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Topological aspects of generalized Sklyanin algebras

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Plan

1. Sklyanin algebras (SA) and generalized Sklyanin algebras (GSA)
2. Detour: Topology of intersections of real quadrics
3. Topology of Casimir levels of Sklyanin algebra
4. Jacobian Poisson Structures (JPS)
5. Compatibility problem for GSA and JPS
6. Poisson homology of JPS and singularity theory

(Cf.: G.Kh. “On one class of exact Poisson structures”, Proceedings of A.Razmadze Mathematical Institute, Vol. 119, 1999. //Abstract. We introduce a class of Poisson structures defined by a simple explicit formula and establish basic properties of these structures. It is shown that this construction gives non-trivial examples of exact Poisson structures on affine algebraic varieties. For such a structure on hypersurface in 3d, the zero-dimensional Poisson homology is computed in terms of the Milnor number of Casimir polynomials. Some generalizations and open problems are also discussed)

Two papers on relations between GSA and JPS

- R.Prszybysz, On one class of exact Poisson structures, J. Math. Phys. 42, 2001.

Abstract. We discuss some properties of a natural class of Poisson structures on Euclidean spaces and abstract manifolds. In particular, it is proved that such structures may be reconstructed from their Casimir functions. The dimension of 0^{th} Poisson homology of these structures is computed in terms of the Milnor number of their Casimir functions. We also analyze some concrete examples of such structures in low dimensions.

- G.Kh., R.Prszybysz, On Certain Super-Integrable Hamiltonian Systems, J. Dyn. Contr. Syst. 8, 2002.

Abstract. We establish a natural connection between maximally super-integrable Hamiltonian systems with polynomial first integrals and algebraic Poisson structures of rank two. Basic properties of the both classes of objects are presented, with a special emphasis on related topological aspects. It is shown that the signature technique for computing topological invariants enables one to compute effectively the Euler characteristics of Casimir levels of arising Poisson structures and determine the combinatorial structure of the Liouville foliation. The crucial points of our approach are illustrated in the case of Sklyanin algebras, in particular, we present a complete computation of their Euler graphs and invariant polynomials.

Ad hoc comment: The mentioned connection between JPS and maximally super-integrable systems relies on a classical theorem on a global last multiplier proved by Ch.-J. de la Vallée Poussin (à propos: le premier président élu de la toute nouvelle Union mathématique internationale). - «Nothing new under the sun»

Some further papers on GSA and JPS
(caveat: “Post hoc non ergo propter hoc”)

- A.Odesskii, V.Rubtsov, Polynomial Poisson algebras with regular structure of symplectic leaves, arXiv:math/0110032v1, 2001.
- Ph.Monnier, Poisson cohomology in dimension two, Israel J. Math. 129, 2002.
- I. Cruz, H. Mena-Matos , Normal Forms for Two Classes of Exact Poisson Structures in Dimension Four, J. Lond. Math. Soc. 73, 2006.
- A.Pichereau, Poisson (co)homology and isolated singularities, J. Algebra 299, 2006.
- S.Pelap, Poisson (co)homology of polynomial Poisson algebras in dimension four: Sklyanin’s case, J. Algebra 322, 2009.
- P.Damianou, F.Petalidou, Poisson brackets with prescribed Casimirs, Can. J. Math. 64, 2012.
- J.Ahn, S.Oh, S.Park, Poisson brackets determined by Jacobians, J. Chungcheong Math. Society 26, 2013.
- P. Damianou, Poisson brackets after Jacobi and Plücker, arXiv, 2019.

From Sklyanin algebras to n-Sklyanin algebras

- "In principio erat" **Sklyanin Algebra** (**SA**) in 4d (Funkts. Anal. Pril. 16, no.4, 1982):

(1) $\{x_i, x_0\} = J_{jk}x_j x_k$, $\{x_j, x_k\} = -2x_0 x_i$; i,j,k ∈ {1, 2, 3} (cyclic), $J_{jk} = J_j - J_k$ ∈ K (= **R, C**)

Casimirs: $K_0 = \sum_{i=1}^{3} x_i^2$, $K_1 = x_0^2 + \sum_{i=1}^{3} J_i x_i^2$; (quadratic generators of the center)

- Next: **generalized Sklyanin algebra (GSA)** in 4d (Vanhaecke ca.1999):

(2) $\{x_i, x_j\} = c_{km}x_k x_m$, i,j,k,m ∈ {1, 2, 3, 4}, c_{km} ∈ K, $c_{12}c_{34} - c_{13}\, c_{24} + c_{14}c_{23} = 0$ (Plücker rel.)

Casimirs: $f_1 = c_{34}x_2^2 - c_{24}x_3^2 + c_{23}x_4^2$, $f_2 = c_{23}x_1^2 - c_{13}x_2^2 + c_{12}x_3^2$; (generators of the center)

Pecularities: quadratic brackets, two quadratic Casimirs, 2d Casimir levels (rank = 2)

- Further generalization: **n-Sklyanin algebra** (Panov 2002, Damianou 2006):

(3) $\{x_i, x_j\} = c_{ij}x_1 \ldots \widehat{x_i} \ldots \widehat{x_j} \ldots x_n$, c_{ij} ∈ K (= **R, C**); $c_{ij}c_{kl} - c_{ik}\, c_{jl} + c_{jk}c_{il} = 0$, 1≤i<j<k<l≤n

Casimirs: $f_{ijk} = c_{jk}x_i^2 - c_{ik}x_j^2 + c_{ij}x_k^2$; (quadratic Casimirs, 2d Casimir levels)

Two important observations (G.Kh., 1998): (1) topology of Casimir levels of GSA can be effectively studied using the so-called **signature formulae for topological invariants** (see below)

(2) all the above brackets belong to the class of **Jacobian Poisson Structures** defined below.

Sklyanin algebra (SA) in more detail

- **SA** is a quadratic Poisson algebra S(J) on $\mathbf{K}^4$ (K= **R**, **C**) with coordinates $x_0, \dots, x_3$ defined by the relations:
- $\{x_1, x_0\} = J_{23}x_2x_3$, $\{x_2, x_0\} = J_{31}x_1x_3$, $\{x_3, x_0\} = J_{12}x_2x_3$, $\{x_1, x_2\} =$ -2x_0x_3, $\{x_2, x_3\} =$ -2x_0x_4, $\{x_3, x_1\} =$ -2x_0x_2 , J_{ij} = J_i - J_j, J_i ∈ **K** (=**R**, **C**), i =1, 2, 3.
- It has two quadratic Casimirs $K_0 = \sum_{i=1}^{3} x_i^2$, $K_1 = x_0^2 + \sum_{i=1}^{3} J_i x_i^2$, which generate the whole center of S(J)
- This is a three-dimensional family of exact affine Poisson structures
- According to Sklyanin, the knowledge of topology of Casimir levels of SA was helpful for constructing representations of the quantized algebra. This suggested that it made sense to investigate the topology of quadratic mapping defined by Casimirs of GSA, which was a very well studied topic of nonlinear analysis
- **NB**: in GSA case it suffices to compute the Euler characteristics of Casimir levels

Detour: **Topology of intersections of real quadrics**

- (Co)homology of “polyquadrics” (intersections of real quadrics): d'après C.T.Wall, A.Agrachev-R.Gamkrelidze, J.Lopez de Medrano, A.Lerario, V.Krasnov (specifically, “Agrachev spectral sequence”, cf. A.Agrachev, “Quadratic cohomology” in Arnold Math. J. v.1, no.1, 2013)
- Criteria for properness, surjectivity and convexity of images of real quadratic mappings (mostly classical or folklore)
- Topological classification of low-dimensional biquadrics and triquadrics (J.Lopez de Medrano, V.Krasnov), topology of fibers in the case of distance-squared mappings in higher dimensions (S.Ichiki, T.Nishimura)
- **NB:** For GSA, the signature formula for the Euler characteristic given below yields detailed results on the topology of Casimir levels (also for moduli spaces of planar pentagons and spider-linkages)

Signature formula for the Euler characteristic

- Let $f_1, \dots, f_m$ be real polynomials in n variables of algebraic degree d such that the algebraic variety X = $\{f_i$ = 0, i=1,…,m} is compact. The Euler characteristic $\chi(X)$ can be computed as follows.
- Put h_i:= $({x_o}^{d+1})\, f_i(x_1/x_0,\dots, x_n/x_0)$, F := $\sum_{i=1}^{m} h_i^2 - \sum_{i=0}^{n} x_i^{2d+4}$.
- **Theorem.** Polynomial F has an isolated critical point at the origin and

$2\chi(X)$ = $(-1)^n$ - deg_0 *grad* F = $(-1)^n$ – *sig* Q_F, where Q_F is the Gorenstein quadratic form on the Milnor algebra of F and *sig* denotes its signature.

Details can be found in my monograph “Signature formulae for topological invariants” (Proc. A.Razmadze Math. Institute v. 125, 2001). For SA and GSA we have n=4, m=2, d=2, and the existing computer programs for the mapping degree easily yield the values of Euler characteristics of all Casimir levels.

Topology of Casimir levels of SA

- A simple example of application of the known results on intersections of real quadrics is concerned with the topology of Casimir levels of the original SA
- Consider the quadratic mapping Q: $\boldsymbol{R^4} \rightarrow \boldsymbol{R^2}$, defined by the above two Casimirs of SA. The discriminant is easily seen to consist of three parallel lines, the Euler characteristic of non-singular fiber over each chamber (component of the complement to discriminant) computed by the mentioned signature formula attains one of the three values: 0, 2, 4. It follows that the non-singular fibers are diffeomorphic either to a disjoint union of two spheres or to two-torus, while the bifurcation between these two types consists in pinching two meridians of a torus, which gives ”two bananas with common tips”. It seems remarkable that such a fairly simple “shift of paradigm” yielded a correction of an error in the original paper of Sklyanin (see a fragment of that paper in the next two slides).

Теорема 1. *Центр* $Z_{\mathcal{P}^*}$ *алгебры* $\mathcal{P}^*$ *состоит из функций от двух квадратичных центральных функций* K_0 *и* K_1:

$$K_0 = \sum_{\alpha=1}^{3} S_\alpha^2, \quad K_1 = S_0^2 + \sum_{\alpha=1}^{3} J_\alpha S_\alpha^2 \tag{21}$$

(заметим, что при сдвиге всех $J_\alpha \mapsto J_\alpha + c$ *функция* K_1 *заменяется на* $K_1 + cK_0$*).*

Доказательство. Пусть $f \in Z_{\mathcal{P}^*}$. Это означает, что $\{f, S_a\} = 0$, или, в силу (13),

$$\begin{aligned} \{f, S_0\} &= \sum_{\alpha=1}^{3} \frac{\partial f}{\partial S_\alpha} J_{\beta\gamma} S_\beta S_\gamma = 0, \\ \{f, S_\alpha\} &= -\frac{\partial f}{\partial S_0} J_{\beta\gamma} S_\beta S_\gamma + \frac{\partial f}{\partial S_\beta} S_0 S_\gamma - \frac{\partial f}{\partial S_\gamma} S_0 S_\beta = 0. \end{aligned} \tag{22}$$

Общее решение системы однородных линейных уравнений (22) имеет вид (при условии (8)) $df = f_0 dK_0 + f_1 dK_1$, откуда немедленно вытекает утверждение теоремы.

Задача. Доказать, что центр $Z_{\mathcal{P}}$ алгебры $\mathcal{P}$ состоит из *полиномов* от K_0 и K_1.

Следствие. *Пусть* $\Gamma(K_0, K_1)$ *— двумерное алгебраическое многообразие, задаваемое уравнениями* (21) *при фиксированных значениях* K_0, K_1. *Тогда АСП* $\mathcal{P}_\Gamma$, *определяемая как сужение алгебры* $\mathcal{P}$ *на* Γ *невырождена в смысле определения* 2.

Топология многообразия Γ существенно зависит от выбора параметров K_0, K_1. Заметим, что для выбранной нами параметризации (6) справедливы неравенства $J_1 \geqslant J_2 \geqslant J_3$, следовательно S_0^2 лежит в отрезке $[K_1 - K_0 J_1, K_1 - K_0 J_3]$. Возможны 3 случая:

а) $K_1 \in (K_0 J_1, \infty)$, б) $K_1 \in (K_0 J_3, K_0 J_1)$, в) $K_1 \in (-\infty, K_0 J_3)$.

В случае а) многообразие Γ гомеоморфно сфере $\mathbb{S}^2$, в случае б) Γ гомеоморфно несвязному объединению двух сфер (которые при $J_2 = J_3$ сливаются в тор) и, наконец, в случае в) вещественных решений нет.

Comments to the previous two slides

- Unfortunately, I could not get the English translation of this paper but there seem to be many colleagues in the auditory who can hopefully confirm that the previous slide, in the sixth line from above, contains an erroneous statement that a regular fiber can be homeomorphic to the two-sphere, which is not the case as was shown in a joint paper with R.Przybysz in J. Dyn. Control Syst. v.8, 2002. In fact, we gave there a topological classification of Casimir fibers for the whole five-dimensional family of generalized Sklyanin algebras given by formulas (2) above.
- For GSA of the form (2) these results can also be derived from the classification of real biquadrics given by L. de Medrano. However, to the best of my knowledge, such a classification is absent for intersections of more than three real quadrics. So for n-Sklyanin algebras and general JPS one has to use other approaches, in particular, the aforementioned signature formulae for the mapping degree and Euler characteristic developed in the papers of D.Eisenbud with H.Levine, G.Kh., J.Bruce, Z.Szafraniec and N.Dutertre (cf. my monograph indicated above)

Jacobian Poisson structures

- Consider a simple formula defining the so-called **Jacobian Poisson brackets** (aka **Nambu-Poisson brackets**):

(*) $\{f, g\} = J(f, g, h_1, \dots, h_{n-2})$,

where f, g, h_j are assumed to be polynomials in n indeterminates over the field of real or complex numbers (h_j are fixed) and J stands for the usual Jacobian (one can extend this formula to convergent or formal power series, rational functions, smooth functions, manifolds with given volume form, quaternion coefficients etc.).

The fact that such brackets satisfy Jacobi identity was known but (kind of) unattended. In particular, it was mentioned and proved in several papers on the Nambu-Poisson brackets (Nambu 1973, Grabowski-Marmo-Perelomov 1993, Takhtajan 1994, Marmo-Vilasi-Vinogradov 1998), and n-ary Lie algebras (Filippov 1985, Panov 1996, 2002). In this way we obtain a class of Poisson structures on Euclidean spaces nowadays usually referred to as **Jacobian Poisson structures.** Polynomials h_j are Casimirs of the above brackets and may be considered as "Hamiltonians to be" in the Nambu approach to integrable systems.

My encounter with Sklyanin algebras

- My encounter with these concepts and results was caused by the necessity to suggest a topic for a Ph.D. thesis in mathematical physics, which was a “conditio sine qua non” for getting a temporary position at the University of Lodz in 1999. Having some background in real algebraic geometry and singularity theory I was attracted by the title of a book of Pol Vanhaecke “Integrable systems in the realm of algebraic geometry” (LNM, 1996). After browsing it I realized that SA, GSA and n-Sklyanin algebras can be defined by the Jacobian formula (*) and, moreover, some relevant topological results on Sklyanin algebras and certain other JPS can be obtained using the signature formula presented above. A simple instance of such kind is described in slides 9-12. Several further problems which naturally arised in the context of JPS have been studied in a series of papers of several authors some of which are mentioned in slides 2, 3.

Further problems on GSA and JPS

There are several further problems in the context of JPS of which, in this talk, I will only touch upon the two largely unexplored ones:

- Compatibility problem for GSA and JPS;
- Poisson (co)homology of JPS and induced Poisson structures on the complete intersections with isolated singularities.

Compatibility criterion for two JPS

- As was proved in [Kh.-Prz. 2002] by direct computation, compatibility of two GSA of the form (2) in 4d is equivalent to the relation

  $c_{12}c'_{34}$ - $c_{13}c'_{24}$ + $c_{14}c'_{23}$ + $c_{23}c'_{14}$ - $c_{24}c'_{13}$ + $c_{34}c'_{12}$ = 0.

As was shown in a 2019 preprint of P.Damianou, this condition has a nice interpretation in terms of the Plücker embedding, which suggested several perspectives and open problems formulated in loc. cit.

An appropriate set of analogous conditions yields a criterion of compatibility of two n-Sklyanin brackets (Ph.D. Thesis of R.Przybysz, Lodz University, 2002).

Up to my updated knowledge, no explicit criteria of compatibility have been given for two JPS in n dimensions for n>4. For n=4, a sufficient condition of compatibility was given by R.Prszybysz in terms of the functional dependence of the four Casimirs in question.

Poisson (co)homology of JPS in qh case

Notice that each JPS yields a Poisson structure on the common zero-set Z of its Casimirs. So there arise two problems concerned with GSA and JPS:

(1) compute Poisson (co)homology of JPS on the underlying Euclidean space;

(2) compute Poisson (co)homology of the induced JPS on Z.

The second problem has an easy and instructive answer in 3d if the Casimir h is quasihomogenous (qh). From definitions follows that PH_o is isomorphic to the factor of the local algebra over the ideal {A, A} + (h). For a qh polynomial h, this ideal is known to coincide with the ideal (∇h) generated by the partial derivatives of F. The dimension of the factor over (∇h) is equal to the Milnor number of h and can be easily computed through the qh weights and qh degree of h. Below is given a page from [G.Kh., 1999] containing this result.

As is well-known, for a commutative algebra A with the Poisson structure τ_P defined by restricting to X of Poisson bracket $\{\cdot,\cdot\}_P$ on C^3 in the above way, one may consider its Poisson homology $HP_*(A,\tau_P)$ [2] which is an important invariant in Poisson geometry. We will only consider the zero-dimensional homology HP_0 which is by definition the factor-algebra $A/\{A,A\}$, where in the denominator stays the ideal generated by all pairwise brackets of elements of A.

Recall that an important invariant of an isolated singular point $p \in X$ of algebraic hypersurface X in C^n is the Milnor number of X at p which is defined as the number of so-called vanishing cycles of the Milnor fiber of X at p [6].

Theorem 2. *For a homogeneous polynomial $P \in C_3$ with an isolated critical point at the origin, the dimension of zero-dimensional Poisson homology $HP_0(A,\tau_P)$ is equal to the Milnor number of hypersurface $\{P=0\}$ at the origin.*

Poisson homology and Tyurina number

- The latter two problems concerned with Poisson (co)homology of JPS have gained some attention in the last two decades. In particular, in qh case similar results in terms of the Milnor number were given in the mentioned papers of A.Pichereau (3d) and S.Pelap (for SA and some GSA in 4d).
- At the same time no results seem to be available without assuming that Casimirs are qh with the same system of weights. This suggests further problems and conjectures. For example, in non-qh 3d case it is easy to see that the role of Minor number of Casimir h is played by the **Tyurina number** of h (equal to the dimension of the base of miniversal unfolding of h). A plausible **conjecture** is that the same holds true for a complete intersection with isolated singularity.

Thank you for your attention!

გმადლობთ ყურადღებისთვის!