Universiteit
Antwerpen

Seminar day at the University of Liège
lecture room R.7, building B28
November 28th, 2019

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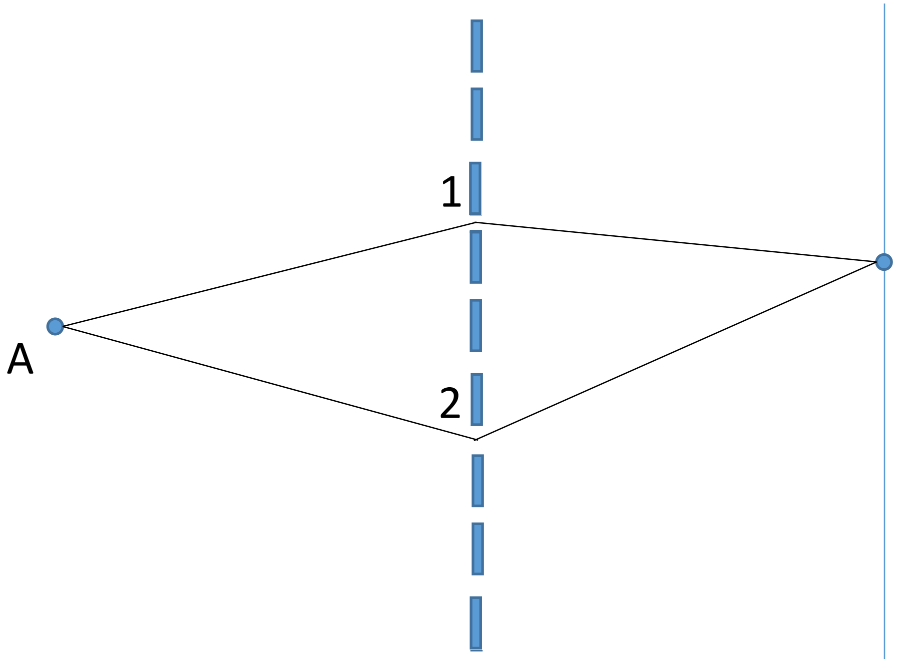
Theory of
Quantum and
Complex systems

Functional integral description of a superfluid Fermi gas

Path-integral outline of this talk:

Part I: path integrals for a quantum particle

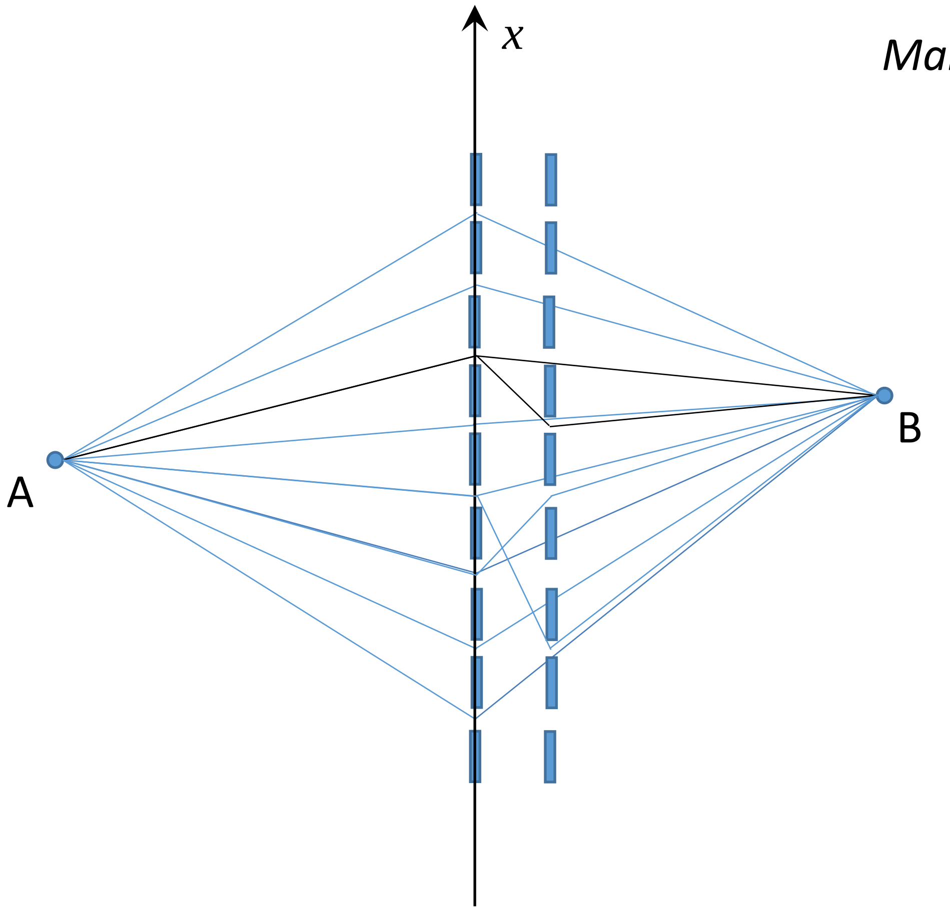


Two alternatives: add the amplitudes

$$\phi_{AB} = \phi_1 + \phi_2$$

$$P_{AB} = |\phi_{AB}|^2 = |\phi_1 + \phi_2|^2$$

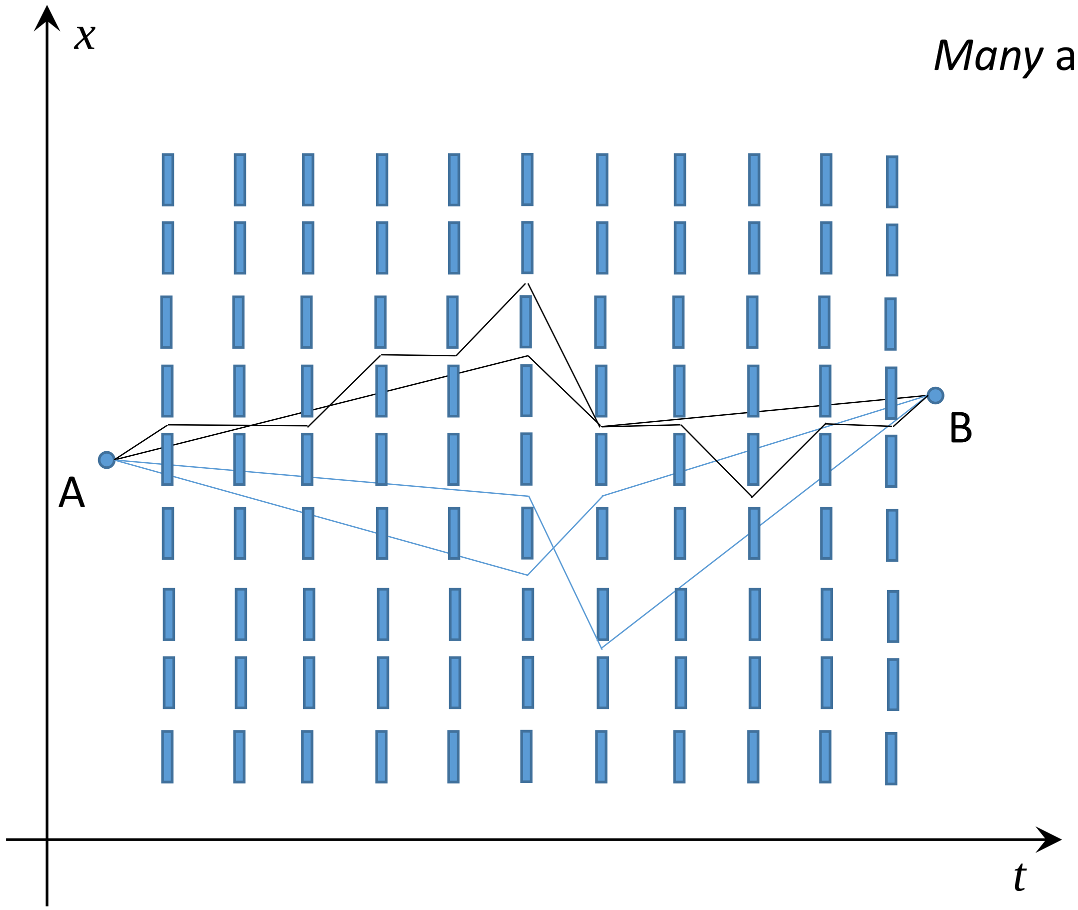
Introduction: Path integrals for a quantum particle



Many alternatives: add the amplitudes

$$\phi_{AB} = \sum_{x_1} \sum_{x_2} \phi_{x_1,x_2}$$

Introduction: Path integrals for a quantum particle

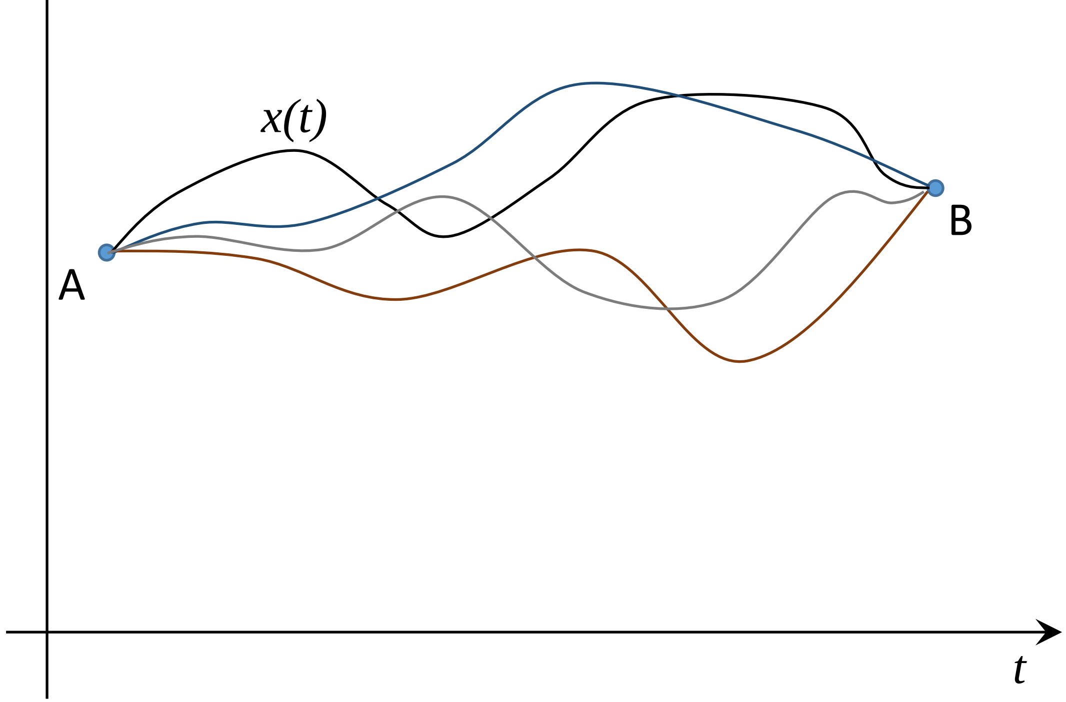


Many alternatives: add the amplitudes

$$\phi_{AB} = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_N} \phi_{x_1, x_2, \ldots, x_N}$$

$$\phi_{AB} = \langle r_B(T) | r_A(0) \rangle = K(r_B, T | r_A, 0)$$

is called the path integral propagator



Many alternatives: add the amplitudes

$$\phi_{AB} = \int \mathcal{D}x \; \phi\left[x(t)\right]$$

The amplitude corresponding to a given path *x*(*t*) is

$$\phi\left[x(t)\right] = \exp\left\{\frac{i}{\hbar} S[x(t)]\right\}$$

Here, S is the action functional:

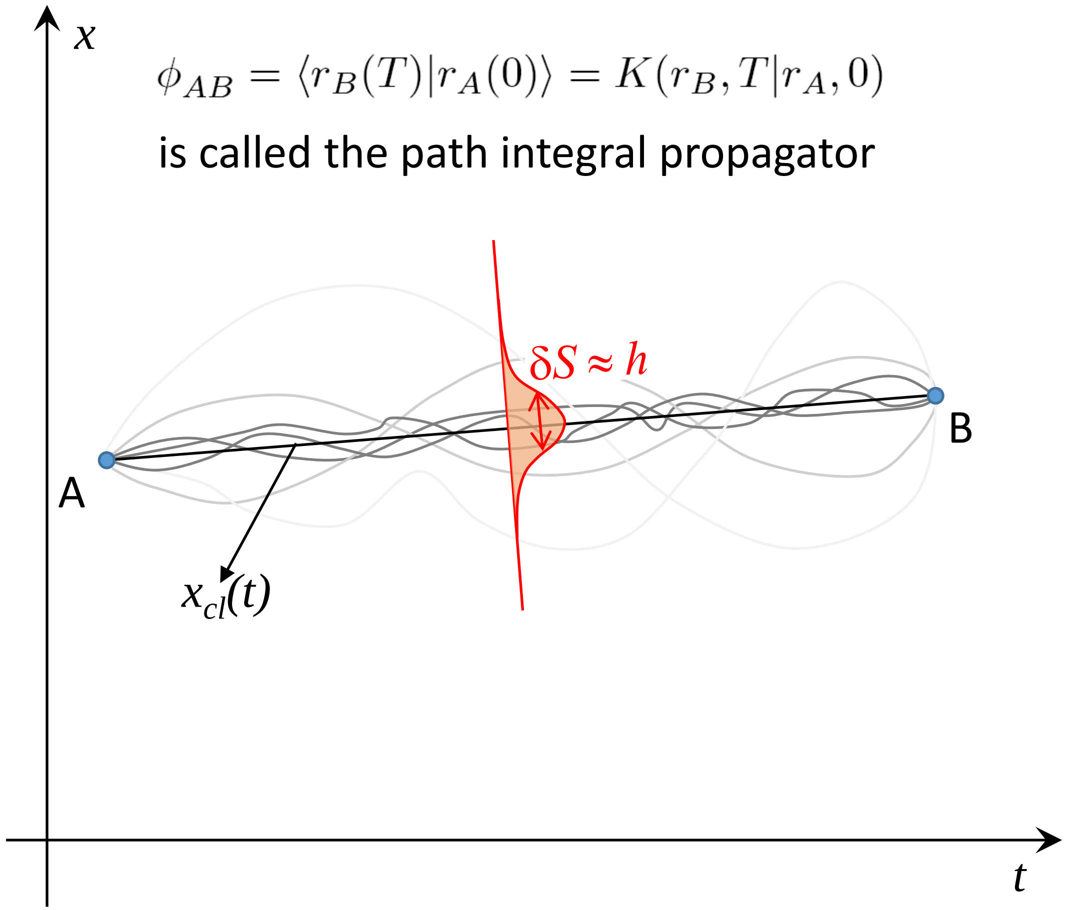
$$S[x(t)] = \int_0^T L(x, \dot{x}, t) dt$$

With L the Lagrangian, eg.

$$L(x, \dot{x}, t) = \frac{m}{2}\dot{x}^2 - V(x)$$

H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009)

Introduction: Path integrals for a quantum particle



$$\phi_{AB} = \langle r_B(T) | r_A(0) \rangle = K(r_B, T | r_A, 0)$$

is called the path integral propagator

No operators any more! $\hat{x} \to x(t)$

Many alternatives: add the amplitudes

$$\phi_{AB} = \int \mathcal{D}x \; \phi\left[x(t)\right]$$

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Introduction: Path integrals for quantum statistical physics

Thusfar, we discussed the path integral propagator

$$K\left(x_B, t | x_A, 0\right) = \langle x_B(t) | x_A(0) \rangle = \left\langle x_B \left| e^{-i\hat{H}t/\hbar} \right| x_A \right\rangle$$

To study the phases of the atomic Fermi gas and its thermodynamics, we need the density matrix:

$$\rho(x_B, \beta | x_A) = \langle x_B | \hat{\rho} | x_A \rangle = \left\langle x_B \left| e^{-\beta\hat{H}} \right| x_A \right\rangle$$

From the above, it is clear that the density matrix can be expressed as an analytic continuation of the propagator:

$$\rho(x_B, \beta | x_A) = \int_{\{x_A,0\}}^{\{x_B,\beta\}} \mathcal{D}x \ \exp\left\{ -\frac{1}{\hbar} \underbrace{\int_0^\beta \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] d\tau}_{\text{Euclidean action } S[x(\tau)]} \right\}$$

$\tau = it$

$t = -i\hbar\beta \Rightarrow \tau = \beta$

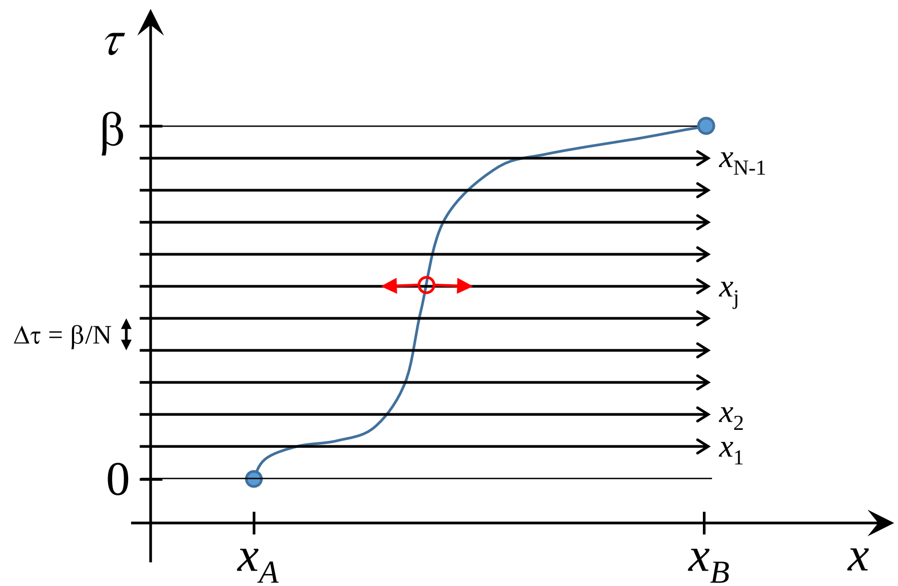
Euclidean action $S[x(\tau)]$

H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009)

Introduction: Path integrals for quantum statistical physics

$$\rho(x_B,\beta|x_A) = \int_{\{x_A,0\}}^{\{x_B,\beta\}} \mathcal{D}x \; \exp\left\{-\frac{1}{\hbar}\int_0^\beta \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right] d\tau\right\}$$

Sum over all paths $x(\tau)$ going from x_A to x_B , as τ goes from 0 to β.

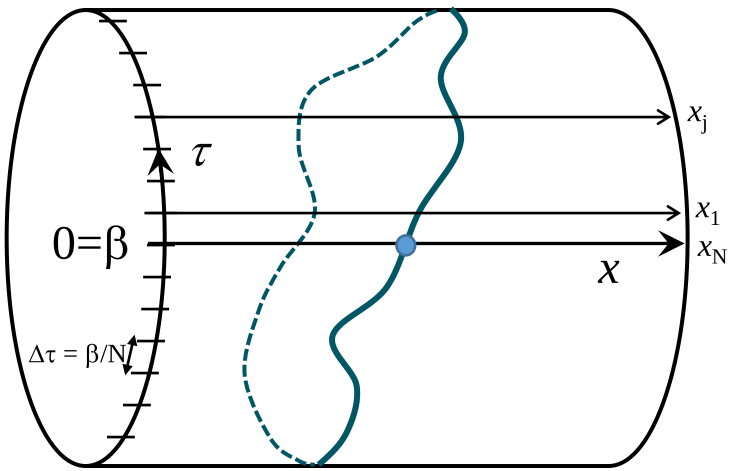


$$\lim_{N\to\infty}\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_{N-1}$$

H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th ed. (World Scientific, Singapore, 2009)

$$\mathcal{Z} = \int_{x(\beta)=x(0)} \mathcal{D}x \ \exp\left\{ -\frac{1}{\hbar}\int_0^\beta \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right] d\tau \right\}$$

Sum over all periodic paths $x(\tau)$ as τ goes from 0 to β.



$$\lim_{N\to\infty} \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dx_N$$

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Part II: path integrals in field theory

From particle paths to fields

Newtonian particles have action functionals from which the equation of motion can be derived for the (unique) classical path followed by the particle.

Classical fields also have action functionals from which the field equations can be derived yielding the (unique) classical field configuration.

Example: electrostatics

$$S[\phi(\vec{r})] = \int \left(\frac{\varepsilon}{2} (\nabla\phi)^2 - \rho\phi \right) d^3\vec{r}$$

permittivity — electrostatic potential — charge density

extremizing this action yields the field equation:

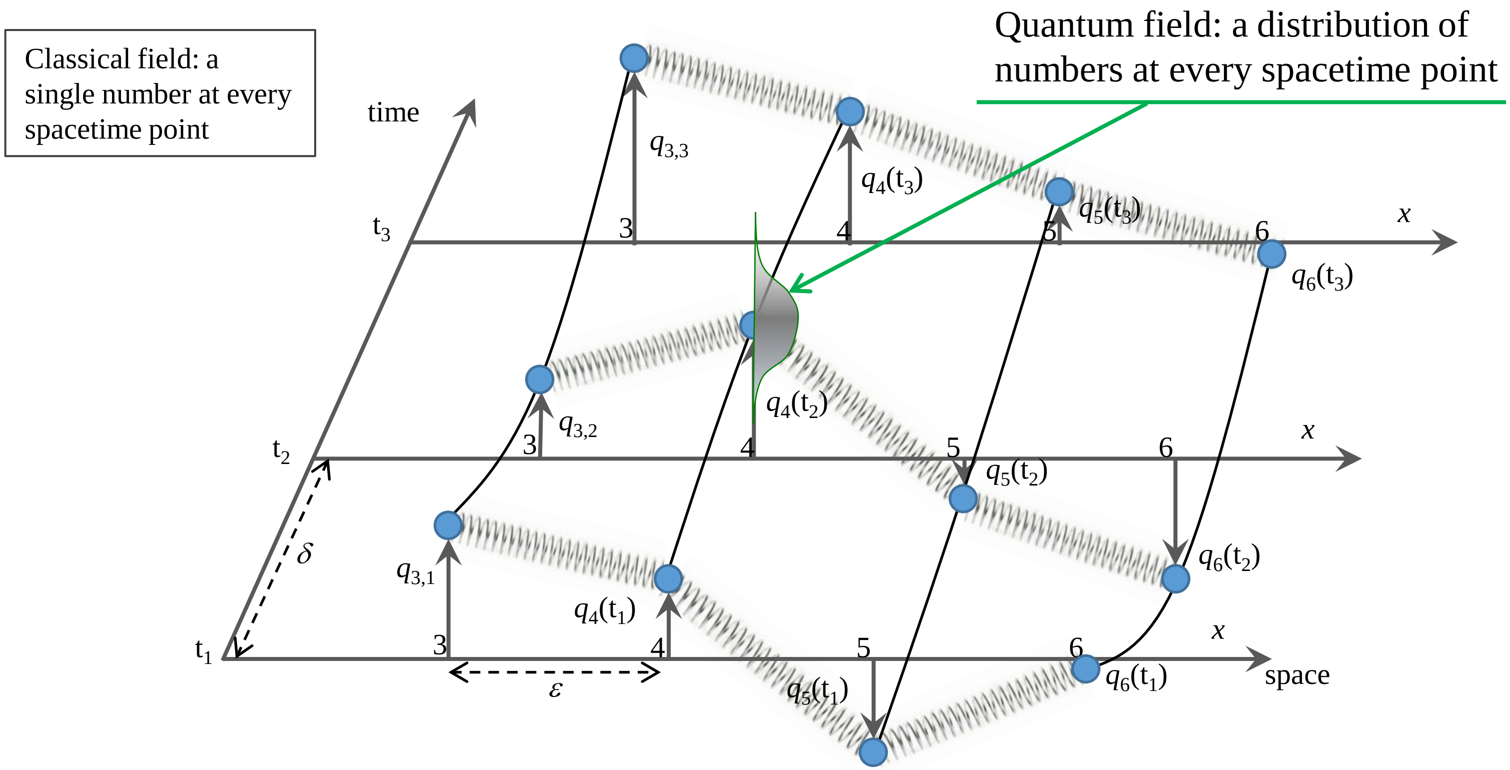
$$\Delta\phi = -\rho/\varepsilon$$

Poisson equation of electrostatics

From particle paths to fields

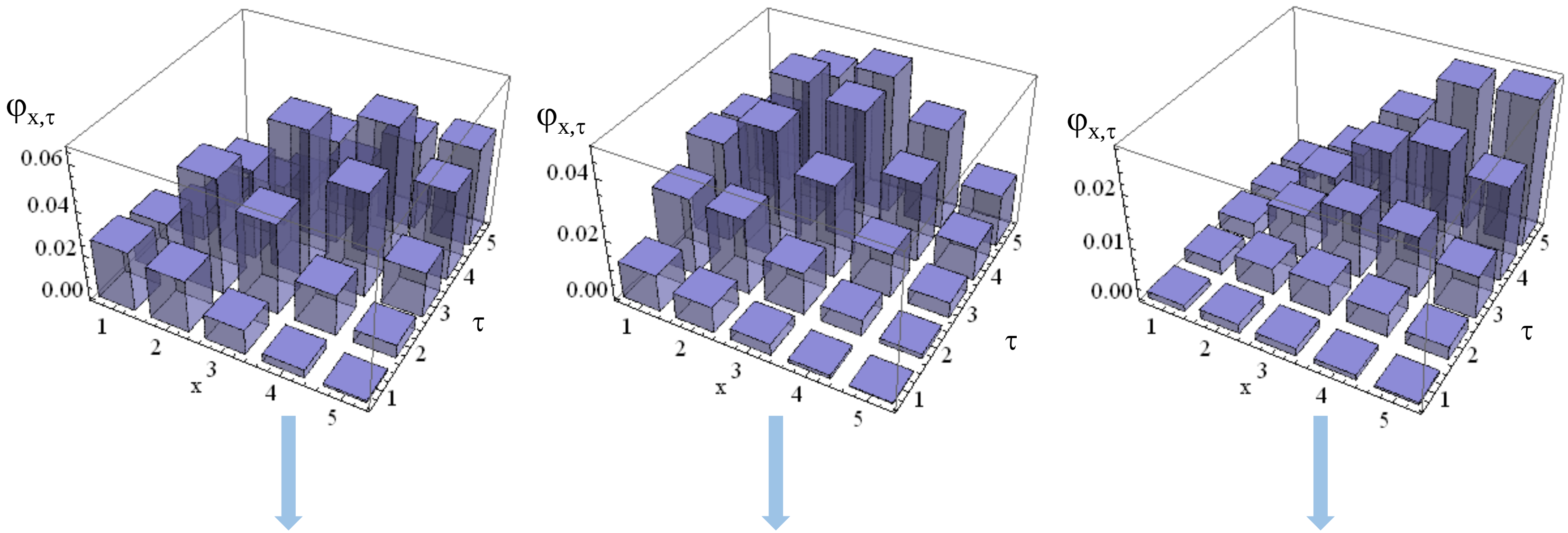
For quantum particles, all paths must be taken into account, weighted by the exponent of the particle action.

For quantum fields, all field configurations must be taken into account, weighted by the exponent of the field action.



From particle paths to fields

The “path” integral prescription is to average over all possible field configurations For $\varphi_{x,\tau}$ giving each field configuration a weight $\exp\{-S[\varphi]/\hbar\}$.



$\exp\{-S[\varphi_{x,\tau}]/\hbar\}$ + $\exp\{-S[\varphi_{x,\tau}]/\hbar\}$ + $\exp\{-S[\varphi_{x,\tau}]/\hbar\}$ +…

This “path integral” sum is again denoted by $\int \mathcal{D}\varphi \exp\left\{-\frac{1}{\hbar}S[\varphi]\right\}$

Side note: fields for fermions

We require that the “numbers” living on the spacetime points anticommute, i.e.

$$\psi_a \psi_b = -\psi_b \psi_a$$

No ordinary algebra ($\mathbb{R}$, $\mathbb{C}$,...) does this. Imposing this rule for the multiplication leads to a new algebra, the Grassmann algebra $\mathbb{G}$.

As a consequency, functions become really simple:

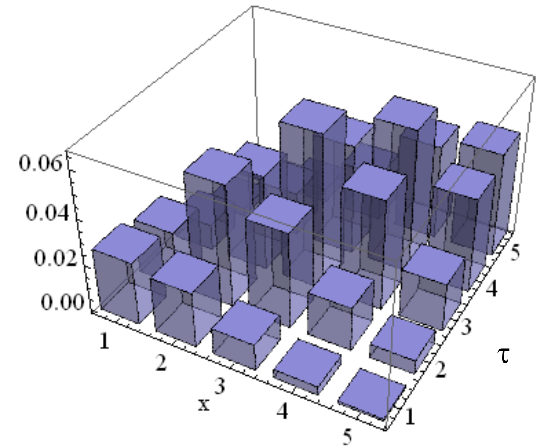
$$f(\psi_a, \psi_b) = c_0 + c_1 \psi_a + c_2 \psi_b + c_3 \psi_a \psi_b$$

example → $$\exp\{-A\psi_a\psi_b\} = 1 - A\psi_a\psi_b$$

Integrations simplify as well:

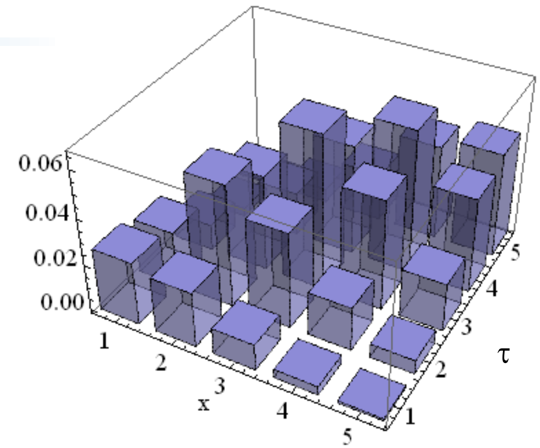
$$\int \psi_a d\psi_a = 1 \qquad \int d\psi_a = 0$$

example → $$\int d\psi_a \int d\psi_b \exp\{-A\psi_a\psi_b\} = A$$



“[Perhaps] a time will come when it will be drawn forth from the dust of oblivion and the ideas laid down here will bear fruit. ”

Analytically Solvable Path Integrals for fields



Sadly, there is basically only one integral that we can do analytically, namely that for **quadratic action functionals**.

$$\int \mathcal{D}\varphi \exp\left\{-\sum_{j,\ell} \varphi_j A_{j,\ell} \varphi_\ell\right\} = \frac{1}{\sqrt{\det(A)}}$$

re-label:

$$\varphi_{x,\tau} \to \varphi_j \qquad j = 1, \ldots, 25$$

$$\int \mathcal{D}\phi \ \exp\left\{-\sum_{j,\ell} \phi_j^* A_{j,\ell} \phi_\ell\right\} = \frac{1}{\det(A)}$$

2 real fields (=one complex field) per spacetime point:

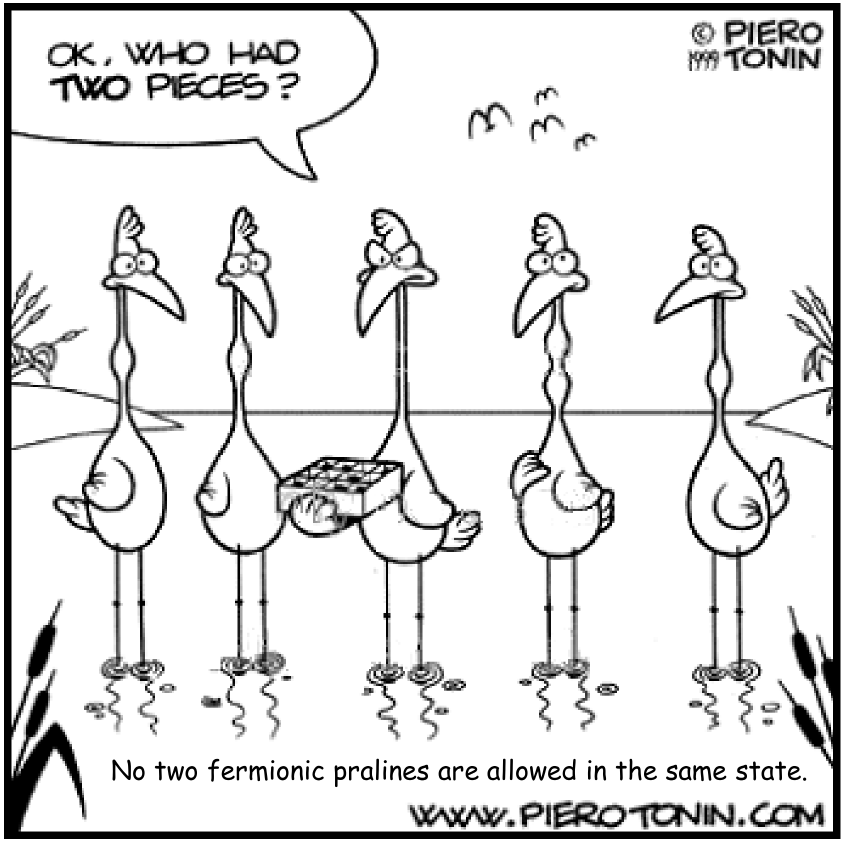
$$\varphi_j^{(\mathrm{Re})}, \varphi_j^{(\mathrm{Im})} \to \phi_j = \varphi_j^{(\mathrm{Re})} + \mathrm{i}\varphi_j^{(\mathrm{Im})}$$

$$\int \mathcal{D}\psi \ \exp\left\{-\sum_{j,\ell} \bar{\psi}_j A_{j,\ell} \psi_\ell\right\} = \det(A)$$

2 Grassmann fields per spacetime point:

$$\bar{\psi}_j, \psi_j$$

Part III: The ultracold atomic Fermi gas

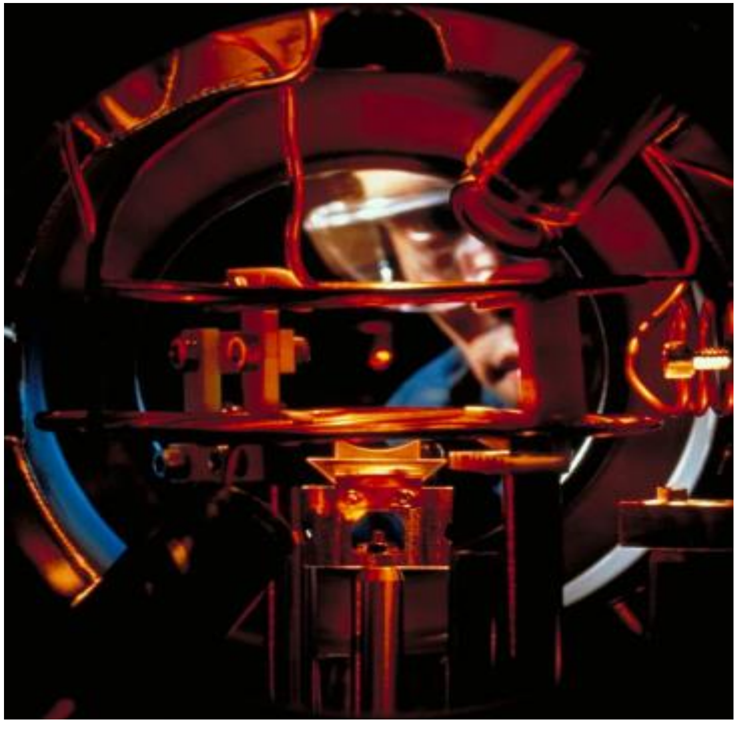


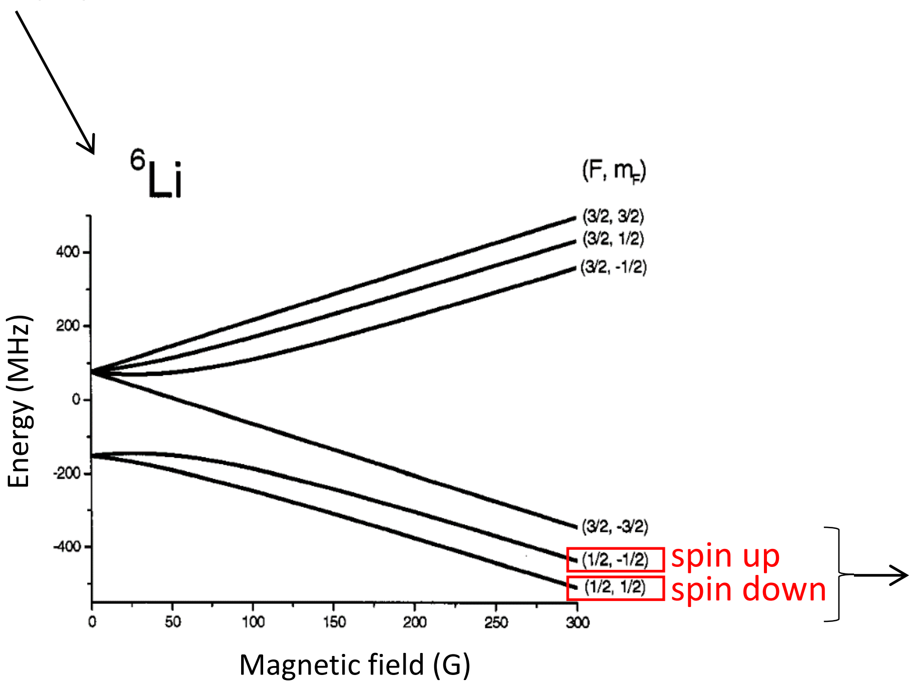
Ultracold atomic Fermi gases

Quantum gases are optically cooled, trapped collections of ultracold atoms.

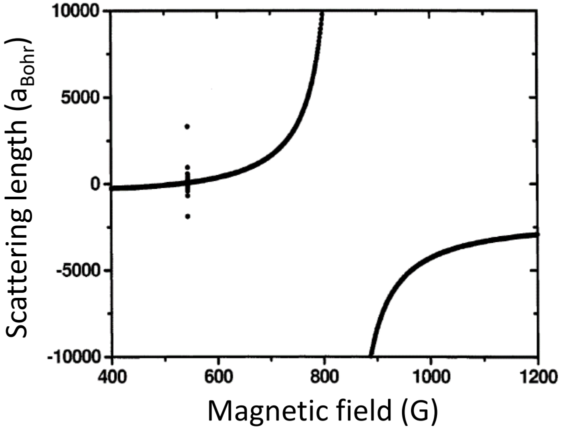
Typically: 10^5-10^6 atoms at nanokelvin temperatures.

Common fermionic species in experiment:
^6Li and ^{40}K





Interactions: s-wave contact interactions, only between opposite spin fermions.



Graphs: thesis G.B. Partridge (promotor R. Hulet), "Pairing of fermionic lithium-6 throughout the BEC-BCS crossover", Rice Univerisy 2007

Superfluidity and the BEC-BCS crossover

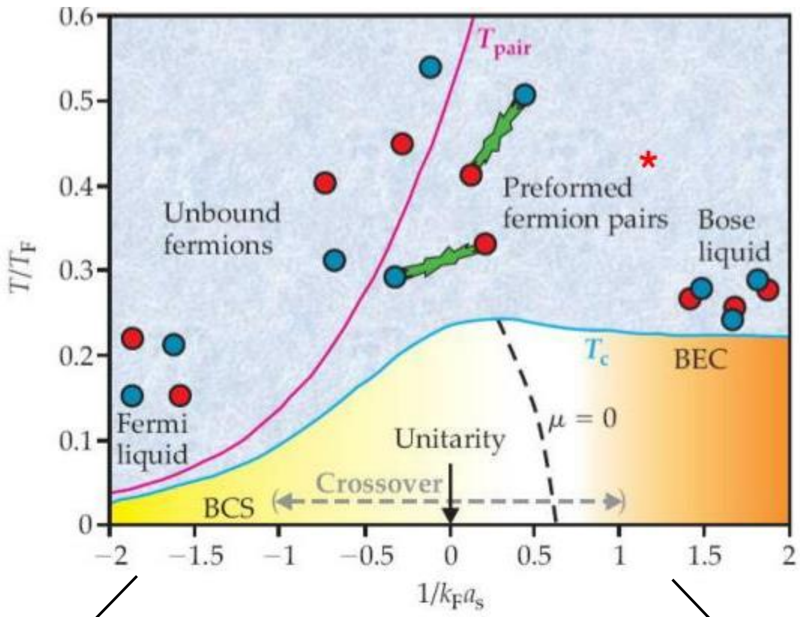
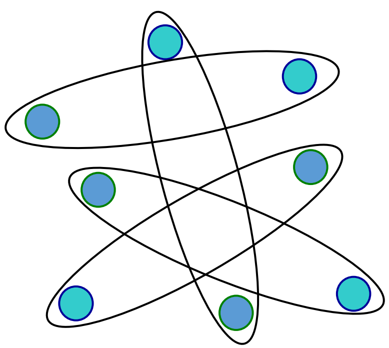
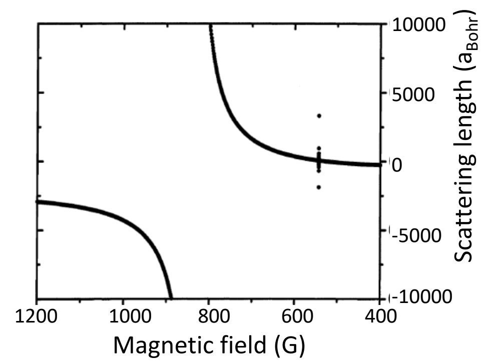


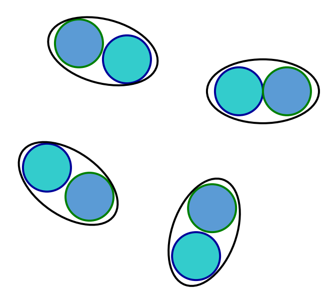
Image source:
C.A.R.Sa de Melo,
Physics Today, oct.2008

BCS side of the resonance





BEC side of the resonance



Constructing the action functional

The Hamiltonian of the Fermi gas interacting through a contact potential

$V(\mathbf{x}-\mathbf{x}') = g\delta(\mathbf{x}-\mathbf{x}')$ with $g = \dfrac{4\pi\hbar^2 a_s}{m}$ is:

$$\hat{H} = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int d\mathbf{x}\ \hat{\psi}^\dagger_{x,\sigma} \left(-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} - \mu\right) \hat{\psi}_{x,\sigma} + g \int d\mathbf{x}\ \hat{\psi}^\dagger_{x,\uparrow}\hat{\psi}^\dagger_{x,\downarrow}\hat{\psi}_{x,\downarrow}\hat{\psi}_{x,\uparrow}$$

For the path integral version, trade the operators for Grassmann fields:

$$\mathcal{H}[\psi] = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int d\mathbf{x}\ \bar{\psi}_{x,\sigma} \left(-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} - \mu\right) \psi_{x,\sigma} + g \int d\mathbf{x}\ \bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow}\psi_{x,\downarrow}\psi_{x,\uparrow}$$

The field Lagrangian corresponding to this Hamiltonian is

$$\mathcal{L}[\psi] = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int d\mathbf{x}\ \bar{\psi}_{x,\sigma} \left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} - \mu\right) \psi_{x,\sigma} + g \int d\mathbf{x}\ \bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow}\psi_{x,\downarrow}\psi_{x,\uparrow}$$

Constructing the action functional

The partition sum is

$$\mathcal{Z} = \int \mathcal{D}\psi \ \exp\left\{-\mathcal{S}\left[\psi\right]/\hbar\right\}$$

where action for the fermionic field is given by

$$\mathcal{S}\left[\psi\right] = \sum_{\sigma\in\{\uparrow,\downarrow\}} \int dx \ \bar{\psi}_{x,\sigma}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla_{\mathbf{x}}^2 - \mu_\sigma\right)\psi_{x,\sigma} + g\int dx \ \bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow}\psi_{x,\downarrow}\psi_{x,\uparrow}$$

$$\int dx = \int_0^\beta d\tau \int d\mathbf{x}$$

Fix number of up and down fermions separately

Constructing the action functional

The partition sum is

$$\mathcal{Z} = \int \mathcal{D}\psi \;\; \exp\left\{-\mathcal{S}\left[\psi\right]/\hbar\right\}$$

where action for the fermionic field is given by

$$\mathcal{S}\left[\psi\right] = \underbrace{\sum_{\sigma\in\{\uparrow,\downarrow\}} \int dx \;\bar{\psi}_{x,\sigma}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} - \mu_\sigma\right)\psi_{x,\sigma}}_{\text{Quadratic action ☑}} + \underbrace{g\int dx\;\bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow}\psi_{x,\downarrow}\psi_{x,\uparrow}}_{\text{Quartic term ☒}}$$

Quadratic action ☑

Quartic term ☒

Trick #1 : completing the square

Remember the following Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = e^{b^2/(4a)}\int_{-\infty}^{\infty} e^{-a[x-b/(2a)]^2}dx = \sqrt{\frac{2\pi}{a}}e^{b^2/(4a)}$$

It has a counterpart for complex variables:

$$\int_{\mathbb{C}} e^{-a|z|^2+bz^*+b^*z}dz = \sqrt{\frac{2\pi}{a}}e^{|b|^2/(4a)}$$

"b" can be a product of Grassmann variables:

$b = \bar{\psi}_{x,\uparrow}\psi_{x,\uparrow}$ "direct" channel (b ≡ fermion density)

$b = \bar{\psi}_{x,\uparrow}\psi_{x,\downarrow}$ "exchange" channel (b ≡ magnetization density)

$b = \psi_{x,\uparrow}\psi_{x,\downarrow}$ "anomalous" channel (b ≡ pair field)

This leads to the "Hubbard-Stratonovic Transformation" (for the anomalous channel):

$$\exp\left\{-g\int dx\ \bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow}\psi_{x,\downarrow}\psi_{x,\uparrow}\right\} = \int\mathcal{D}\Delta\exp\left\{\int dx\frac{|\Delta_x|^2}{g} + \int dx\left(\Delta_x\bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow} + \Delta_x^*\psi_{x,\downarrow}\psi_{x,\uparrow}\right)\right\}$$

The Hubbard-Stratonovic transformation

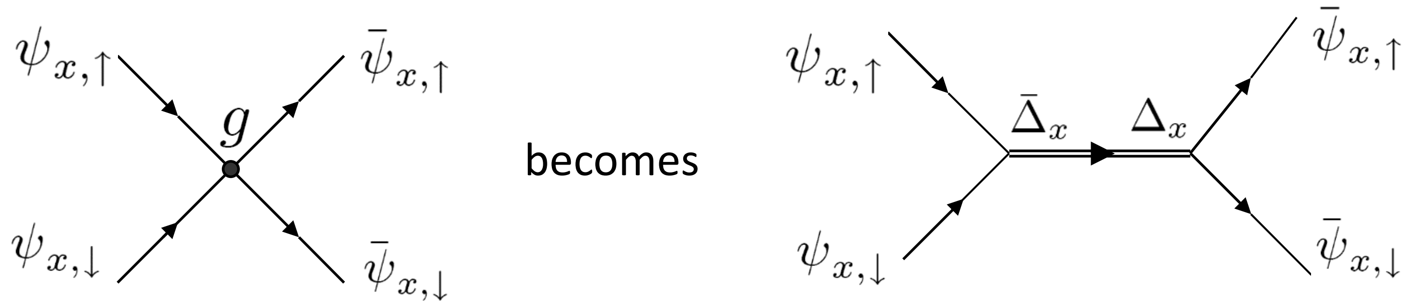
The partition sum is

$$\mathcal{Z} = \int \mathcal{D}\psi \int \mathcal{D}\Delta \;\exp\left\{-\mathcal{S}\left[\psi,\Delta\right]\right\}$$

where action for the fermionic field is given by

$$\mathcal{S}\left[\psi,\Delta\right] = \int dx \left[\sum_{\sigma\in\{\uparrow,\downarrow\}} \bar{\psi}_{x,\sigma}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla_{\mathbf{x}}^2 - \mu_\sigma\right)\psi_{x,\sigma} - \frac{\left|\Delta_x\right|^2}{g} - \left(\Delta_x\bar{\psi}_{x,\uparrow}\bar{\psi}_{x,\downarrow} + \Delta_x^*\psi_{x,\downarrow}\psi_{x,\uparrow}\right)\right]$$

Quadratic action in the Grassmann fields ☑



The Hubbard-Stratonovic transformation

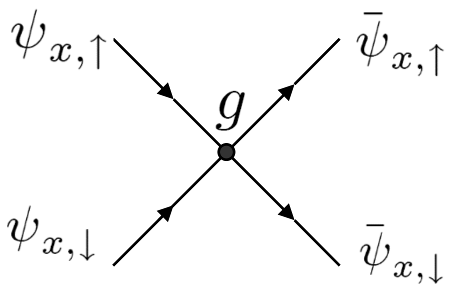
The partition sum is

$$\mathcal{Z} = \int \mathcal{D}\Delta \ \exp\left\{-\mathcal{S}\left[\Delta\right]\right\}$$

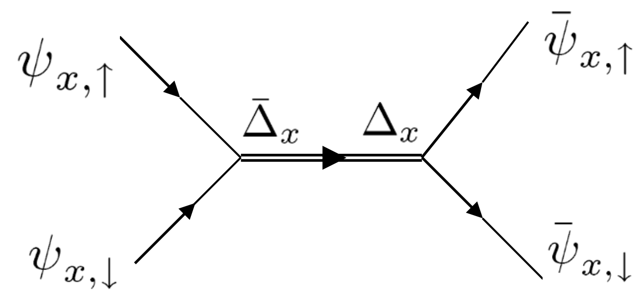
where action for the fermionic field is given by

$$\mathcal{S}\left[\Delta\right] = \int dx \left\{ -\frac{\left|\Delta_x\right|^2}{g} - \underbrace{\mathrm{Tr}_\sigma \left[\log \begin{pmatrix} -\partial_\tau + \nabla_{\mathbf{x}}^2 + \mu_\uparrow & \Delta_x \\ \Delta_x^* & -\partial_\tau - \nabla_{\mathbf{x}}^2 - \mu_\downarrow \end{pmatrix} \right] \right\}}$$

Not quadratic at all (in fact, all powers of Δ)

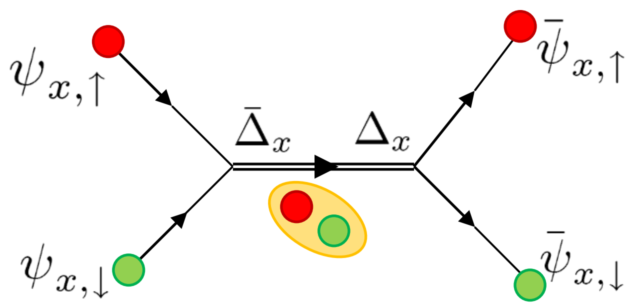


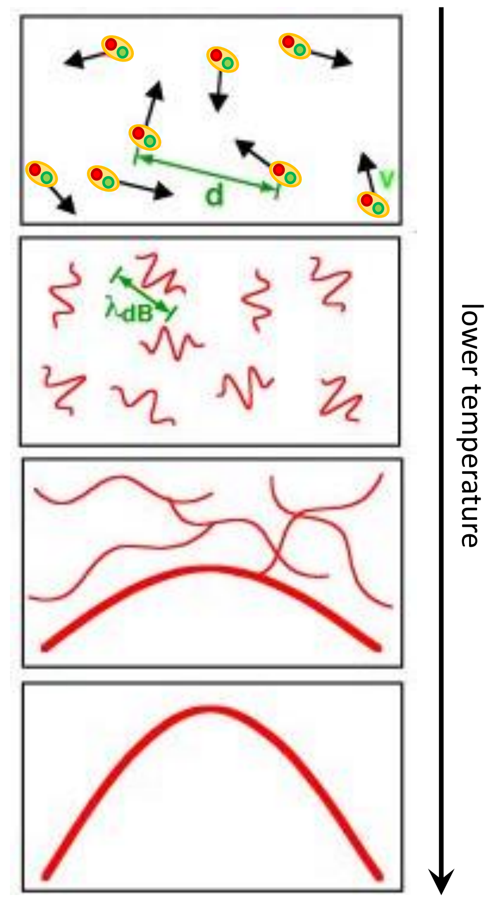
becomes



Pairs condense in the superfluid state

The introduction of new fields is used in particle physics to renormalize divergent diagrams. Some argue that these ‘new’ particles are not real, but in our case we have a clear interpretation.





The (bosonic) Δ-field represents the field of the fermionic pairs.

Application of path integral description to BEC-BCS crossover, see:
C.A.R. Sa de Melo, M. Randeria, and J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
Additional details can be found for example in Stoof, Dickerscheid & Gubbels, *Ultracold Quantum Fields* (Springer, 2009).

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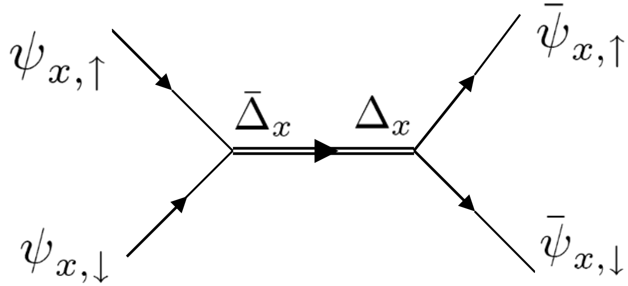
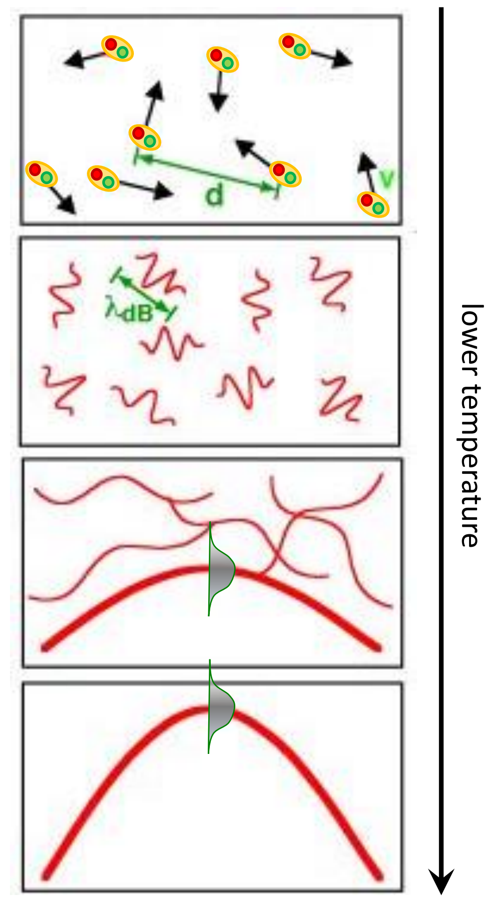
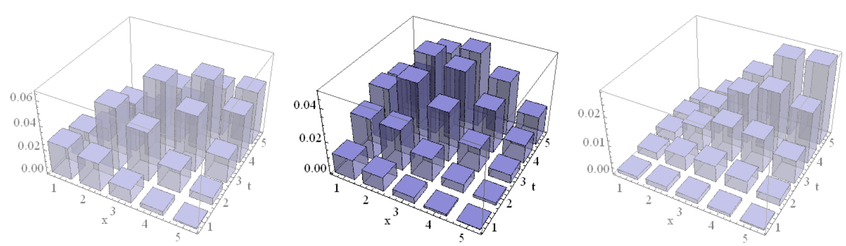




The (bosonic) Δ-field represents the field of the fermionic pairs. When these pairs Bose condense to form a superfluid, the macroscopic occupation of a single mode makes the pair field more “classical”, i.e. dominated by a single realization.



$$\Delta_{\mathbf{x},\tau} = \Delta$$

Application of path integral description to BEC-BCS crossover, see:
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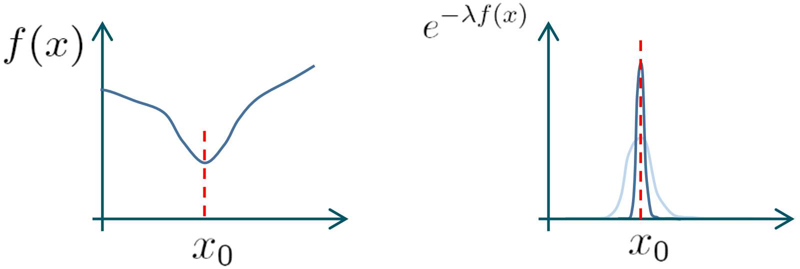
Trick #2 : the saddle-point expansion

Consider the ordinary integral

$$\mathcal{I} = \int_{-\infty}^{\infty} e^{-\lambda f(x)} dx$$



For large λ, only points close to x_0 will matter: $f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots$

$$\mathcal{I} \approx e^{-\lambda f(x_0)} \int_{-\infty}^{\infty} e^{-\lambda f''(x_0)(x-x_0)^2/2} dx$$

Restricting the integral to Gaussian fluctuations makes it analytically integrable:

$$\mathcal{I} \approx \sqrt{\frac{2\pi}{\lambda f''(x_0)}} e^{-\lambda f(x_0)}$$

Gaussian fluctuation expansion

$$\Delta_{\mathbf{x},\tau} = \Delta + \eta_{\mathbf{x},\tau}$$

saddle point

$$S[\Delta_{\mathbf{x},\tau}] = S[\Delta] + \delta S[\eta_{\mathbf{x},\tau}] + \delta^2 S\left[\eta_{\mathbf{x},\tau}\right] + \ldots$$

(the term $\delta S[\eta_{\mathbf{x},\tau}]$ is crossed out)

saddle point value

$$\int \mathcal{D}\Delta \;\exp\left\{-S[\Delta_{\mathbf{x},\tau}]\right\} \approx \exp\left\{-S[\Delta]\right\} \times \int \mathcal{D}\eta \;\exp\left\{-\frac{1}{\hbar}\delta^2 S[\eta_{\mathbf{x},\tau}]\right\}$$

$$x = x_0 + (x - x_0)$$

saddle point

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots$$

saddle point value

$$\int_{-\infty}^{\infty} e^{-\lambda f(x)} dx \approx e^{-\lambda f(x_0)} \times \int_{-\infty}^{\infty} e^{-\lambda f''(x_0)(x-x_0)^2/2} dx$$

The saddle point value

The saddle-point contribution to the partition sum is

$$\mathcal{Z}_{\mathrm{sp}} = e^{-\beta\Omega_{\mathrm{sp}}(T,V,\mu_\sigma)} = \exp\left\{-\mathcal{S}_{\mathrm{sp}}\left[\Delta\right]\right\}$$

with

$$\mathcal{S}\left[\Delta\right] = \int dk \left\{-\frac{\left|\Delta\right|^2}{g} - \mathrm{Tr}_\sigma\left[\log\begin{pmatrix} -\partial_\tau + \nabla_{\mathbf{x}}^2 + \mu_\uparrow & \Delta \\ \Delta & -\partial_\tau - \nabla_{\mathbf{x}}^2 - \mu_\downarrow \end{pmatrix}\right]\right\}$$

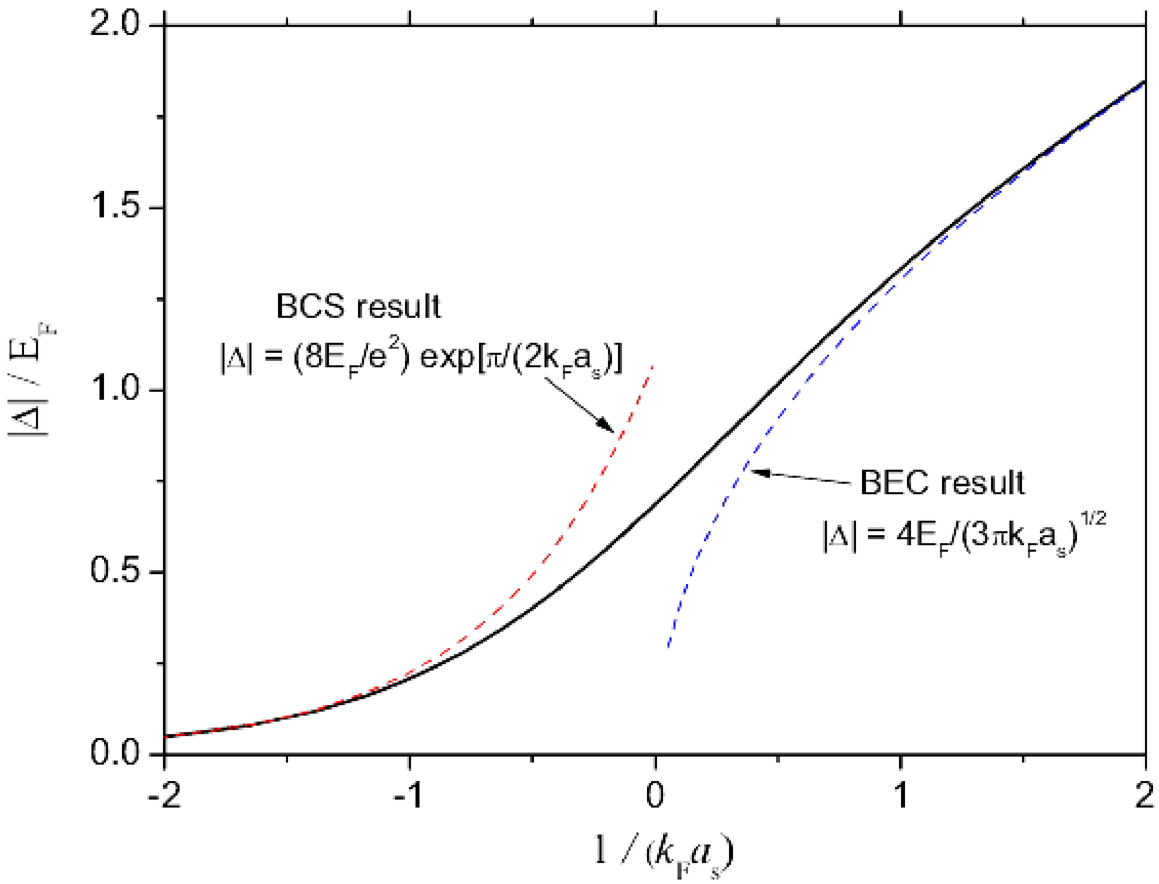
Performing the remaining integrations, we obtain the saddle point free energy

$$\Omega_{\mathrm{sp}} = -\frac{\left|\Delta\right|^2}{8\pi k_F a_s} - \int \frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\beta}\ln\left[2\cosh(\beta E_k) + 2\cosh(\beta\zeta)\right] - \xi_{\mathbf{k}} - \frac{\left|\Delta\right|^2}{2k^2}\right]$$

Energy dispersions:
$$\begin{cases} \xi_k = k^2 - \mu \\ E_k = \sqrt{\xi_k^2 + \left|\Delta\right|^2} \end{cases}$$

Chemical potentials:
$$\begin{cases} \mu = \left(\mu_\uparrow + \mu_\downarrow\right)/2 \\ \zeta = \left(\mu_\uparrow - \mu_\downarrow\right)/2 \end{cases}$$

The pair condensate order parameter Δ

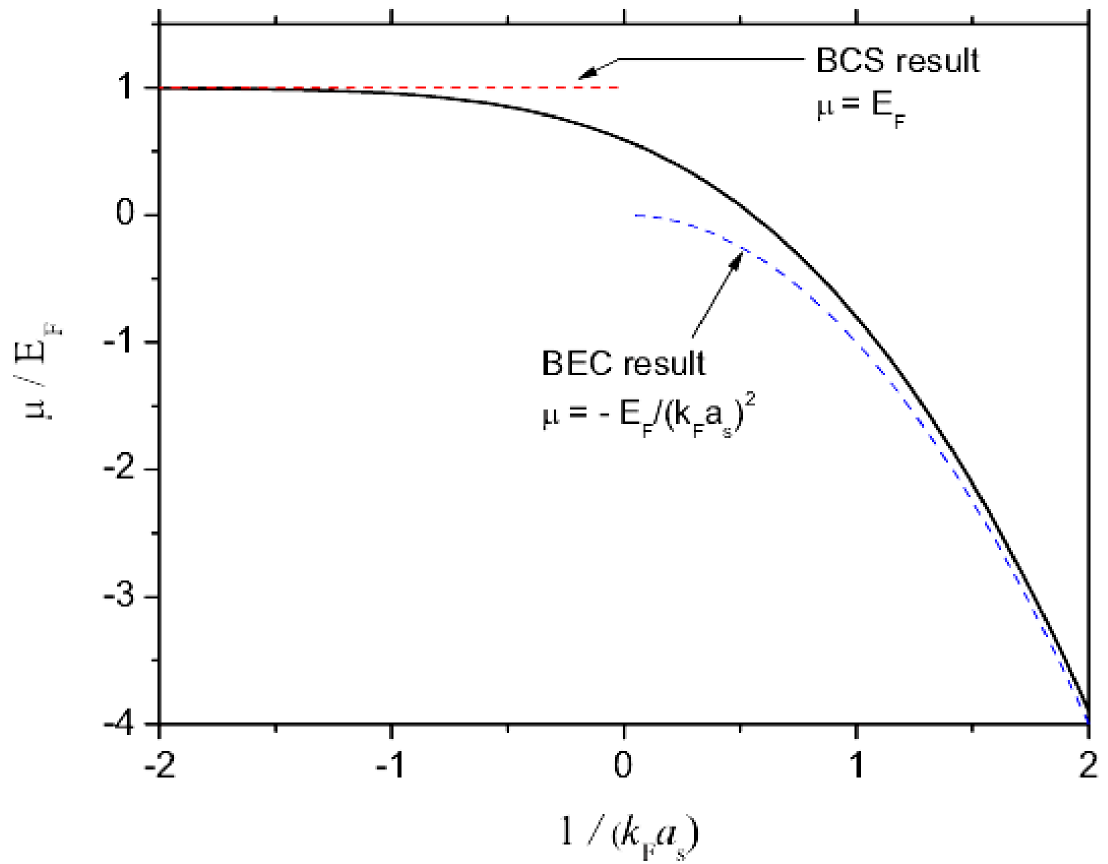


From the gap equation:

$$\frac{\partial \Omega_{\rm sp}}{\partial \Delta} = 0$$

C. A. R. Sa de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
J. R. Engelbrecht, M. Randeria, and C. A. R. Sa de Melo, Phys. Rev. B **55**, 15153 (1997).

The chemical potential (spin-balanced case):



From the gap equation:

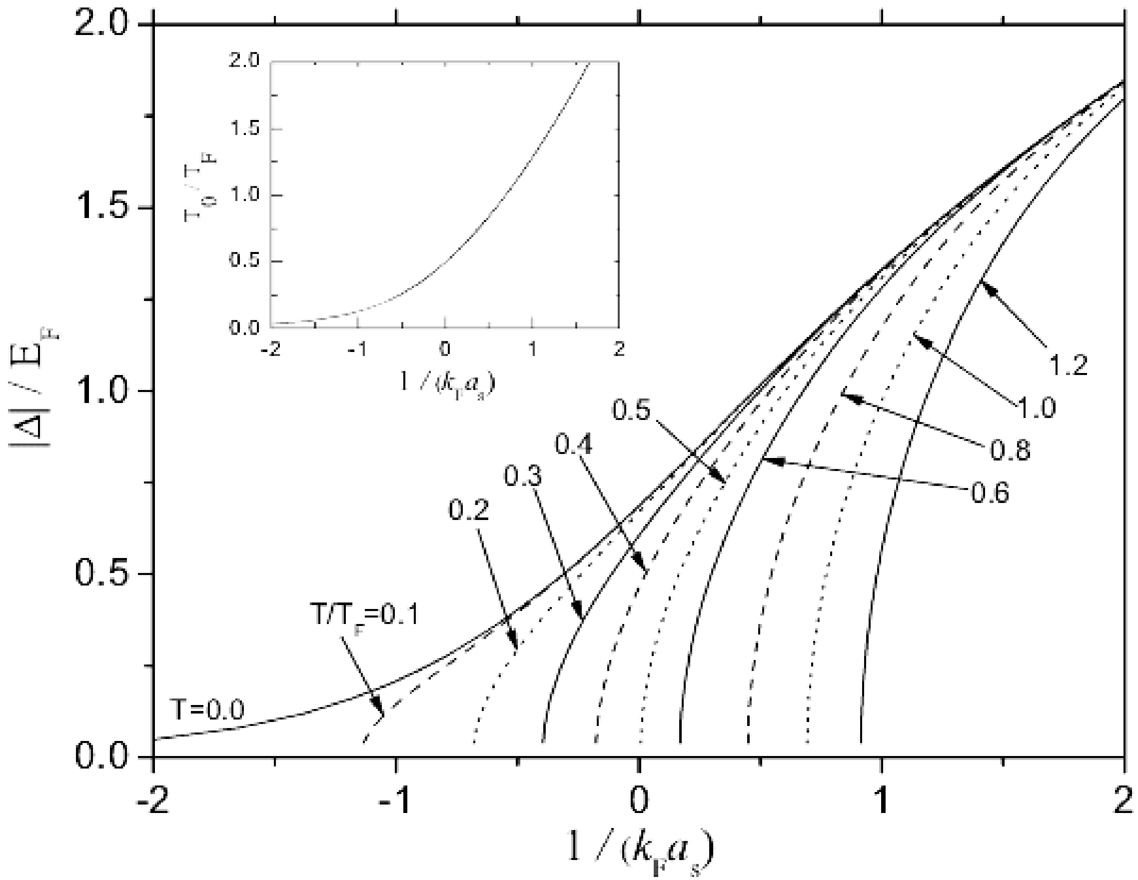
$$\frac{\partial \Omega_{\rm sp}}{\partial \Delta} = 0$$

and the number equation:

$$n_\sigma = -\frac{\partial \Omega_{\rm sp}}{\partial \mu_\sigma}$$

C. A. R. Sa de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
J. R. Engelbrecht, M. Randeria, and C. A. R. Sa de Melo, Phys. Rev. B **55**, 15153 (1997).

Temperature dependence of the pairing order parameter



From the gap equation:

$$\frac{\partial \Omega_{\rm sp}}{\partial \Delta} = 0$$

and the number equation:

$$n_\sigma = -\frac{\partial \Omega_{\rm sp}}{\partial \mu_\sigma}$$

The temperature at which the pairing is broken is not equal to the critical temperature for superfluidity! In the BEC region, pairs are robust and phase fluctuations destroy superfluidity.

Gaussian fluctuations

Expand the action up to quadratic order in the fluctuation field

$$\Delta_{\mathbf{x},\tau} = \Delta + \delta_{\mathbf{x},\tau} \qquad \delta_{\mathbf{x},\tau} = a_{\mathbf{x},\tau} e^{i\theta_{\mathbf{x},\tau}}$$

phase fluctuations ($\theta_{\mathbf{x},\tau}$)

amplitude fluctuations ($a_{\mathbf{x},\tau}$)

This yields the Gaussian fluctuation expansion

$$\int \mathcal{D}\Delta \ \exp\{-S[\Delta_{\mathbf{x},\tau}]\} \approx \exp\{-\mathcal{S}_{\text{sp}}[\Delta]\} \times \int \mathcal{D}\delta \ \exp\{-\mathcal{S}_{\text{fl}}[\delta_{\mathbf{x},\tau}]\}$$

This we obtained previously ($\mathcal{S}_{\text{sp}}$)

The fluctuation action ($\mathcal{S}_{\text{fl}}$)

The fluctuation action can be written as

$$\mathcal{S}_{\text{fl}}[\delta_{\mathbf{q},m}] = \int dq \begin{pmatrix} \theta_q & a_q \end{pmatrix} \cdot \underbrace{\begin{pmatrix} M_{++}(q) & -iM_{+-}(q) \\ -M_{+-}(q) & M_{--}(q) \end{pmatrix}} \cdot \begin{pmatrix} \theta_q \\ a_q \end{pmatrix}$$

The "fluctuation matrix" $\mathbb{M}(i\omega_n, \mathbf{q})$

Remember the one bosonic field integral we can do: $\int \mathcal{D}\phi \ \exp\left\{-\sum_{j,\ell} \phi_j^* A_{j,\ell} \phi_\ell\right\} = \frac{1}{\det(A)}$

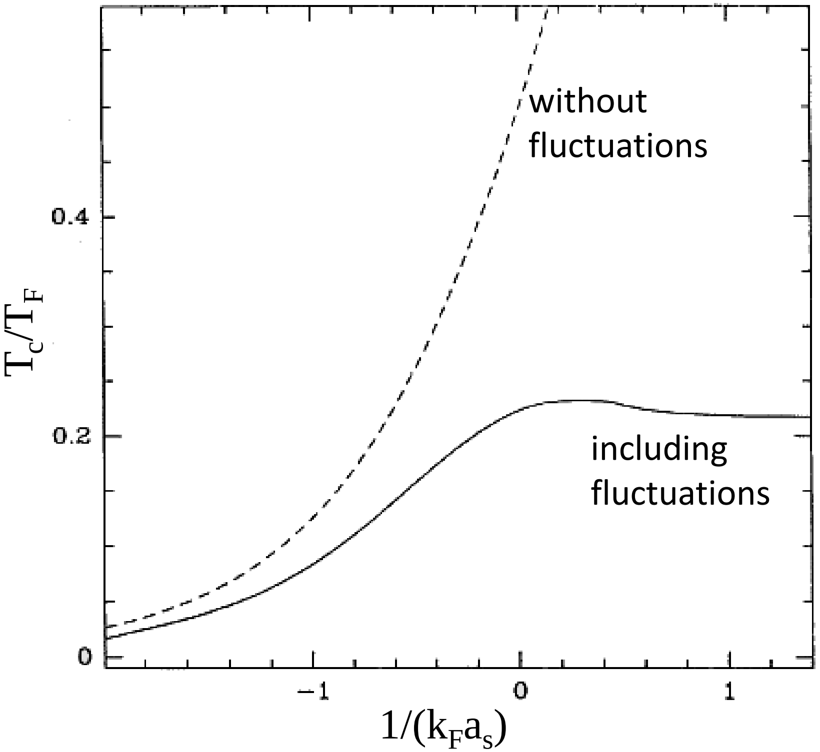
$$\Rightarrow \qquad \mathcal{Z}_{\text{fl}} = \int \mathcal{D}\delta \ \exp\{-\mathcal{S}_{\text{fl}}[\delta_{\mathbf{x},\tau}]\} = \prod_q \frac{1}{\det[\mathbb{M}(q)]}$$

The fluctuation contribution to the free energy

Solving the gap and number equations for $\Delta \to 0_+$ yields T_c :

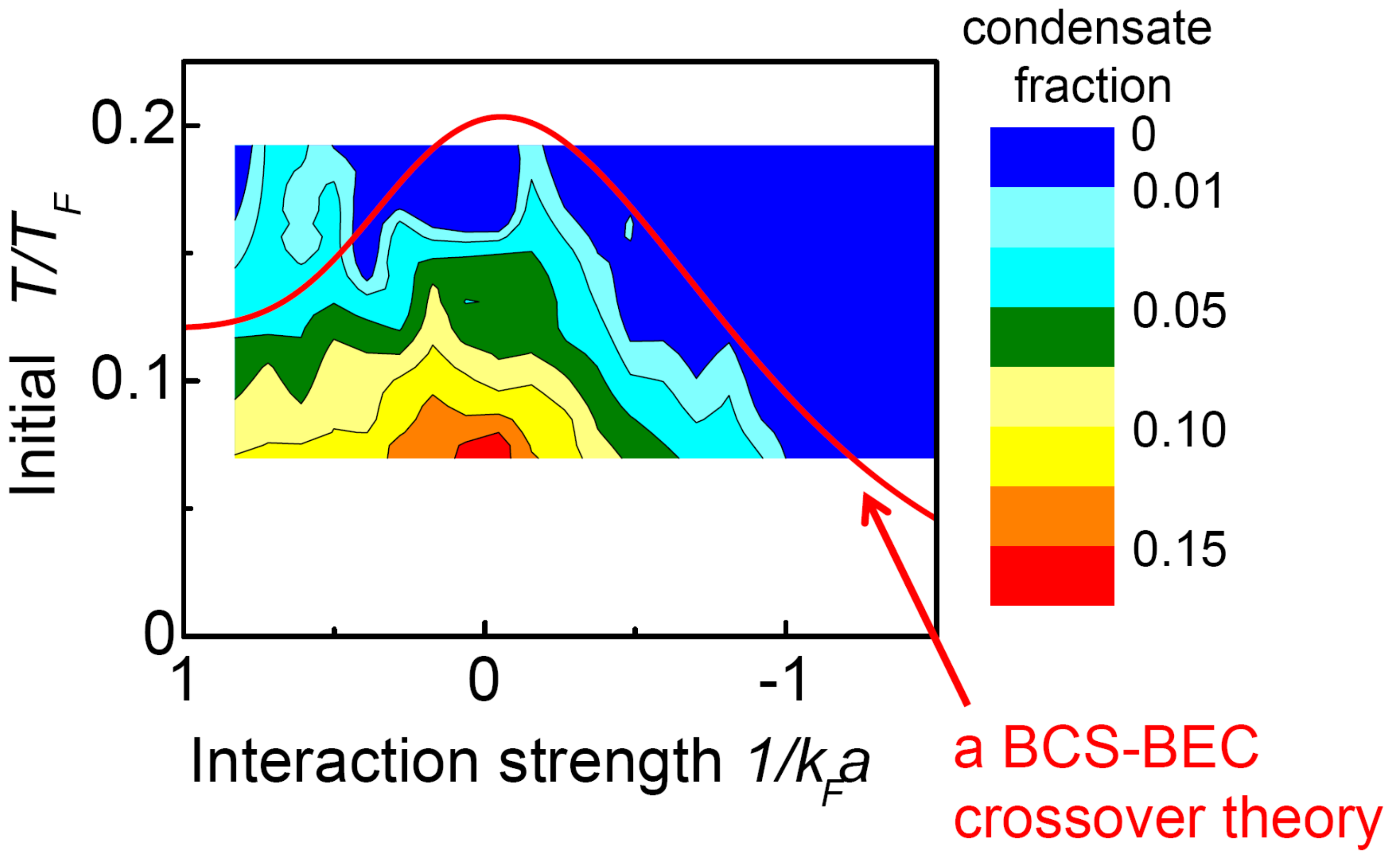
$$\frac{\partial \Omega_{\text{sp}}}{\partial \Delta} = 0$$

$$n_\sigma = -\frac{\partial \Omega_{\text{sp}}}{\partial \mu_\sigma} - \frac{\partial \Omega_{\text{fl}}}{\partial \mu_\sigma}$$



C. A. R. Sa de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
J. R. Engelbrecht, M. Randeria, and C. A. R. Sa de Melo, Phys. Rev. B **55**, 15153 (1997).

The critical temperature



C.A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004)

The fluctuation matrix

The action functional for the excitations of the fermion pairs:

$$\mathcal{S}_{\text{fl}}\left[\delta_{\mathbf{q},m}\right] = \int dq \begin{pmatrix} \theta_q & a_q \end{pmatrix} \cdot \underbrace{\begin{pmatrix} M_{++}(q) & -iM_{+-}(q) \\ -M_{+-}(q) & M_{--}(q) \end{pmatrix}} \cdot \begin{pmatrix} \theta_q \\ a_q \end{pmatrix}$$

The “fluctuation matrix” $\mathbb{M}(i\omega_n, \mathbf{q})$

- A tremendous amount of work has been done to fully understand the fluctuation matrix

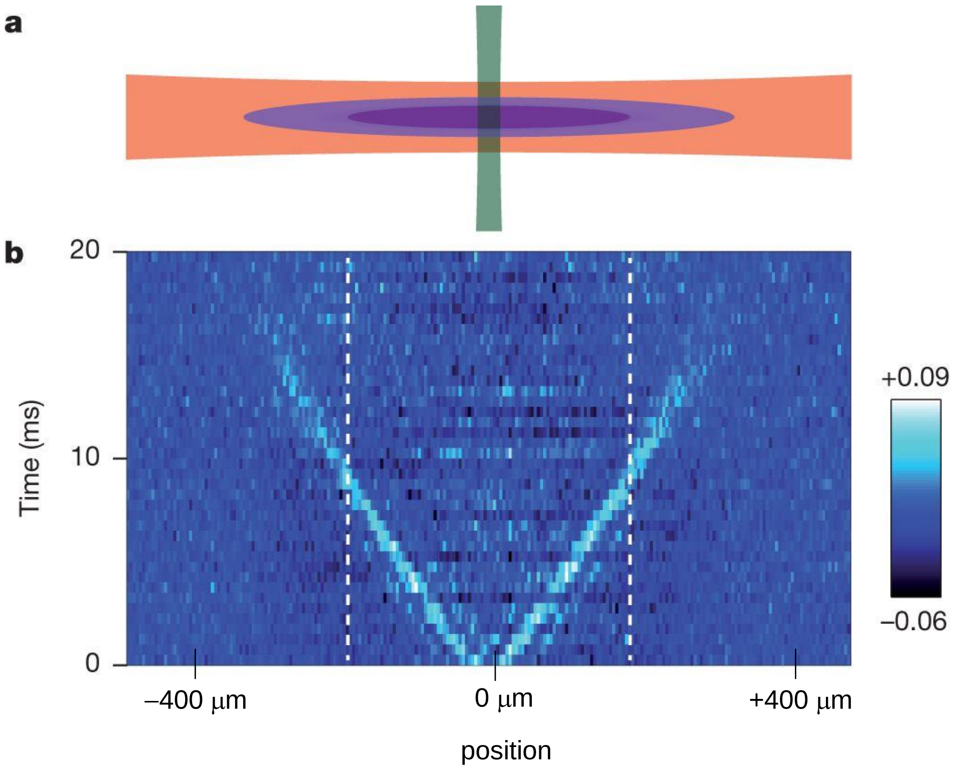
[1] J. Engelbrecht, M. Randeria, C.A.R. Sa de Melo, Phys. Rev. B **55**, 15153 (1997).
[2] Y. Ohashi and A. Griffin, Phys. Rev. A **67**, 063612 (2003).
[3] R. Combescot, M. Yu. Kagan, S. Stringari, Phys. Rev. A **74**, 042717 (2006).
[4] D.-S. Lee, C.-Y. Lin, and R. J. Rivers, Phys. Rev. Lett. **98**, 020603 (2007).
[5] G. Bighin, L. Salasnich, P. A. Marchetti, and F. Toigo, Phys. Rev. A **92**, 023638 (2015).
[6] D. Pekker and C.M. Varma, Annual Review of Condensed Matter Physics **6**, 269 (2015).
[7] H. Kurkjian, Y. Castin, and A. Sinatra, Phys. Rev. A **93**, 013623 (2016).
... and many more ... and even more for these modes in superconductors.

The fluctuation matrix

The action functional for the excitations of the fermion pairs:

$$\mathcal{S}_{\text{fl}}\left[\delta_{\mathbf{q},m}\right]=\int dq\,\begin{pmatrix}\theta_q & a_q\end{pmatrix}\cdot\begin{pmatrix}M_{++}(q) & -iM_{+-}(q)\\ -M_{+-}(q) & M_{--}(q)\end{pmatrix}\cdot\begin{pmatrix}\theta_q\\ a_q\end{pmatrix}$$

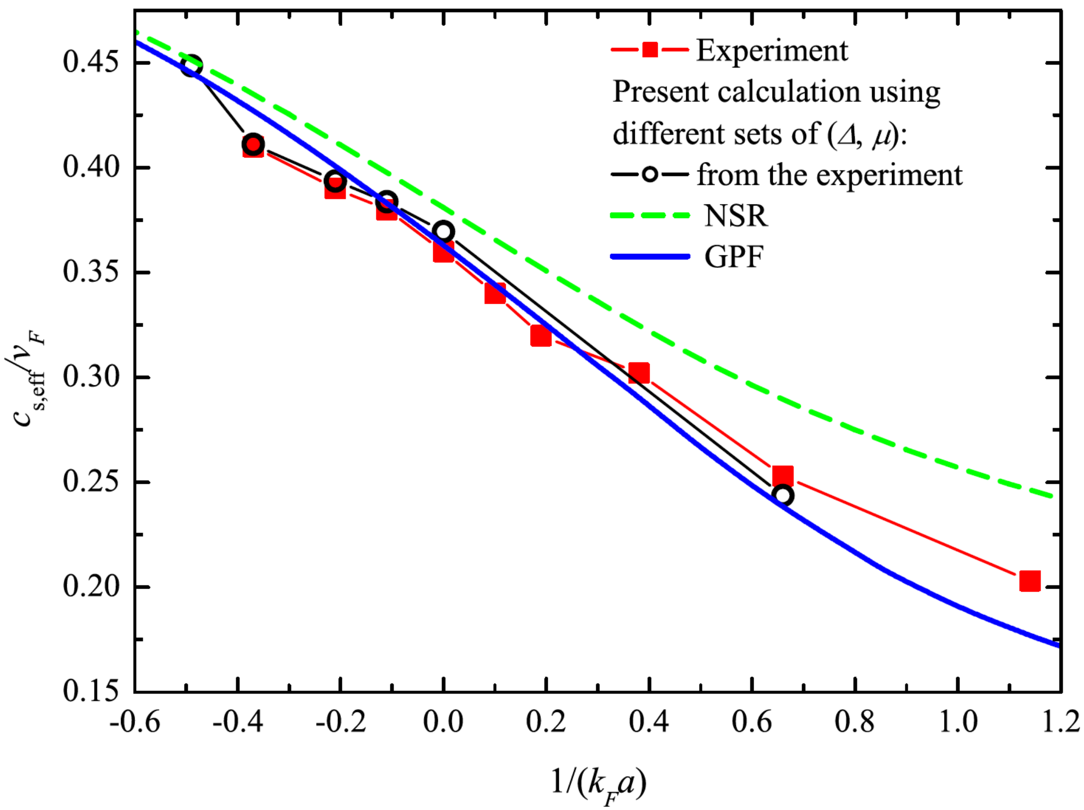
- The inverse fluctuation matrix $\mathbb{M}^{-1}$ can be interpreted as a propagator for coupled amplitude-phase modes, and its poles (the zeros of $\det(\mathbb{M})$) reveal the dispersion and lifetime of the bosonic excitations of the superfluid.



Sidorenkov et al., Nature **498**, 78–81(2013)

Example: sound velocity in a Fermi superfluid

- The poles of the propagator, or equivalently the zeroes of $\det(\mathbb{M})$, reveal the dispersion and lifetime of the bosonic excitations of the superfluid.



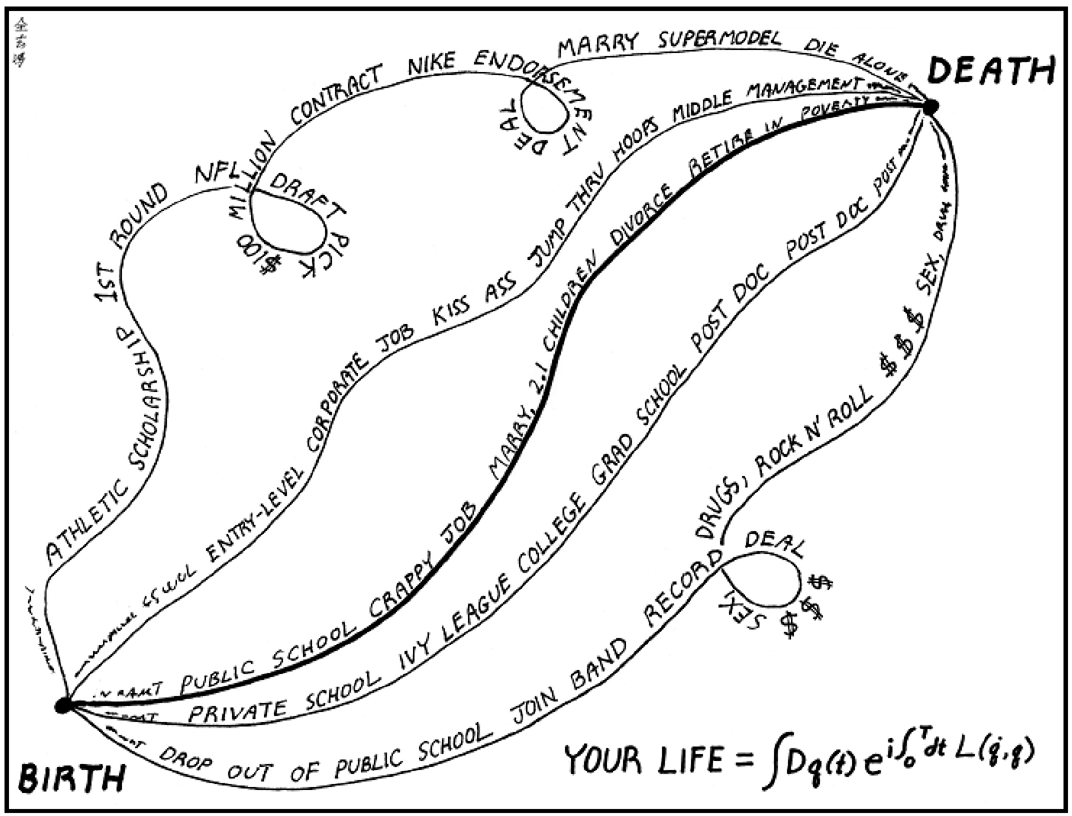
S.N. Klimin, H. Kurkjian, J. Tempere, Journ. Low. Temp. Phys. **196**, 102 (2019).

Conclusions

The path integral or functional integral technique is well suited to describe superfluid Fermi gases, at any temperature or any interaction strength.

Two tricks (and key steps) to keep in mind:

- the Hubbard-Stratonovic decomposition introduces the pair field,
- the fluctuation expansion around the classical ("pair condensate") field.



The Path Integral Formulation of Your Life

Title

Title

The saddle point value

The saddle-point contribution to the partition sum is

$$\mathcal{Z}_{\text{sp}} = e^{-\beta\Omega_{\text{sp}}(T,V,\mu_\sigma)} = \exp\left\{-\mathcal{S}_{\text{sp}}[\Delta]\right\}$$

with

$$\mathcal{S}_{\text{sp}}[\Delta] = \int dk \left\{ -\frac{|\Delta|^2}{g} - \text{Tr}_\sigma \left[\log \begin{pmatrix} -i\omega_n + k^2 + \mu_\uparrow & \Delta \\ \Delta & -i\omega_n - k^2 - \mu_\downarrow \end{pmatrix} \right] \right\}$$

$$\int dk = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \times \int d\mathbf{k}$$

$$\omega_n = (2n+1)\pi/\beta$$

The fluctuation contribution to the free energy

$$\int \mathcal{D}\Delta \; \exp\{-S[\Delta_{\mathbf{x},\tau}]\} \approx \exp\{-\mathcal{S}_{\text{sp}}[\Delta]\} \times \int \mathcal{D}\delta \; \exp\{-\mathcal{S}_{\text{fl}}[\delta_{\mathbf{x},\tau}]\}$$

Still the same gap equation:

$$\frac{\partial \Omega_{\text{sp}}}{\partial \Delta} = 0$$

$$\mathcal{Z}_{\text{fl}} = \int \mathcal{D}\delta \; \exp\{-\mathcal{S}_{\text{fl}}[\delta_{\mathbf{x},\tau}]\} = \prod_q \frac{1}{\det[\mathbb{M}(q)]}$$

$$e^{-\beta\Omega_{\text{fl}}} = \exp\{-\operatorname{Tr}[\log(\mathbb{M})]\}$$

But now we have different number equations:

$$n_\sigma = -\frac{\partial \Omega_{\text{sp}}}{\partial \mu_\sigma} - \frac{\partial \Omega_{\text{fl}}}{\partial \mu_\sigma}$$

(Nozières & Schmitt-Rink or NSR)

$$n_\sigma = -\frac{\partial \Omega_{\text{sp}}}{\partial \mu_\sigma} - \frac{\partial \Omega_{\text{fl}}}{\partial \mu_\sigma} - \frac{\partial \Omega_{\text{fl}}}{\partial \Delta}\frac{\partial \Delta}{\partial \mu_\sigma}$$

(Hu, Liu & Drummond or GPF)

The fluctuation matrix

$$\mathbb{M} = \begin{pmatrix} M_{++} & -iM_{+-} \\ iM_{-+} & M_{--} \end{pmatrix}$$

The matrix elements are given by:

$$M_{\pm\pm}(i\omega_n, \mathbf{q}) = \sum_{\mathbf{k}} \frac{X(E_+)+X(E_-)}{8E_+E_-}\left(E_+E_- + \xi_+\xi_- \pm \Delta^2\right)\left(\frac{1}{i\omega_n - (E_+ + E_-)} - \frac{1}{i\omega_n + (E_+ + E_-)}\right) + \sum_{\mathbf{k}} \frac{X(E_{\mathbf{k}})}{2E_{\mathbf{k}}}$$
$$+ \sum_{\mathbf{k}} \frac{X(E_+)-X(E_-)}{8E_+E_-}\left(E_+E_- - \xi_-\xi_+ \pm \Delta^2\right)\left(\frac{1}{i\omega_n - (E_+ - E_-)} - \frac{1}{i\omega_n + (E_+ - E_-)}\right),$$

$$M_{+-}(i\omega_n, \mathbf{q}) = \sum_{\mathbf{k}} \frac{X(E_+)+X(E_-)}{8E_+E_-}\left(E_-\xi_+ + E_+\xi_-\right)\left(\frac{1}{i\omega_n - (E_+ + E_-)} + \frac{1}{i\omega_n + (E_+ + E_-)}\right)$$
$$+ \sum_{\mathbf{k}} \frac{X(E_+)-X(E_-)}{8E_+E_-}\left(E_-\xi_+ - E_+\xi_-\right)\left(\frac{1}{i\omega_n - (E_+ - E_-)} + \frac{1}{i\omega_n + (E_+ - E_-)}\right)$$

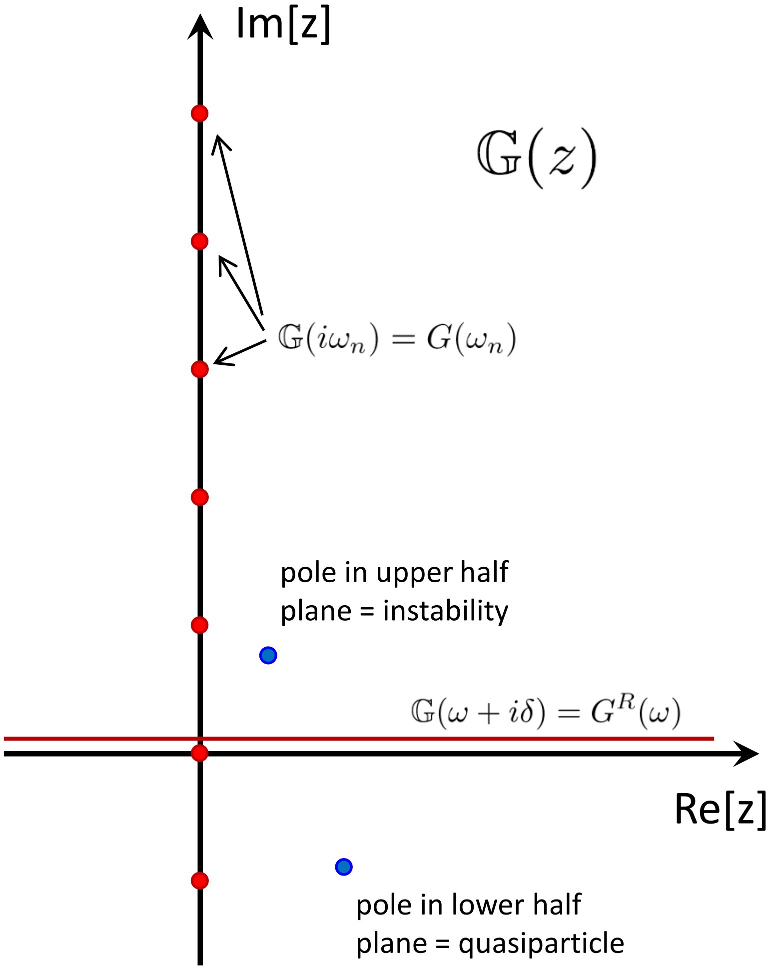
In these expressions, $X(E) = \dfrac{\sinh(\beta E)}{\cosh(\beta E) + \cosh(\beta\zeta)}$ and

$$E_\pm = E_{\mathbf{k}\pm\mathbf{q}/2} = \sqrt{\xi^2_{\mathbf{k}\pm\mathbf{q}/2} + \Delta^2}$$
$$\xi_\pm = \xi_{\mathbf{k}\pm\mathbf{q}/2} = (\mathbf{k} \pm \mathbf{q}/2)^2 - \mu$$

- The fluctuation matrix only depends on Δ, temperature and chemical potentials. These are input parameters, and can be taken from QMC, experimental EOS, NSR theory, …
- The Matsubara frequencies appear in the denominators of integrands, of the form

$F(i\omega_n) = \int d\nu \dfrac{f(\nu)}{i\omega_n - \nu}$ $\rightarrow$ simply replacing $i\omega_n$ by z gives a branch cut on the real axis.

Propagators in the complex frequency plane ($\mathbb{M}(z, \mathbf{q})$ for fixed $\mathbf{q}$)



Experiments measure response functions, given in Kubo's formalism by

$$G^R(t) = -i \left\langle \left[\left\{ \hat{a}(t), \hat{a}^\dagger(0) \right\} \right] \right\rangle \theta(t)$$

$$\rightarrow G^R(\omega) = \int dt \; \mathrm{e}^{\mathrm{i}\omega t} G^R(t)$$

Statistical Field Theory calculates thermal Green's functions

$$G(\tau) = -i \left\langle \mathcal{T} \left[a(\tau) a^\dagger(0) \right] \right\rangle$$

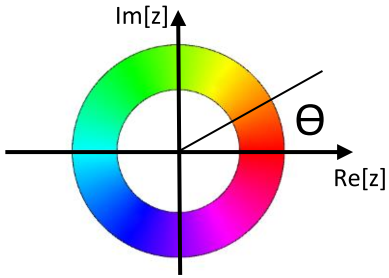
$$\rightarrow G(\omega_n) = \int_0^\beta d\tau \; \mathrm{e}^{\mathrm{i}\omega_n \tau} G(\tau)$$

There exists a unique analytical continuation $\mathbb{G}(z)$ linking them.

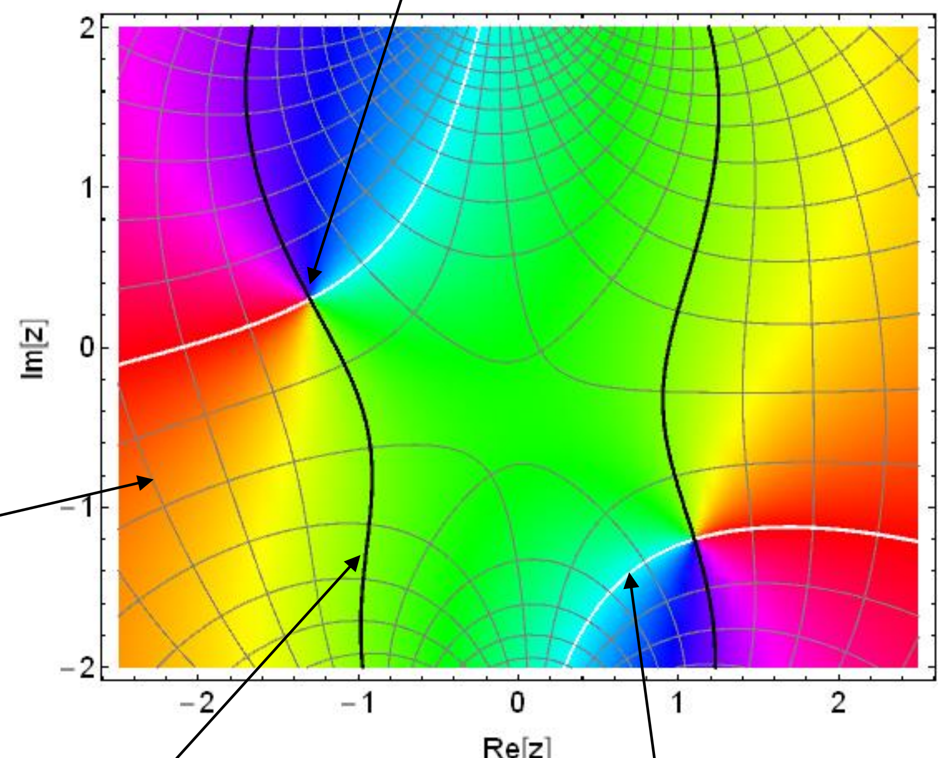
Analytic continuation of the fluctuation matrix

Here we use the following illustration conventions for complex functions:

The phase is shown
on a color circle



a node of the function (f[z]=0)



grey curves: contour lines
for real and imaginary part

thick black curve
Re[f(z)] is zero

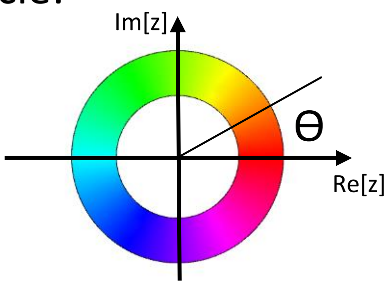
white curve
Im[f(z)] is zero

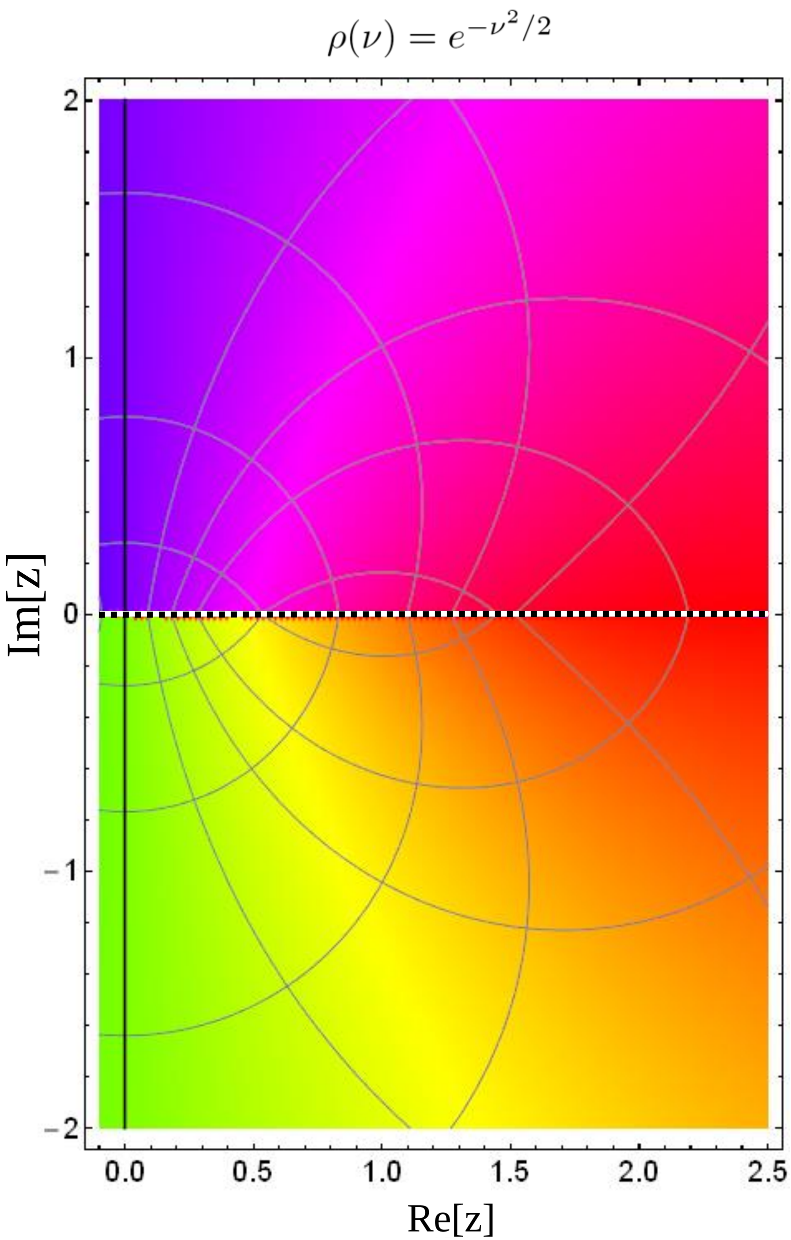
Analytic continuation of the fluctuation matrix

The matrix elements contain terms of the following form:

$$F(z) = \int_{-\infty}^{+\infty} \frac{\rho(\nu)}{z-\nu} d\nu$$

The phase is shown on a color circle:





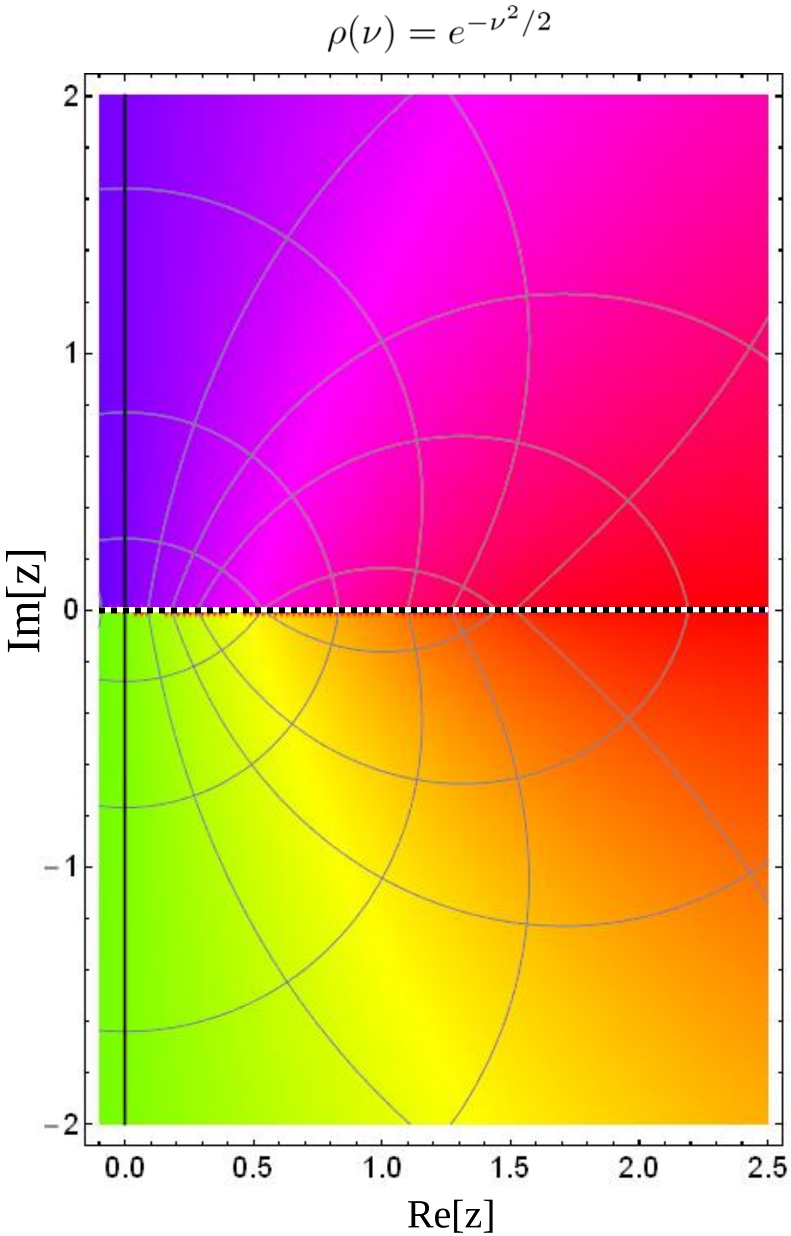
Analytic continuation of the fluctuation matrix

The matrix elements contain terms of the following form:

$$F(z) = \int_{-\infty}^{+\infty} \frac{\rho(\nu)}{z-\nu} d\nu$$

This gives a branch cut at the real axis:

$$\begin{aligned} & F(x+i\varepsilon) - F(x+i\varepsilon) \\ &= \int_{-\infty}^{+\infty} \frac{\rho(\nu)}{x-\nu+i\varepsilon} d\nu - \int_{-\infty}^{+\infty} \frac{\rho(\nu)}{x-\nu-i\varepsilon} d\nu \\ &= 2\pi i \rho(x) \end{aligned}$$



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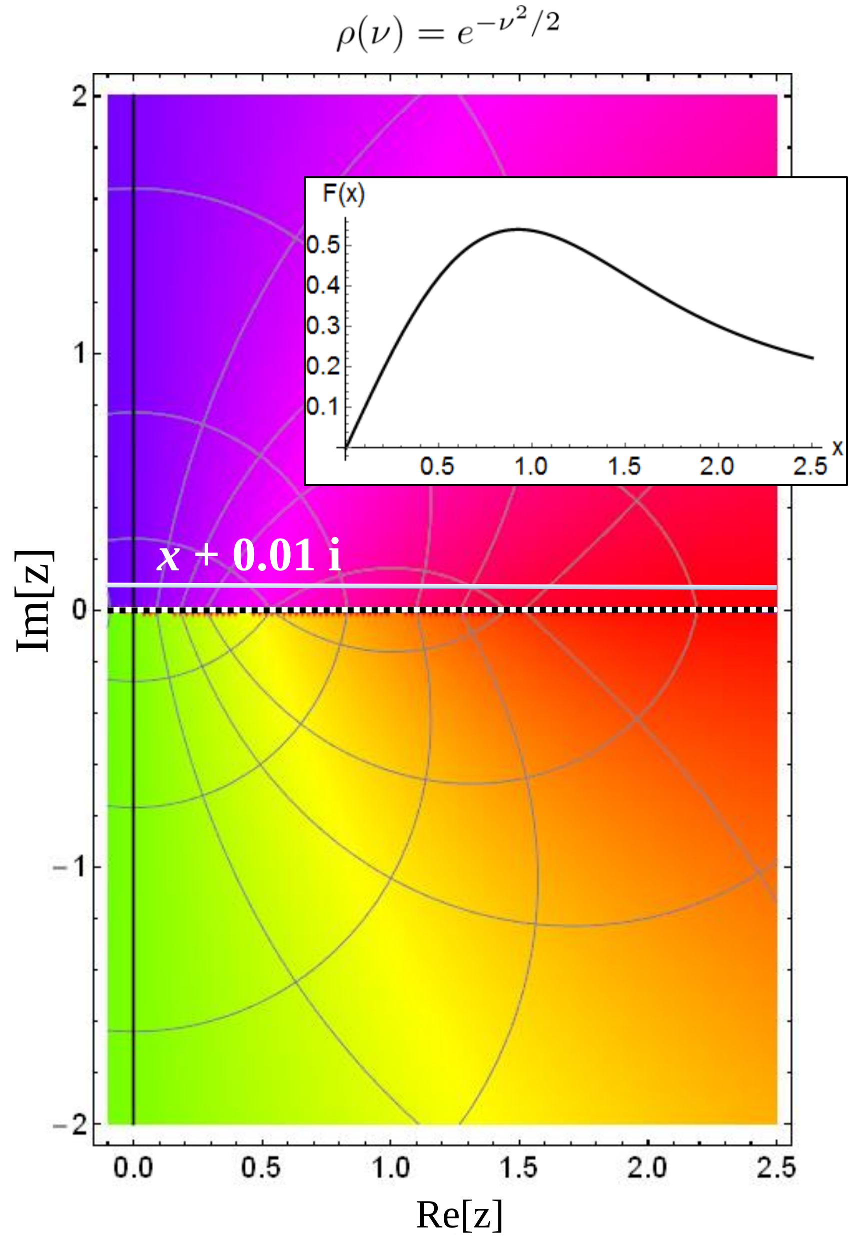

Analytic continuation of the fluctuation matrix

The matrix elements contain terms of the following form:

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Nozières' prescription for the analytic continuation through the branch cut is :

$$\mathrm{Im}\,[z] \quad > \quad 0 : F(z) = \int_{-\infty}^{+\infty} \frac{\rho(\nu)}{z-\nu} d\nu$$

$$\mathrm{Im}\,[z] \quad \leqslant \quad 0 : F(z) = \int_{-\infty}^{+\infty} \frac{\rho(\nu)}{z-\nu} d\nu + 2\pi i \rho(z)$$

$\rho(\nu) = e^{-\nu^2/2}$