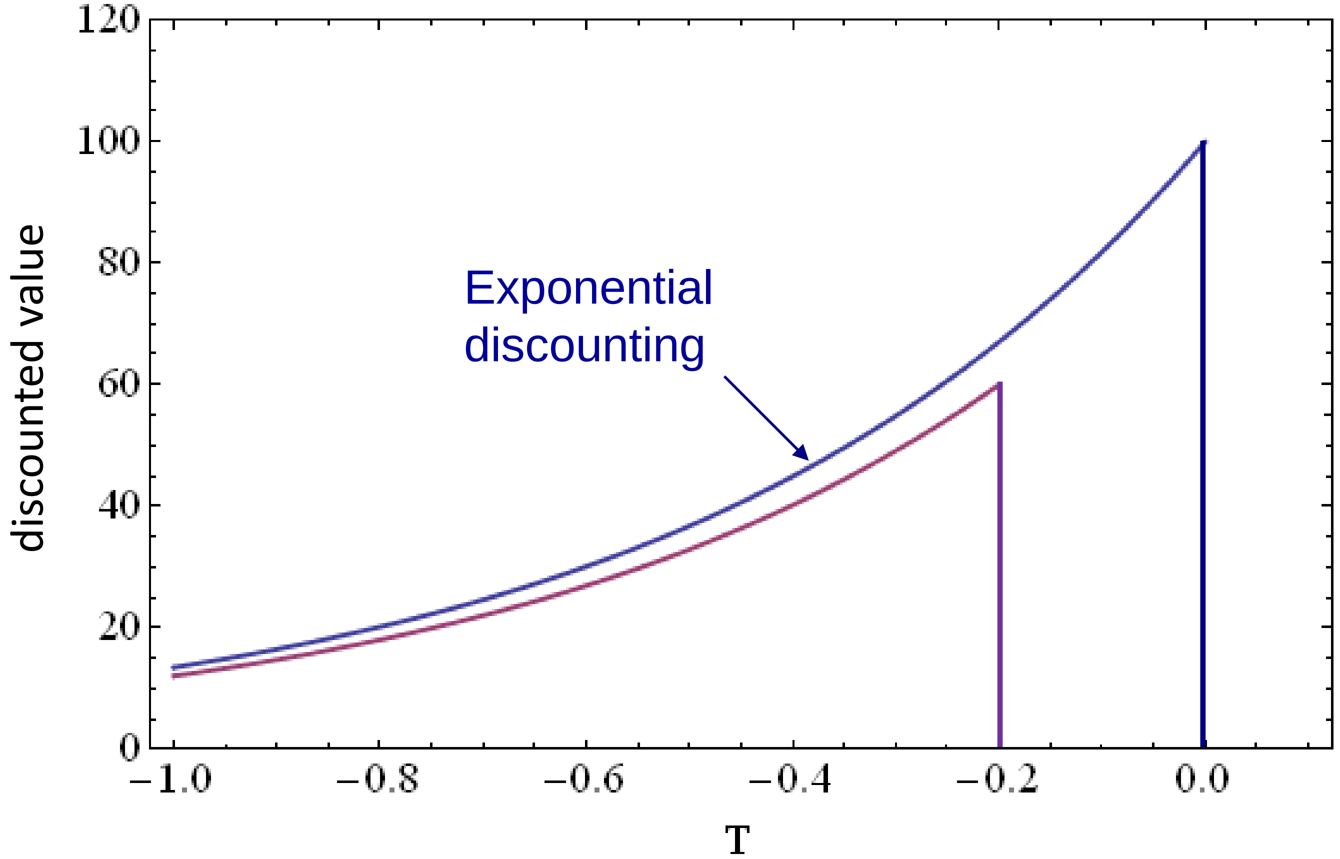
(a) 100 € , one year from now
(b) 0 € , right now

Conclusion: Future money is worth something between 0% and 100% of money now.

Apply bracketing to find the amount where people find (a) and (b) equally attractive.

The rational answer: discount the value at the interest rate for money savings.

value(now) = $e^{-r_e T}$ value(a time T later)



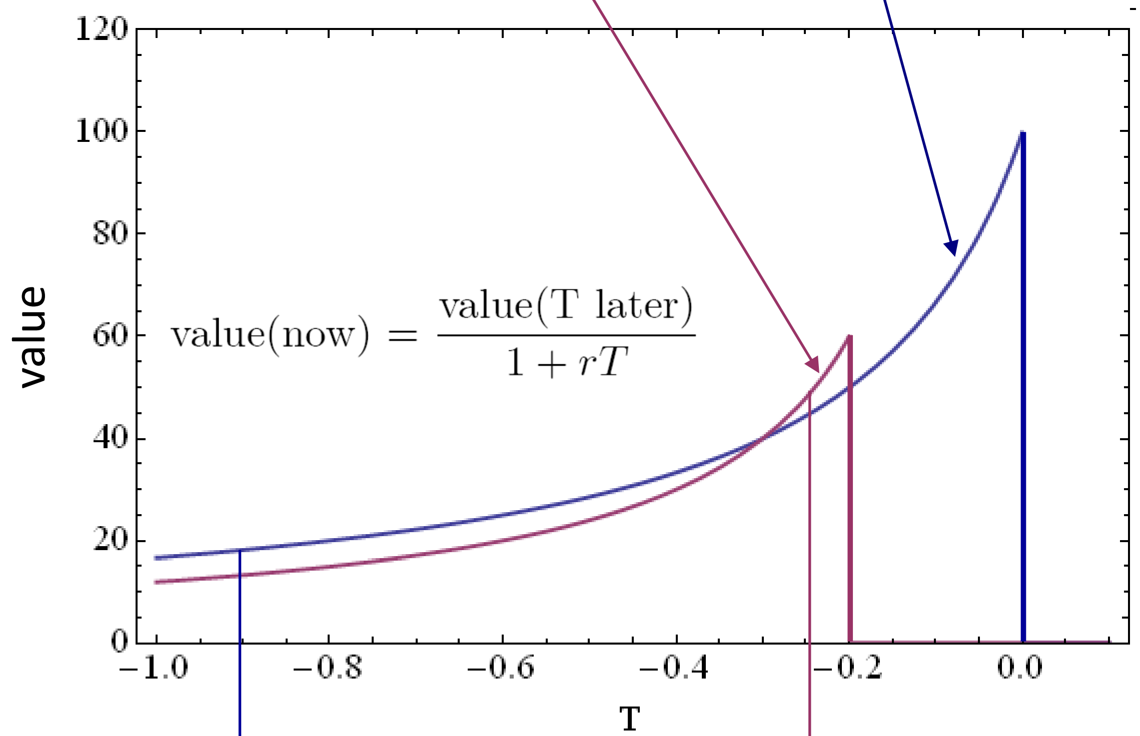
At r = 0.02 , the value now of “100 euro one year from now” is 98.04 euro now.

Homogeneous in time (“time consistency”): the preference of the options do not change when shifting both times

Application to the student – beer relationship

Reward 1: temporary lowering of social barrier

Reward 2: no hangover the next day

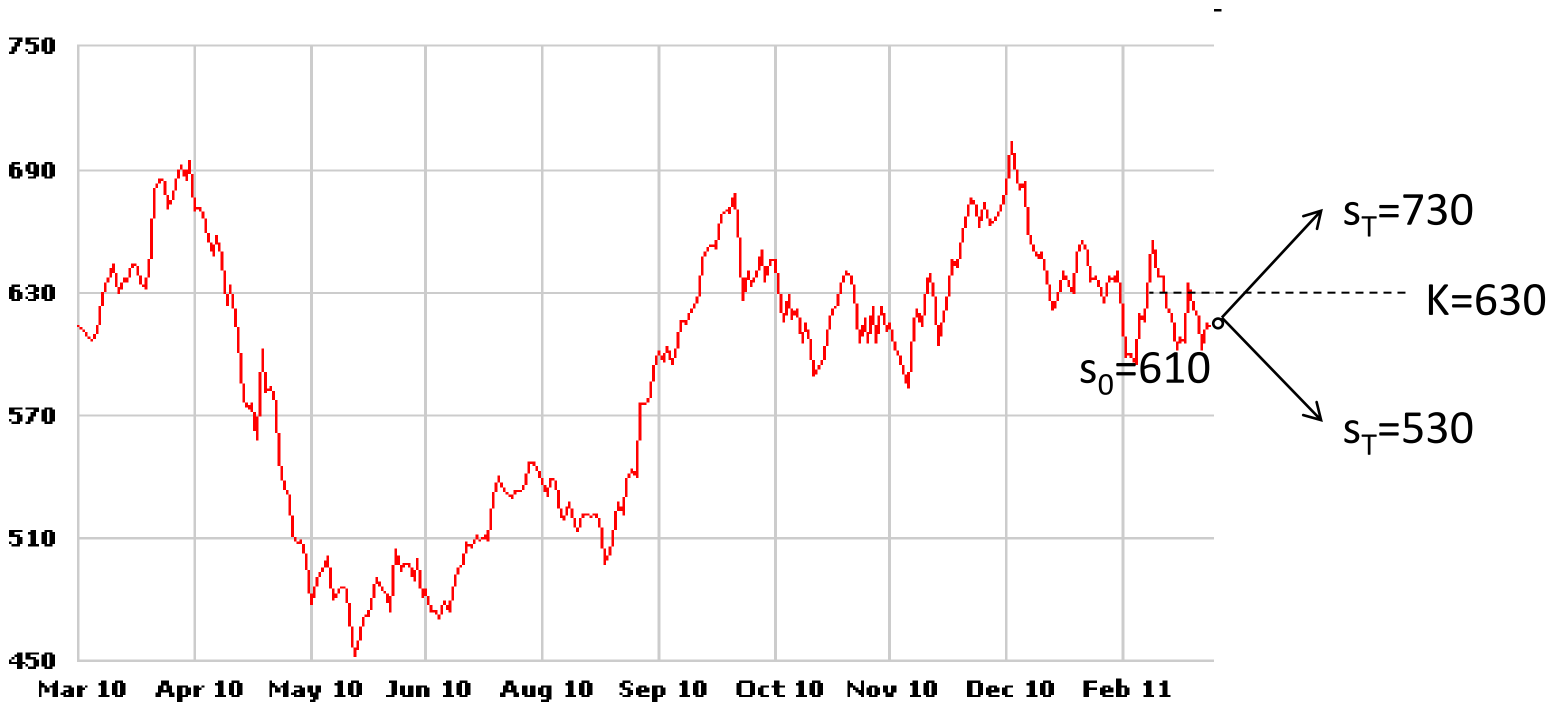


At this stage, the strategy “I will never drink again” wins.

Here the strategy “What the heck, one more won’t hurt” has taken over

The reason exponential discounting of future prices is not valid is **RISK**

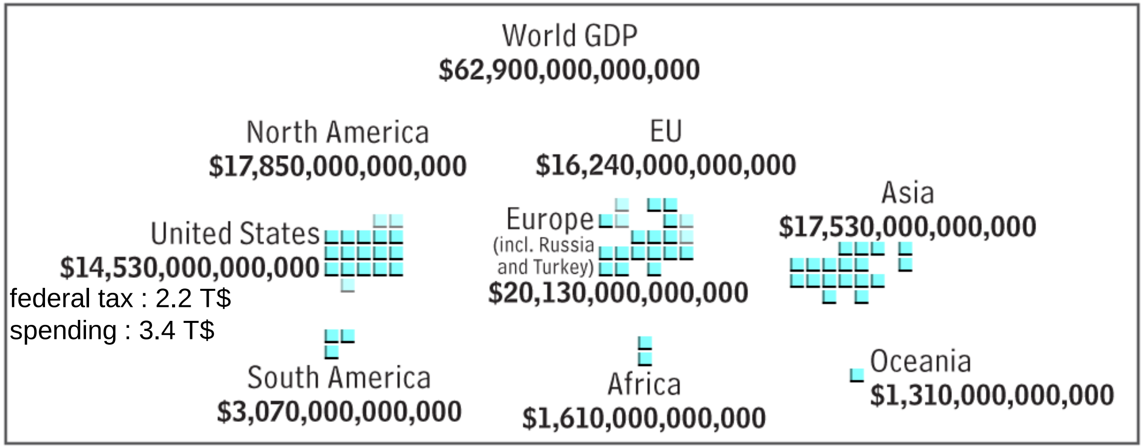
Main question in finance: how much is this 'right to buy in the future' worth?

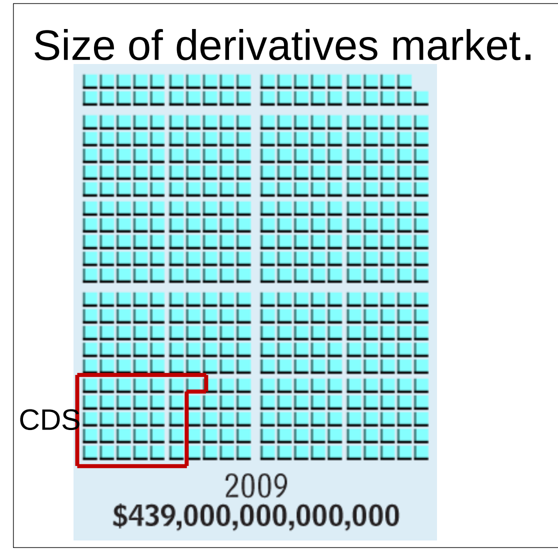


A better deal (for you) is to buy the *RIGHT*, not the obligation to buy steel at 630 EUR/tonne one year from now.

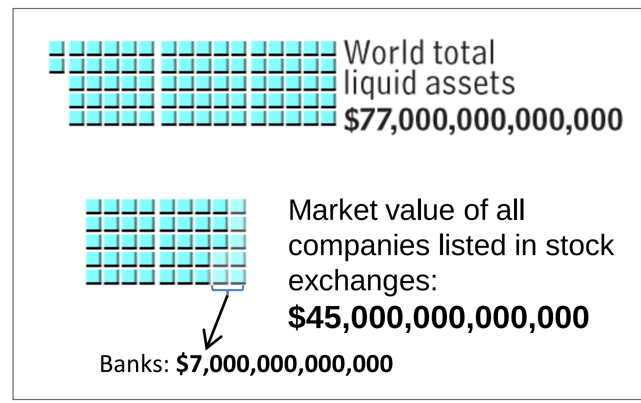
This is called an '*option contract*'. All types exist, eg.also the right to sell, with different times, and for different underlying assets.

2011 data on money. Each block = 1000 billion dollar (10^{12} $)





5 years of derivatives trading corresponds to a market equivalent to the total GDP of the human race since the beginning of civilization.

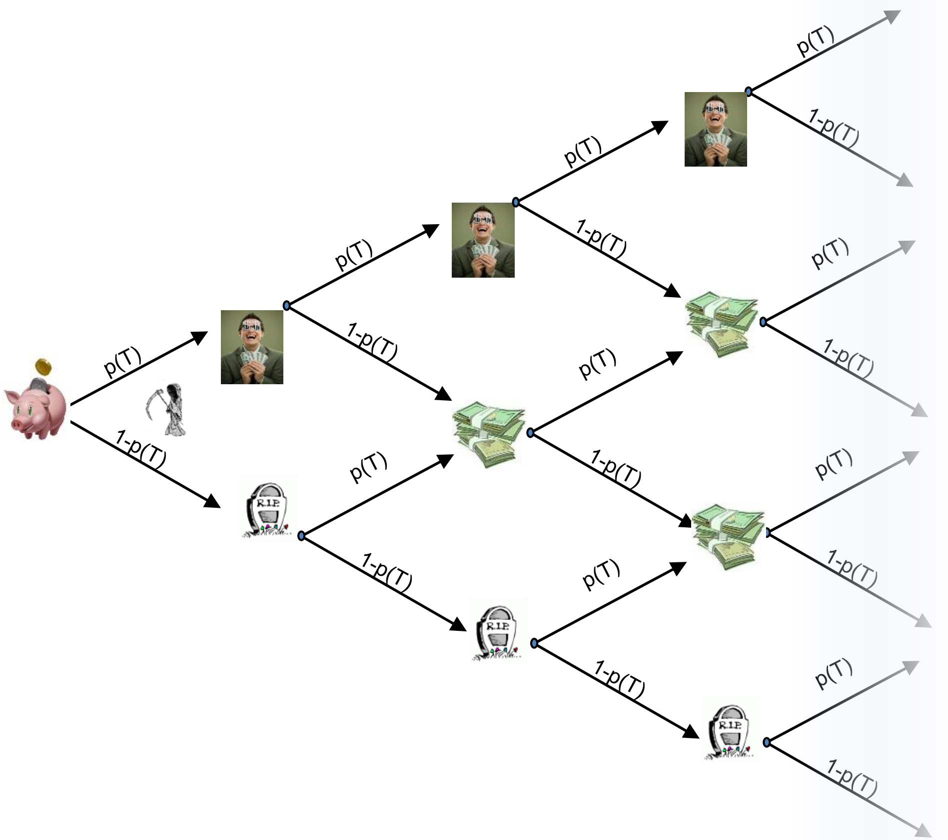


Part III: Path integrals in finance

The 'standard model' of option pricing

Rather than 2 possible futures, there are many – but they can bee seen as a limit of many small 'binomial' steps

$\rightarrow$ according to the central limit theorem, the outcome is Gaussian fluctuations



<u>Black-Scholes-Merton model:</u>

$$ds_t = s_t r dt + s_t \sigma dW$$

leads to exp{rT} cf. "exponential" discounting

adds gaussian fluctuations

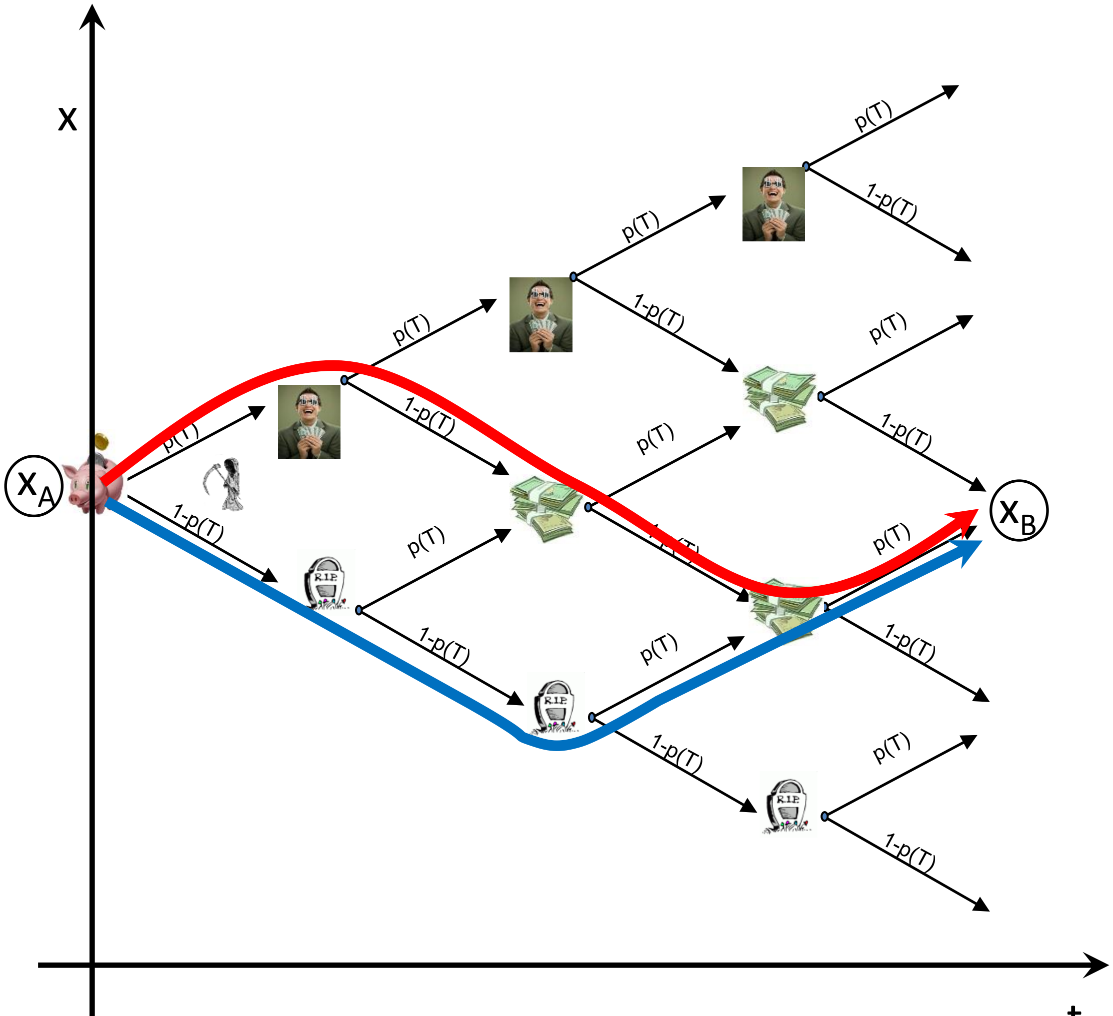
We work with the logreturn $x(t) = \log(s_t/s_0)$ to obtain $dx = \left(r - \sigma^2/2\right) dt + \sigma dW$

This corresponds to a diffusion with diffusion constant σ^2 and drift $(r-\sigma^2/2)$ per unit time.

$$P(x_T) = \frac{1}{\sqrt{4\pi\sigma^2 T}} \exp\left\{ -\frac{\left[x - (r - \sigma^2/2)T\right]^2}{2\sigma^2 T} \right\}$$

The ‘standard model’ of option pricing

The probability to end up in x_T is *also* given by the sum over all paths that end up there, weighed by the probability of these paths



From the stochastic differential equation

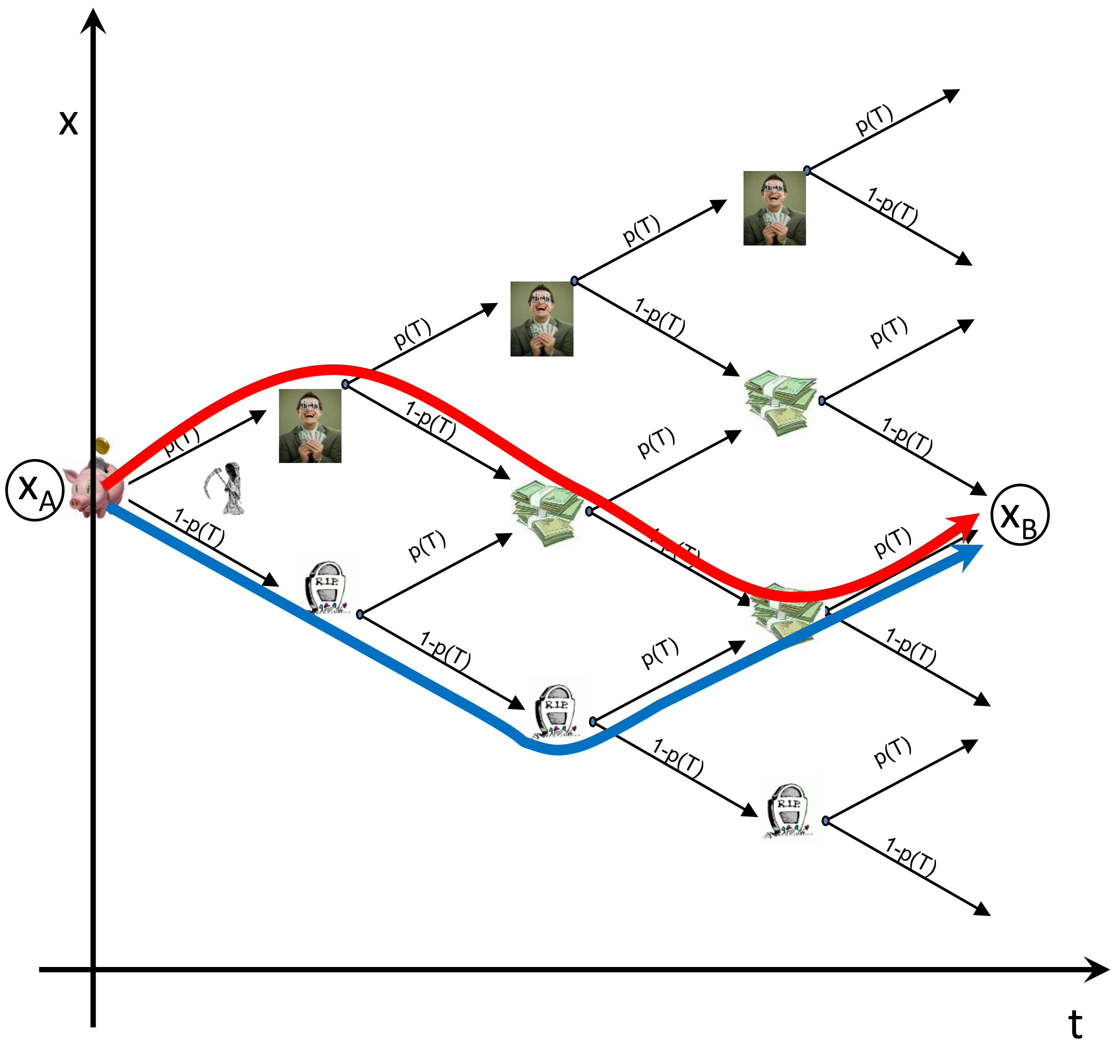
$$dx = A(x,t)dt + B(x,t)dW$$

An infinitesimal time-step propagator is derived:

$$K(x', t+\delta t | x, t) = \frac{1}{\sqrt{2\pi B^2(x,t)\delta t}} \times \exp\left\{ -\frac{[x' - x - A(x,t)\delta t]^2}{2B^2(x,t)\delta t} \right\}$$

The ‘standard model’ of option pricing

The probability to end up in x_T is *also* given by the sum over all paths that end up there, weighed by the probability of these paths



For the Black-Scholes model:

From the stochastic differential equation

$$dx = \left(r - \sigma^2/2\right) dt + \sigma dW$$

An infinitesimal time-step propagator is derived:

$$K(x,t|0,0) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \times \exp\left\{ -\frac{\left[x - (r - \sigma^2/2)t\right]^2}{2\sigma^2 t} \right\}$$

Many refs: Dash, Linetsky, Rosa-Clot, Kleinert,…

Black-Scholes-Merton and black monday

$$\text{option price} = e^{-rT}\mathbb{E}[\text{payoff}] = e^{-rT}\int_{-\infty}^{\infty} \text{payoff}[x_T] K(x_T, T|0,0)\, dx_T$$

$\Leftarrow$

Free particle propagator $\Rightarrow$ Black-Scholes option price

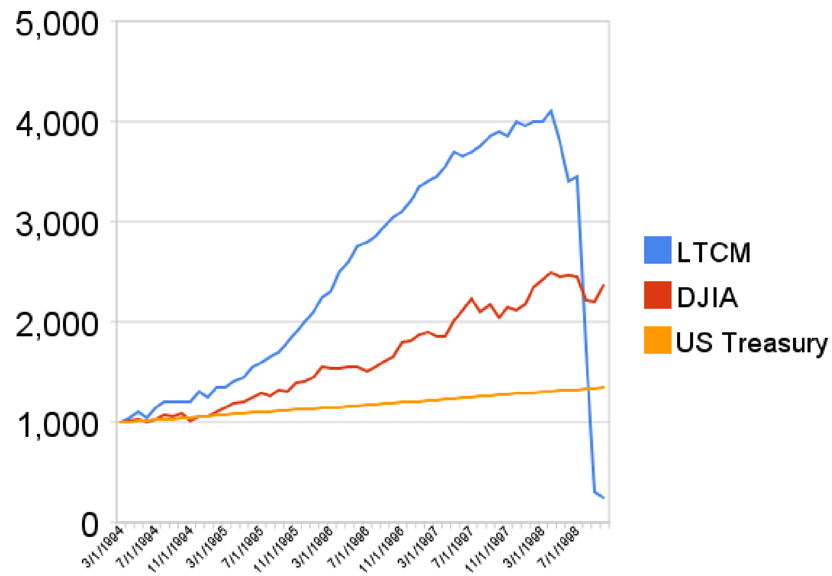
$$K(x_T, T|0,0) = \frac{1}{\sqrt{4\pi\sigma^2 T}} \exp\left\{-\frac{[x-(r-\sigma^2/2)T]^2}{2\sigma^2 T}\right\}$$

1997 nobel prize in economics

Robert C. Merton

Myron S. Scholes

$\rightarrow$ Board members of “Long-Term Capital Management”, which initially made 40% return on investment as first users of the Black-Scholes model. Until in 1987 it lost all of its $4.6 billion.



Fisher Black & Myron Scholes, "*The Pricing of Options and Corporate Liabilities*". Journal of Political Economy **81** (3): 637–654 (1973).
Robert C. Merton, "*Theory of Rational Option Pricing*“, Bell Journal of Economics and Management Science (The RAND Corporation) **4** (1): 141–183 (1973).

Part IV: Beyond the standard model & the need for path integrals

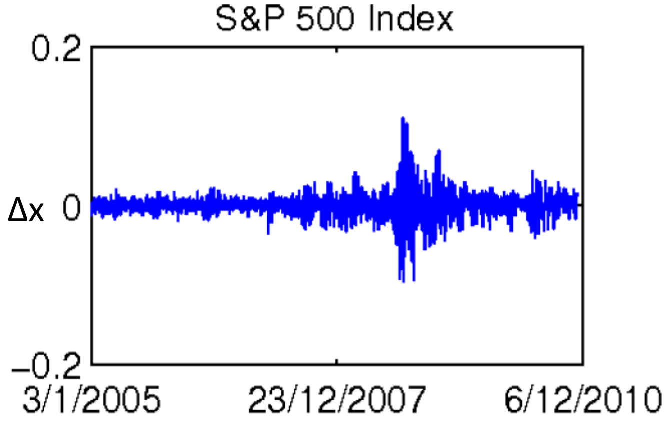
Two problems with the standard model

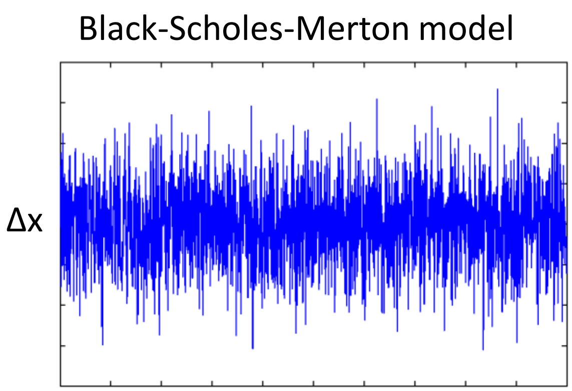
$$\text{option price} = e^{-rT}\mathbb{E}[\text{payoff}] = e^{-rT}\int_{-\infty}^{\infty} \text{payoff}[x_T] K(x_T, T|0,0) dx_T$$

Free particle propagator $\Rightarrow$ Black-Scholes option price

$$K(x_T, T|0,0) = \frac{1}{\sqrt{4\pi\sigma^2 T}} \exp\left\{-\frac{[x-(r-\sigma^2/2)T]^2}{2\sigma^2 T}\right\}$$

Problem 1: The fluctuations are *not Gaussian*





Problem 2: Not all options have a payoff that depends only on x(t=T), many options have a *path-dependent payoff*, i.e. payoff is a functional of x(t).

Two problems with the standard model

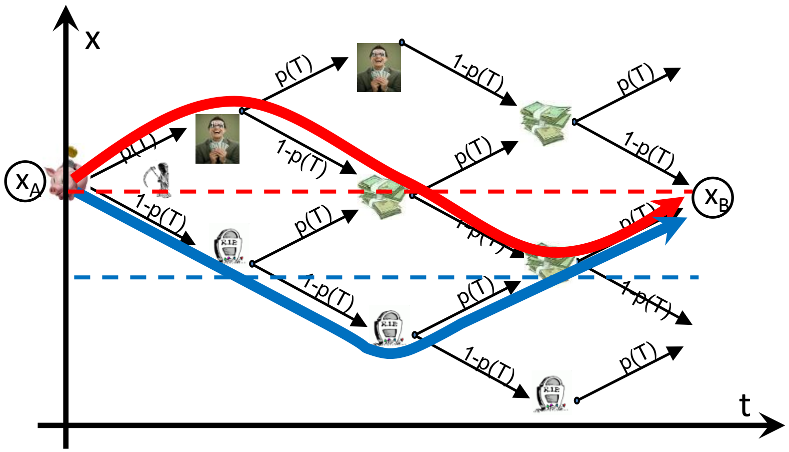
$$\text{option price} = e^{-rT}\mathbb{E}[\text{payoff}]$$
$$= e^{-rT}\int_{-\infty}^{\infty} \text{payoff}[x_T]K(x_T,T|0,0)dx_T$$

Free particle propagator $\Rightarrow$ Black-Scholes option price

$$K(x_T,T|0,0) = \frac{1}{\sqrt{4\pi\sigma^2 T}}\exp\left\{-\frac{[x-(r-\sigma^2/2)T]^2}{2\sigma^2 T}\right\}$$

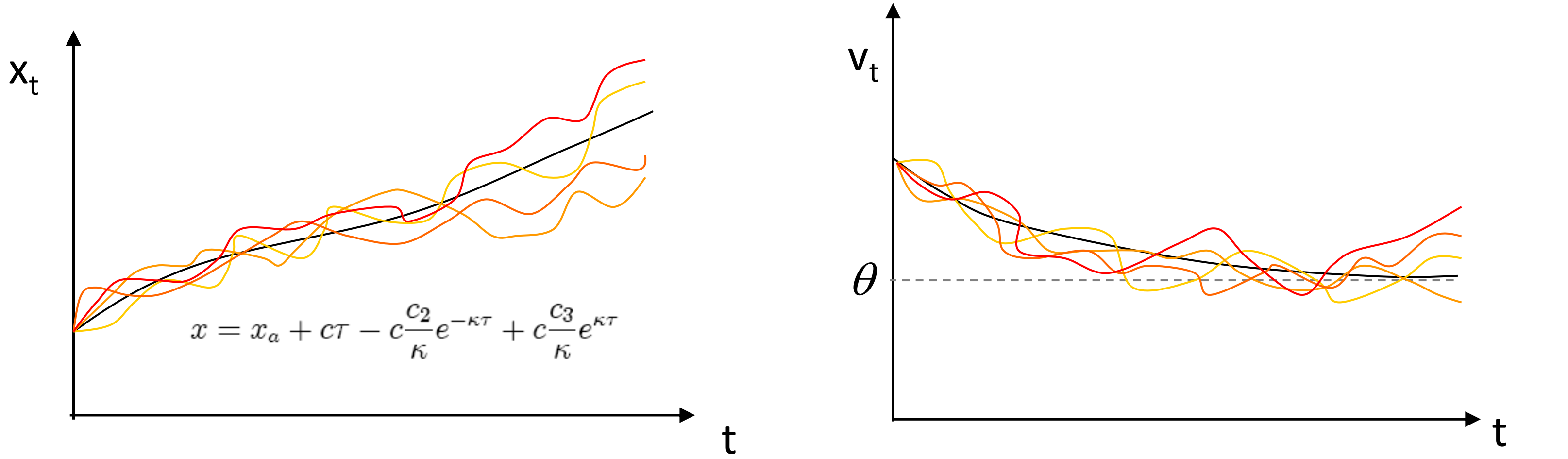
Problem 1: The fluctuations are *not Gaussian*

Problem 2: Not all options have a payoff that depends only on x(t=T), many options have a *path-dependent payoff*, i.e. payoff is a functional of x(t).



Asian option:
payoff is a function of the *average* of the underlying price?

Improving Black-Scholes : stochastic volatility



two particle problem

The Heston model treats the variance as a second stochastic variable, satisfying its own stochastic differential equation:

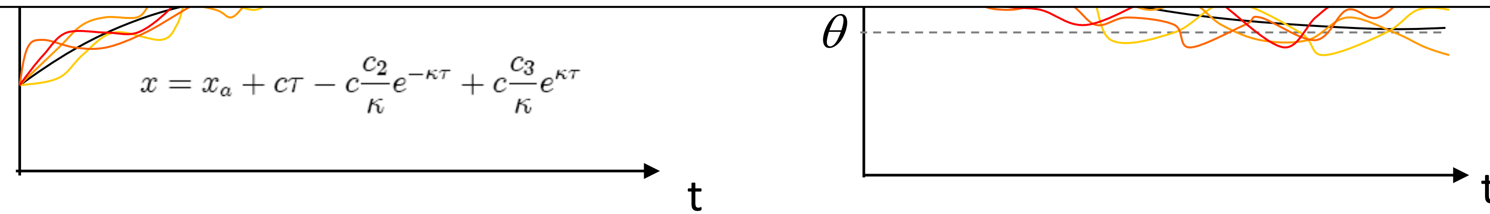
$$\begin{cases} dx = \left(r - \sigma^2/2\right) dt + \sqrt{v}dW_1 \\ dv = \kappa(\theta - v)dt + \sigma_v\sqrt{v}dW_2 \end{cases}$$

mean reversion rate (κ) — mean reversion level (θ) — the 'volatility of the volatility' (σ_v)

Improving Black-Scholes : stochastic volatility

$$\text{call option price} = e^{-rT}\int\int \text{payoff}(x_T)K(x_T,z_T,T|x_0,z_0,0)\,dx_T\,dz_T$$

$$\text{with payoff}(x_T) = \max(s_0 e^{x_T}-k,0)$$



two particle problem

Using the prescription related before, the stochastic differential equations can be rewritten as an infinitesimal-time propagator, from which a Lagrangian can be identified (introducing $z = (v/\theta)^{1/2}$) :

$$\mathcal{K}(x_0,z_0,0|x_T,z_T,T)$$

$$= A\int \mathcal{D}z[t]\int \mathcal{D}x[t]\ \exp\left\{-\int_0^T \frac{\dot{x}^2}{2\theta z^2}dt\right\}\ \exp\left\{-\int_0^T\left(\frac{1}{2m}\dot{z}^2 + az^2 + \frac{b}{z^2}\right)dt\right\}$$

the transition probability for a price and volatility to go from x_0,z_0 at time t=0 to x_T,z_T at T
(details: D.Lemmens, M. Wouters, JT, S. Foulon, Phys. Rev. E **78**, 016101 (2008)).

Other improvements to Black-Scholes

A) Stochastic Volatility

* Heston model:

$$\begin{cases} dx = \left(r - \sigma^2/2\right) dt + \sqrt{v} dW_1 \\ dv = \kappa(\theta - v)dt + \sigma_v \sqrt{v} dW_2 \end{cases}$$

* Hull-White model

$$\begin{cases} dx = \left(r - \sigma^2/2\right) dt + \sigma dW_1 \\ d\sigma = \kappa(\theta - \sigma)dt + \sigma_v dW_2 \end{cases}$$

* Exponential Vasicek model

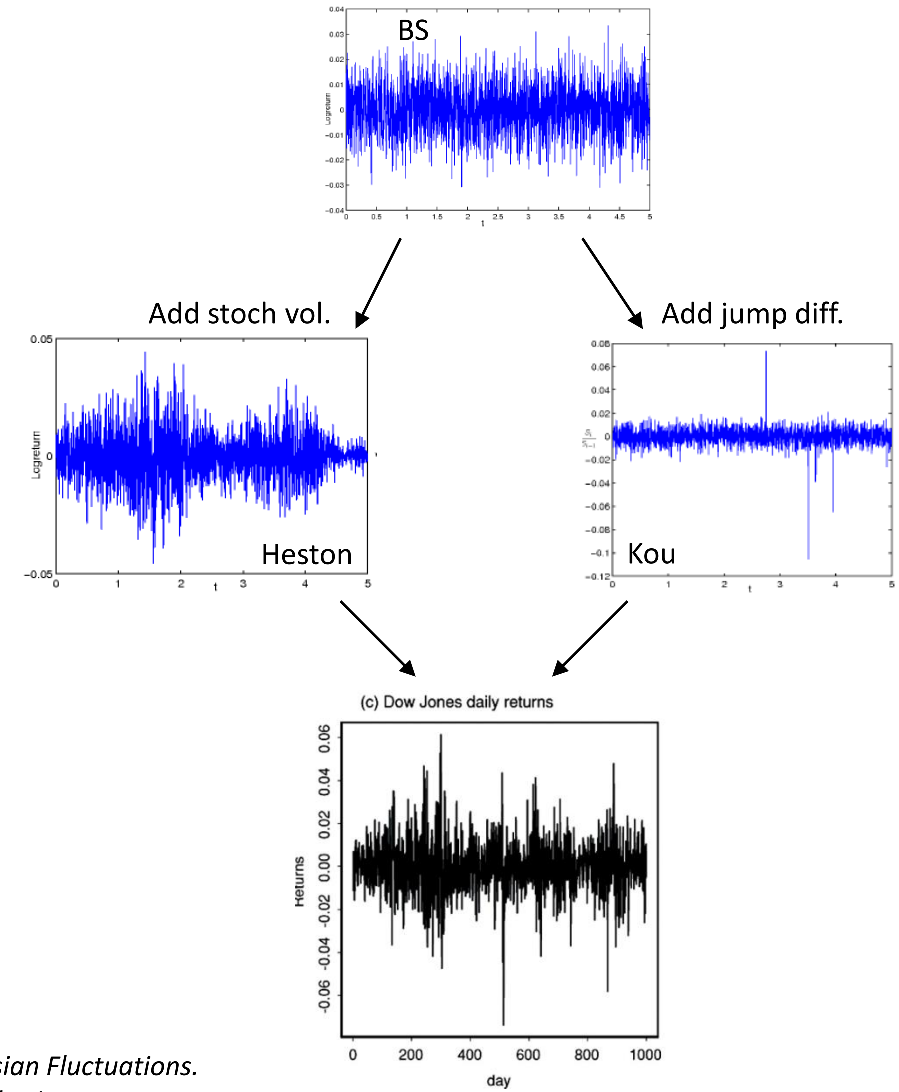
$$\begin{cases} dx = \left(r - \sigma^2/2\right) dt + \sigma dW_1 \\ d\sigma = \sigma \left[\beta \left(a - \ln \sigma\right) + \gamma^2/2\right] dt + \gamma \sigma dW_2 \end{cases}$$

B) Jump Diffusion (and Levy models)

$$ds_t = s_t r dt + s_t \sigma dW + \left(e^J - 1\right) s_t dN_t$$

again a zoo of proposals — poisson process

H. Kleinert , *Option Pricing from Path Integral for Non-Gaussian Fluctuations. Natural Martingale and Application to Truncated Lévy Distributions* , Physica A **312**, 217 (2002).

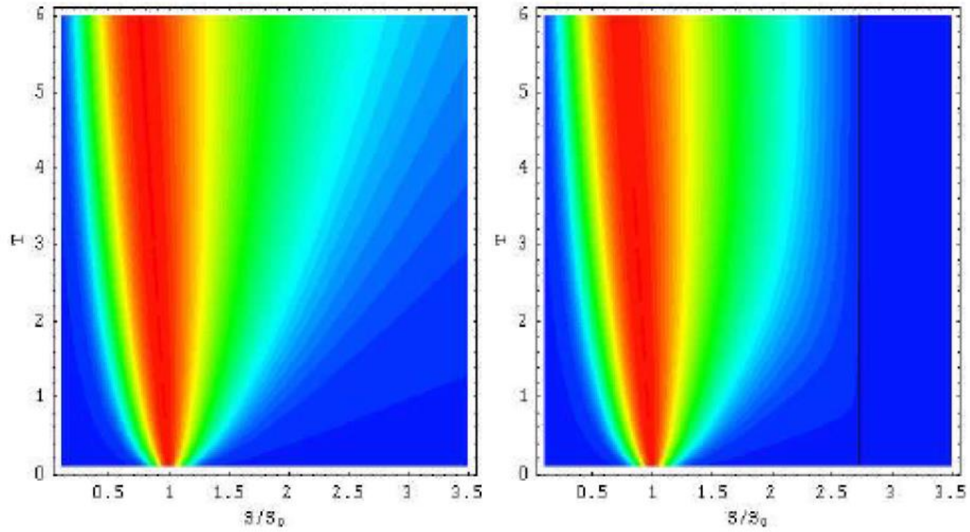


Other models and other tricks

Improvements to Black-Scholes	**...translate into physical actions**	**...to which quantum mechanics solving techniques can be applied**
A) Stochastic Volatility		
1. Heston model	free particle coupled to radial harmonic oscillator	exact solution
B) Stochastic volatiltiy + Jump Diffusion		
2. Exponential Vasicek model	particle in an exponential gauge field generated by free particle	perturbational (Nozieres – Schmitt-Rink expansion)
3. Kou and Merton's models	particle in complicated potential (not previously studied)	variational (Jensen-Feynman variational principle)

References
1. D. Lemmens, M. Wouters, J. Tempere, S. Foulon, Phys. Rev. E 78 (2008) 016101.
2. L. Z. Liang, D. Lemmens, J. Tempere, European Physical Journal B 75 (2010) 335–342.
3. D. Lemmens, L. Z. J. Liang, J. Tempere, A. D. Schepper, Physica A 389 (2010) 5193 – 5207.

More complicated payoffs



- Barrier options: contract becomes void if price goes above/below some value

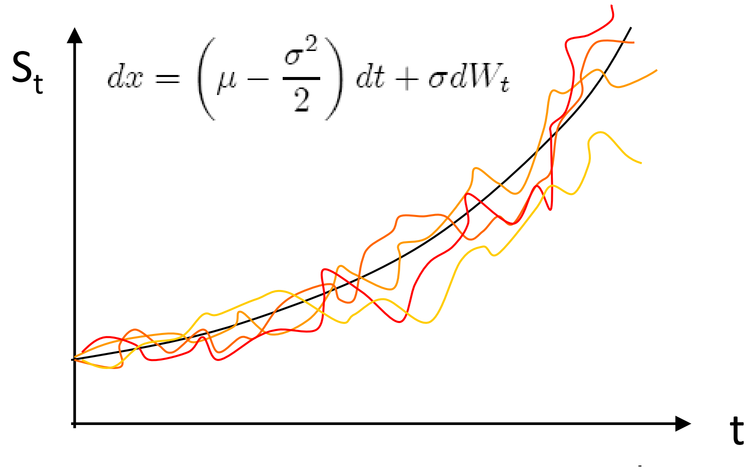
For such options, the price is given by

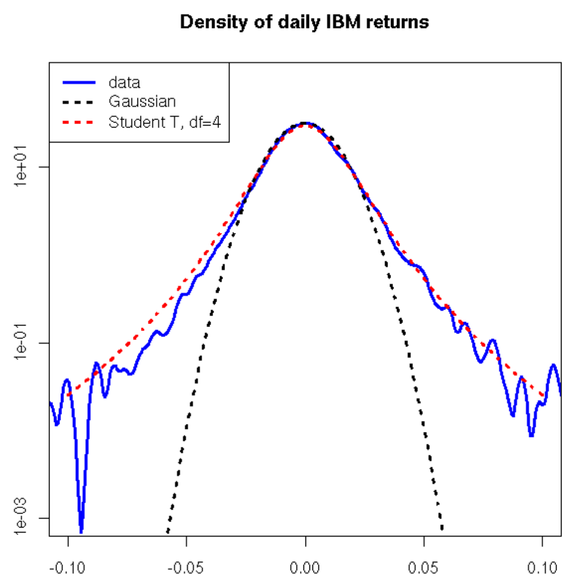
$$\text{option price } = e^{-rT} \int_0^{\infty} \left(\int \mathcal{D}x(t) \text{ payoff}\,[x(t)]\, e^{-\mathcal{S}[x(t)]} \right) dx_T$$

Feynman-Kac ‘interpretation’ $\rightarrow$ include payoff in the path weight:

$$\begin{aligned} \text{option price} &= e^{-rT} \int_0^{\infty} \left(\int \mathcal{D}x(t)\; e^{-\mathcal{S}[x(t)] - \ln(payoff)} \right) dx_T \\ &= e^{-rT} \int_0^{\infty} \left(\int \mathcal{D}x(t)\; e^{-\mathcal{S}_{new}[x(t)]} \right) dx_T \end{aligned}$$

Conclusions

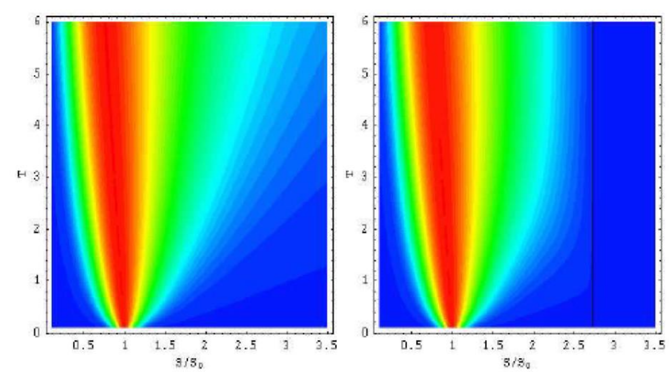




Fluctuating paths in finance are described by stochastic models, which can be translated to Lagrangians for path integration.

Path integrals can solve in a unifying framework the two problems of the 'standard model of finance':

1/ The real fluctuations are not gaussian
2/ New types of option contracts have path-dependent payoffs



D. Lemmens, M. Wouters, JT, S. Foulon, Phys. Rev. E **78**, 016101 (2008);
L. Z. J. Liang, D. Lemmens, JT, European Physical Journal B **75**, 335–342 (2010);
D. Lemmens, L. Z. J. Liang, JT, A. D. Schepper, Physica A **389**, 5193–5207 (2010);
J.P.A. Devreese, D. Lemmens, JT, Physica A **389**, 780-788 (2010);
L. Z. J. Liang, D. Lemmens, JT, Physical Review E **83**, 056112 (2011).

Closing remark #1

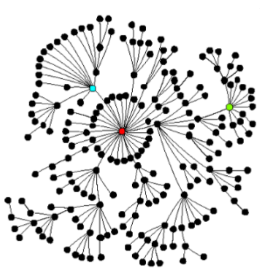
How can we choose between the zoo of models? Various models have been proposed to include various propertie
or specific characteristic of financial time series, such as:

* short memory of the returns
* long memory of the volatility
* heavy-tails in the return distribution
* jumps
* scale-free behavior of logreturns near crash

Approach A: Superstatistics to discriminate between stochastic volatility models

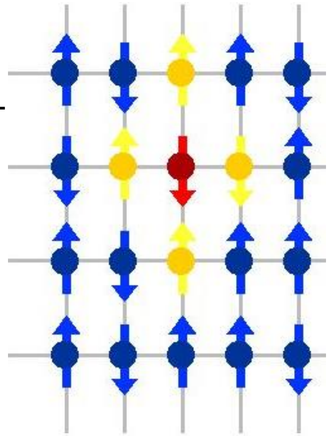
→ a method to “deconvolute” the underlying distribution of volatilities from the time series

Approach B: Game theory on scale-free networks as microscopic theory to generate time series



scale-free networks are more realistic than regular or random lattices

←

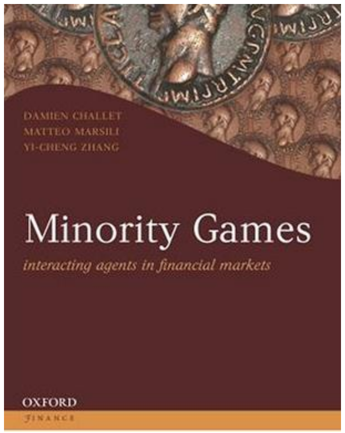


Link to physics: Ising models for 2-state objects interacting on a lattice.

(2 states: buy and sell; price is an emergent property)

→

Ising parameters given by the Nash table from minority games



Closing remark #2

What 'econophysics' is not

"I don't believe talks about econophysics at conferences. Physicists who really worked out a predictive model of the market won't be found at conferences. They'd be on some tropical island keeping quiet about it."
(Larry Schulman, path integral conference Antwerpen 1999)

"Econophysics cannot tell you when exactly the market will crash, just like elasticity theory and tribology cannot predict where exactly on the mountain you will fall if you climb it using beach slippers. Nevertheless elasticity theory and tribology have a lot to say about beach slippers."
(Eugene Stanley, APS March meeting 2008)

TQC

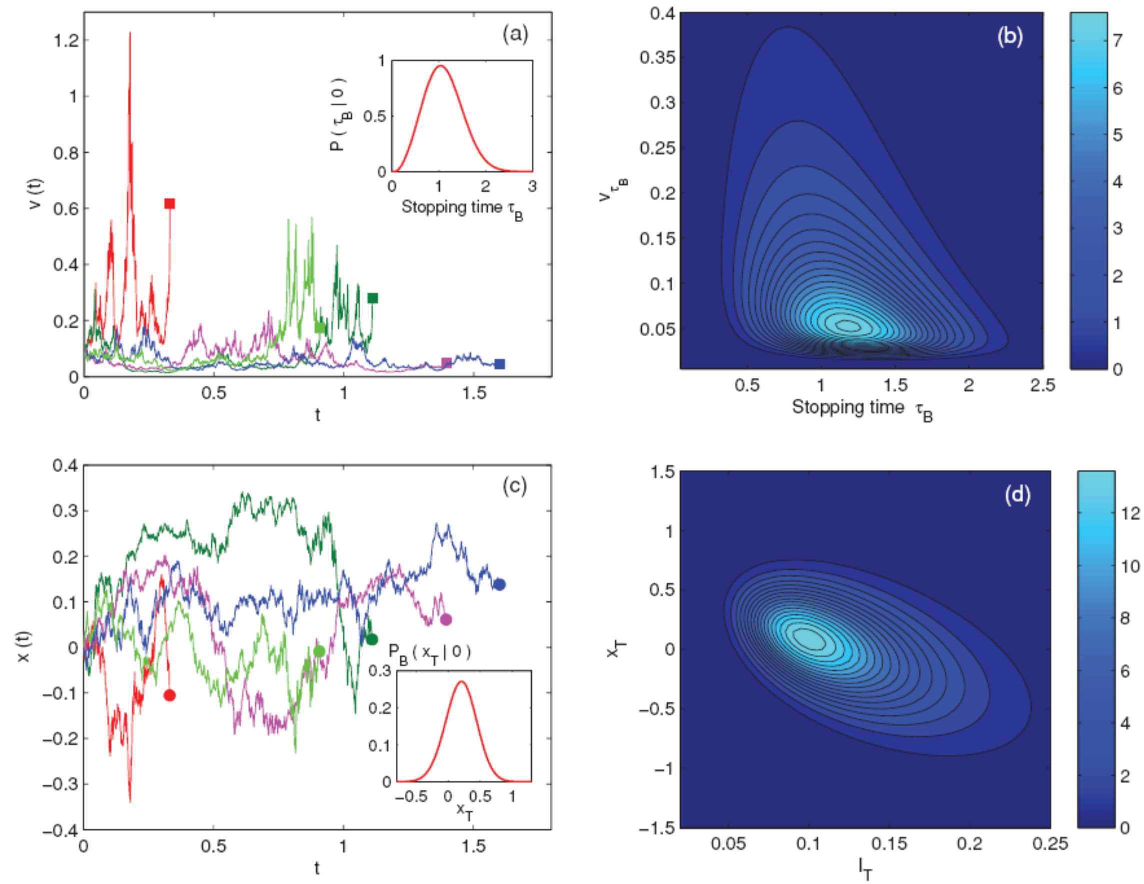
Additional material

More complicated payoffs

Other example: Timer options in the Heston framework

Timer options have an uncertain expiry time, equal to the time at which a certain “variance budget” has been used up)

A Duru-Kleinert transformation allows to introduced a warped spacetime, linking real time and variance budget. This results in a particle in a Kratzer potential, allowing an exact solution.



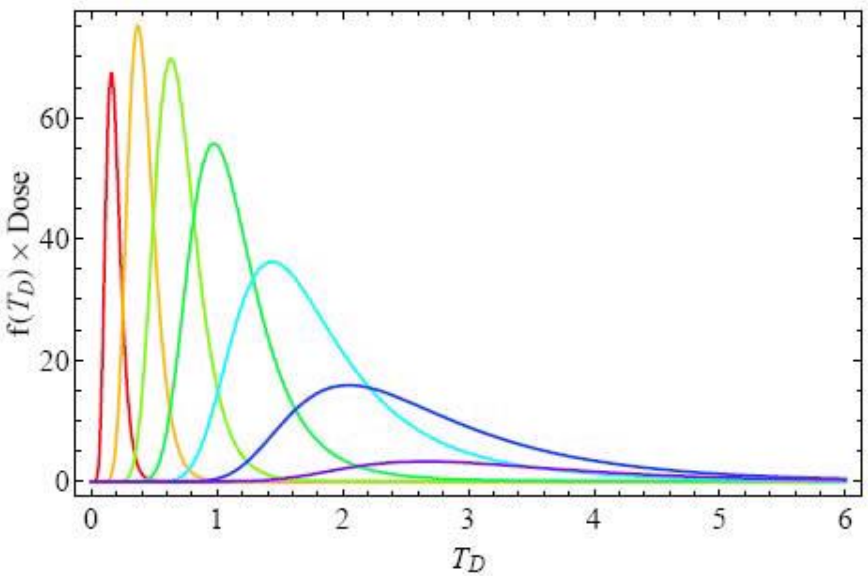
L. Z. J. Liang, D. Lemmens, and J. Tempere, Physical Review E **83**, 056112 (2011).

More complicated payoffs

<u>Other example: Timer options in the Heston framework</u>

Timer options have an uncertain expiry time, equal to the time at which a certain “variance budget” has been used up)

⇨ Leads also to new formulas for estimating the maximum exposure time in an environment with fluctuating radioactivity



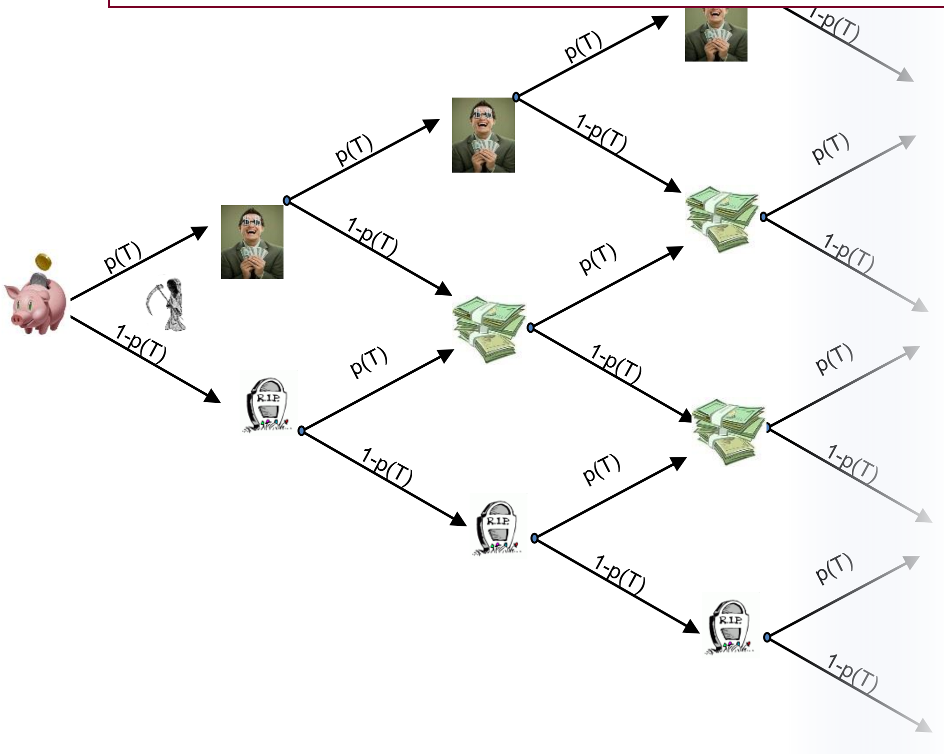
The probability density function for a given dose D to be reached at time T_D, as a function of T_D. The colours correspond to D = 10, .., 60 in steps of 10.

L. Z. J. Liang, D. Lemmens, and J. Tempere, Physical Review E **83**, 056112 (2011).

The ‘standard model’ of option pricing

What is the transition *probability* for a price to go from x_a at time t_a to x_b at t_b ?

$$K(x_T, T|x_0, 0) = \frac{1}{\sqrt{4\pi\sigma^2 T}} \exp\left\{ -\frac{\left[x_T - x_0 - (r - \sigma^2/2)T\right]^2}{2\sigma^2 T} \right\}$$



<u>Black-Scholes-Merton model:</u>

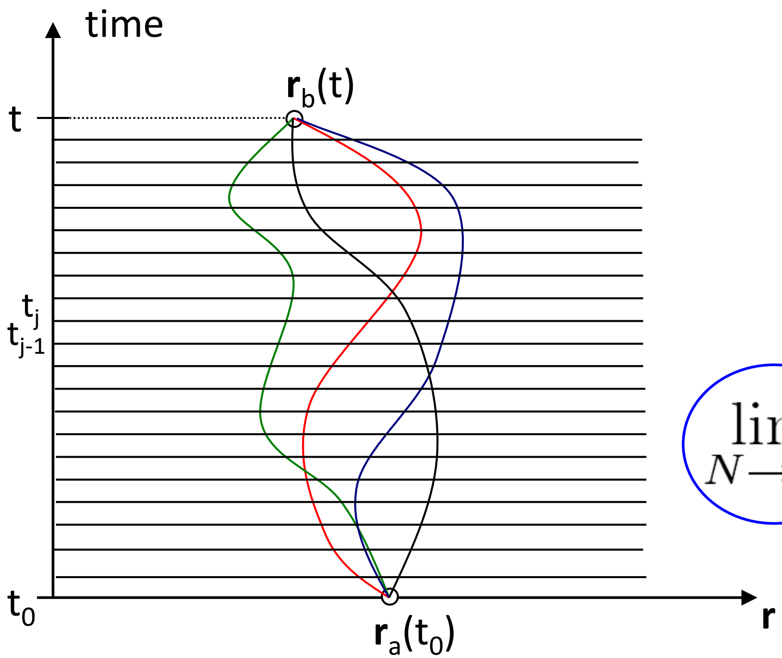
$$dx = \left(r - \sigma^2/2\right) dt + \sigma dW$$

leads to exp{rT}
cf. “exponential”
discounting

adds gaussian
fluctuations

Dash, Linetsky, Rosa-Clot, Kleinert,…

Calculating the path integral: time slicing



$$\int_{r_a(t_0)}^{r_b(t)} \mathcal{D}[\mathbf{r}(t)] \exp\{-i\mathcal{S}[\mathbf{r}(t)]/\hbar\}$$

$$\lim_{N\to\infty} \int_{-\infty}^{+\infty} d\mathbf{r}_1 .... \int_{-\infty}^{+\infty} d\mathbf{r}_{N-1} \exp\{-i\mathcal{S}[\mathbf{r}_0,..,\mathbf{r}_N]/\hbar\}$$

$$\mathbf{r}(t) = \{t_0, \mathbf{r}_0\}, \{t_1, \mathbf{r}_1\}, \{t_2, \mathbf{r}_2\}, ...\{t_N, \mathbf{r}_N\}$$

$$\mathcal{S}[\mathbf{r}_0,..,\mathbf{r}_N] = \sum_{j=1}^{N} \frac{m}{2} \frac{(\mathbf{r}_j - \mathbf{r}_{j-1})^2}{(t_j - t_{j-1})} + \sum_{j=1}^{N} V\left(\frac{\mathbf{r}_j + \mathbf{r}_{j-1}}{2}\right)$$

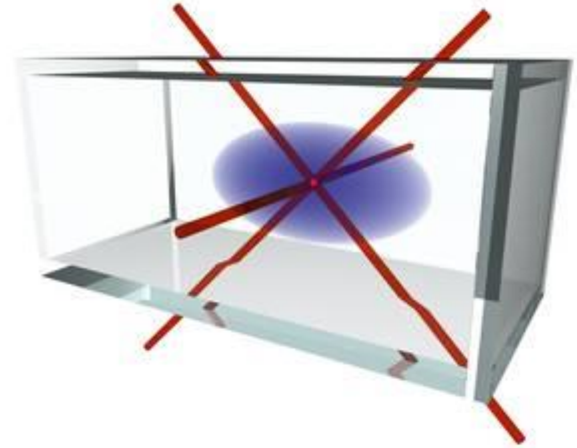
Path integrals and the polaron

Ultracold gases are used as a quantum simulator to model the polaron

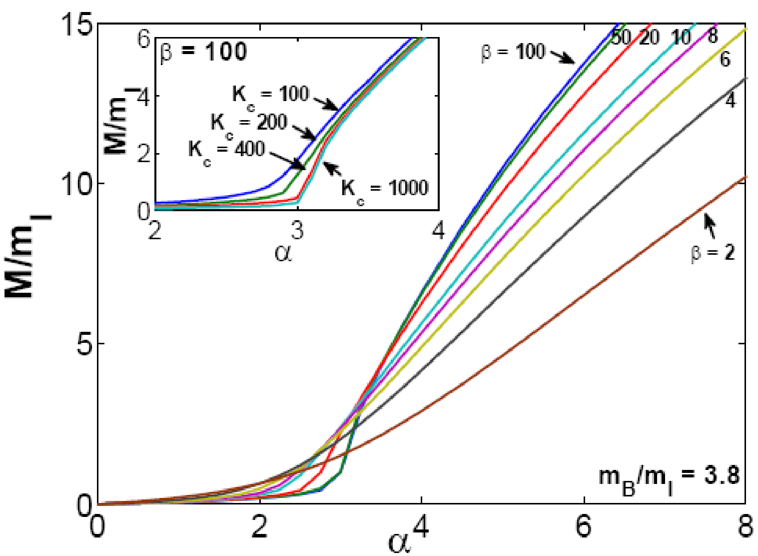
$$\hat{H}_{pol} = \frac{\hat{p}^2}{2m_I} + \sum_{\mathbf{k}\neq 0} \hbar\omega_{\mathbf{k}} \hat{b}^{\dagger}_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \sum_{\mathbf{k}\neq 0} V_{\mathbf{k}} e^{i\mathbf{k}\cdot\hat{\mathbf{r}}} (\hat{b}_{\mathbf{k}} + \hat{b}^{\dagger}_{-\mathbf{k}})$$

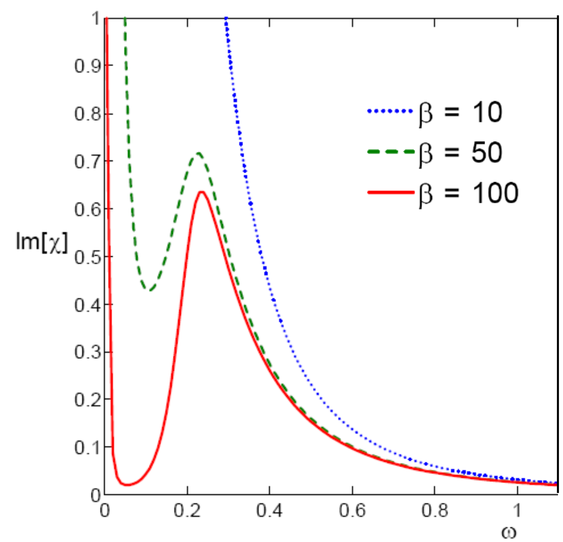
$$\hbar\omega_{\mathbf{k}} = ck\sqrt{1 + (\xi k)^2/2}$$

$$V_{\mathbf{k}} = \sqrt{N_0}\left[\frac{(\xi k)^2}{(\xi k)^2 + 2}\right]^{1/4} g_{IB}$$



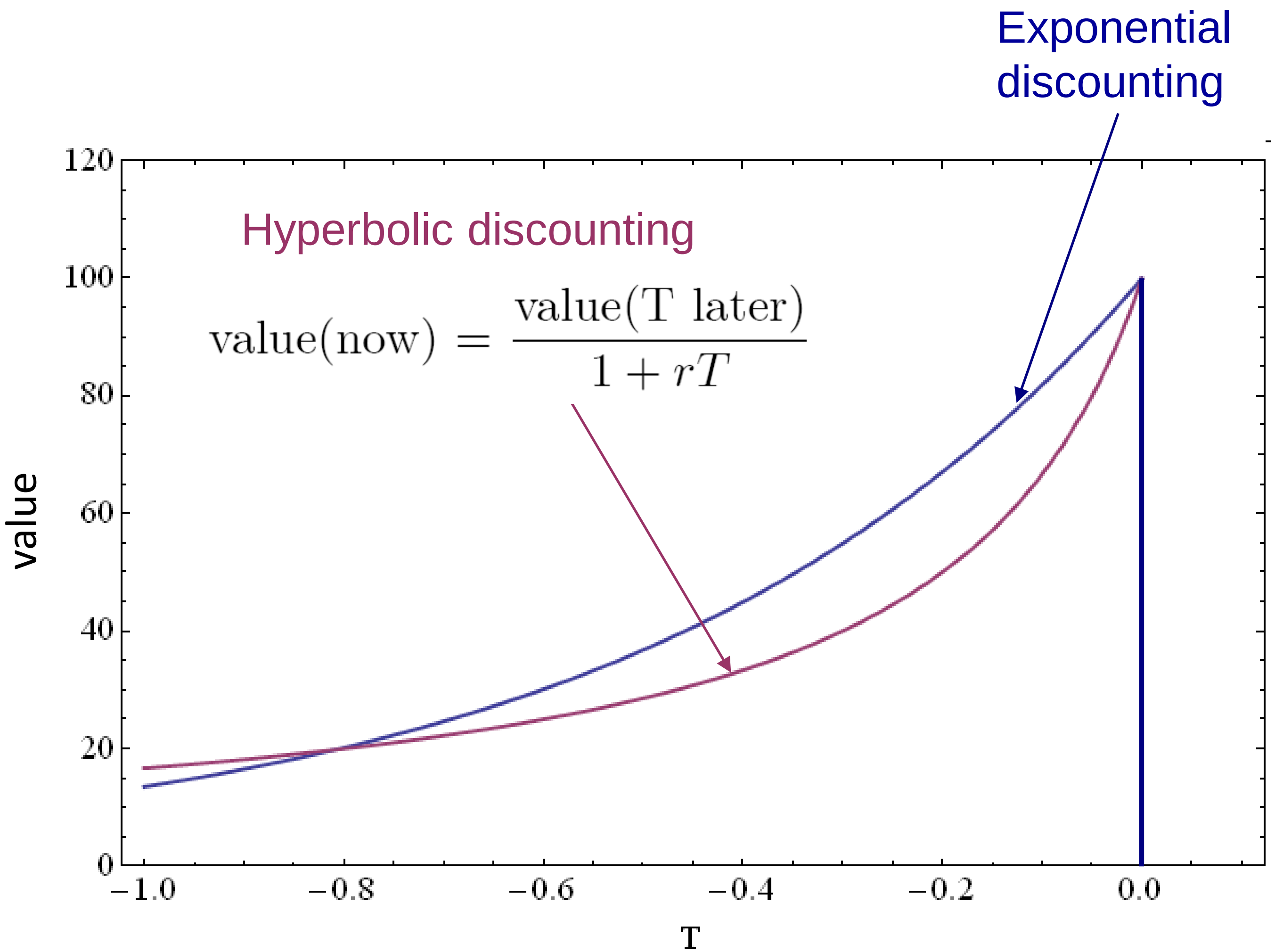
Prof.dr. Widera's research group: probing Cs atoms in Rb condensate





JT, W. Casteels, M. K. Oberthaler, S. Knoop, E. Timmermans, and J. T. Devreese, Phys. Rev. B **80**, 184504 (2009);
W. Casteels, JT, J.T.Devreese, Phys. Rev. A **83**, 033631 (2011).

$$\text{value(now)} = \exp(-rT)\ \text{value(T later)}$$



Compare:

Do you prefer:

(a) 100’000 € in 21 years
(b) 98’039 € in 20 years

← The most prominent choice (a)

to

Do you prefer:

(a) 100’000 € in 1 year
(b) 98’039 € now

← The most prominent choice (b)

⇒ Human psychology does not work with exponential discounting

Improving Black-Scholes : stochastic volatility

For the uncorrelated system, it is worth simplifying the notation again, and introducing $z = (v/\theta)^{1/2}$.

What is the transition probability for a price and volatility to go from x_0,z_0 at time t=0 to x_T,z_T at T ?

$$\mathcal{K}(x_0, z_0, 0|x_T, z_T, T) = \int \mathcal{D}x[t]\mathcal{D}z[t] \ \exp\{-\mathcal{S}\}$$

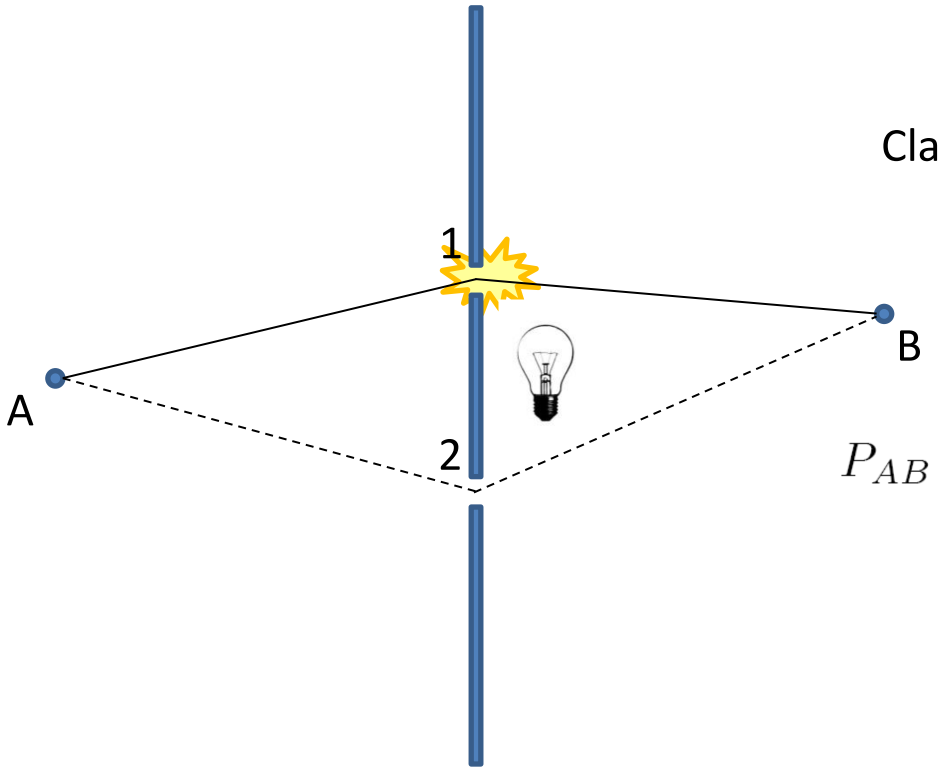
$$\mathcal{K}(x_0, z_0, 0|x_T, z_T, T)$$
$$= A\int \mathcal{D}z[t]\int \mathcal{D}x[t] \ \exp\left\{-\int_0^T \frac{\dot{x}^2}{2\theta z^2}dt\right\} \ \exp\left\{-\int_0^T\left(\frac{1}{2m}\dot{z}^2 + az^2 + \frac{b}{z^2}\right)dt\right\}$$

time-slicing + gaussian integrations

$\{t_1,x_1,z_1\},\ldots,\{t_{N-1},x_{N-1},z_{N-1}\}$

$$\frac{1}{\sqrt{2\pi\theta}}\exp\left(\frac{-x_T^2}{2\theta\int_0^T z^2 d\tau}\right) = \int dk \exp\{ikx_T\}\exp\left\{-k^2\frac{\theta}{2}\int_0^T z^2 dt\right\}$$

Introduction: quantum mechanics with path integrals



Classical: add the probabilities

$$P_{AB} = |\phi_1|^2 + |\phi_2|^2$$

Improving Black-Scholes : stochastic volatility

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$$\mathcal{K}(x_0, z_0, 0 | x_T, z_T, T)$$

$$= \frac{1}{2\pi} \exp\left[\frac{-x_T}{2} + \frac{\kappa^2\theta}{\sigma^2}T\right] \int_{-\infty}^{+\infty} \exp\left[ikx_T + z_0^2\left(\frac{\kappa}{4\sigma^2} - \frac{\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}}{4\sigma}\coth\left(\frac{\sigma}{2}\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}T\right) + \frac{\left(\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2\right)}{4\sinh^2\left(\frac{\sigma}{2}\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}T\right)\left(\kappa+\sigma\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}\coth\left(\frac{\sigma}{2}\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}T\right)\right)}\right)\right]$$
$$\left(\frac{\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}}{\frac{\kappa}{\sigma}\sinh\left(\frac{\sigma}{2}\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}T\right) + \sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}\cosh\left(\frac{\sigma}{2}\sqrt{\frac{1}{4}+\frac{\kappa^2}{\sigma^2}+k^2}T\right)}\right)^{2\frac{\kappa\theta}{\sigma^2}} dk$$

D. Lemmens, M. Wouters, JT, S. Foulon, Phys. Rev. E **78**, 016101 (2008).