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Algebraic constructions of superintegrable systems from commutant

# Abstract

It was discovered how polynomial algebras appear naturally as symmetry algebra of quantum superintegrable quantum systems. They provide insight into their degenerate spectrum, in particular for models involving Painlevé transcendents for which usual approaches of solving ODEs and PDEs cannot be applied. Those algebraic structures extend the scope of usual symmetries in context of quantum systems. They have been connected to other areas of mathematics such as orthogonal polynomials. Among them, the well-known Racah algebra $R(n)$. I will take a different perspective on those algebraic structures which is based on Lie algebras, their related enveloping algebras and commutant. The talk will present explicit examples such as the generic superintegrable systems on the 2-sphere and 3-sphere.

- Rutwig Campoamor-Stursberg, Ian Marquette, Hidden symmetry algebra and construction of quadratic algebras of superintegrable systems, Annals of Physics 424 168378 (2021), arXiv 2020.168378

- Francisco Correa, Mariano del Olmo, I Marquette, Javier Negro, Polynomial algebras from su(3) and a quadratically superintegrable model on the two sphere, J.PhysA. Math. and Theor 54 015205 (2021), arXiv:2007.11163

- Rutwig Campoamor-Stursberg, Ian Marquette, Quadratic algebras as commutants of algebraic Hamiltonians in the enveloping algebra of Schrodinger algebras, Annals of Physics. 437 168694 (2022) arXiv 2021.168694

- Other type of algebraic approaches and context

- Danilo Latini, Ian Marquette, Yao-Zhong Zhang, Construction of polynomial algebras from intermediate Casimir invariants of Lie algebras, J.PhysA Math. and Theor. (2022) arXiv:2204.06840

- Rutwig Campoamor-Stursberg, Ian Marquette, Decomposition of enveloping algebras of simple Lie algebras and their related polynomial algebras, Journal of Lie theory 34 (2024), arXiv:2204.11395

- Other work on subalgebras chains and dynamical symmetries and nuclear physics models, R Campoamor-Stusberg, D Latini, I Marquette, YZ Zhang arXiv 2303.00975, Polynomial algebras from Lie algebra reduction chains $\mathfrak{g} \supset \mathfrak{g}'$, Elliot ( $so(3) \supset su(3)$ ), surfon ( $so(3) \supset so(5)$ )

# Integrability and superintegrability

## Definition

A Hamiltonian system (in n dimensions) with Hamiltonian H

$$H = \frac{1}{2} g^{ik} p_i p_k + V(\vec{x})$$

is integrable if it allows n integrals of motion that are well defined, in involution $\{H, X_a\}_p = 0$, $\{X_a, X_b\}_p = 0$, a,b=1,...,n-1 and functionally independent.

A system is superintegrable if it admits $n + k$ (with $k = 1, ..., n - 1$) functionally independent constants of the motion (well defined). Maximally superintegrable if $k = n - 1$.

QM : $\{H, X_a, Y_b\}$ are well defined quantum mechanical operators and form an algebraically independent set.

# Quadratic superintegrability : systematic approach

- Fris, Mandrosov, Smorodinsky, Uhlir and Winternitz (1965)
- Winternitz, Smorodinsky, Uhlir and I.Fris (1966,1967)
- Makarov, Valiev, Smorodinsky, Winternitz (1968)

$$H = \frac{1}{2}\vec{p}^2 + V(x,y)$$

$$X_j = \sum_{i,k=1}^{2} \{f_j^{ik}(x,y), p_i p_k\} + \sum_{i=1}^{2} g_j^i(x,y) p_i + \phi_j(x,y), j = 1,2$$

- Integrability: 1 such integral $X_1$, four families, related to separation of variables in Cartesian, Polar, Elliptic and Parabolic

- Superintegrability: 2 such integrals $\{X_1, X_2\}$, four cases

- Multiseparability, exact solvability, degenerate spectrum

# Models, properties

$$V_I = \alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}, \quad V_{III} = \frac{\alpha}{r}+\frac{1}{r^2}\left(\frac{\beta}{1+cos(\phi)}+\frac{\gamma}{1-cos(\phi)}\right)$$

- Generalization to magnetic field, spin, curved spaces, Darboux, Dunkl, position mass dependent, pseudo Hermitian
- Classification nondegenerate, conformally flat spaces, ( Miller, Kress, Kalnins, 1998-2008 )
- Their quadratic algebras are interesting object, various connection with Lie algebras and their representations, Casimir invariants
- Relation with special functions/orthogonal polynomials, Askey-Scheme, Inonu-Wigner type, Bocher contraction (Miller, Kalnins 2014-2017 )

# Models, properties

- Classification nondegenerate, conformally flat spaces, ( Miller, Kress, Kalnins, 1998-2008 ), further works Fordy and Huang (2019), to arbitrary dimension by Kress, Schöbel and Vollmer (2020)

- Their quadratic algebras are interesting object, various connection with Lie algebras and their representations, Casimir invariants, algebra $Q(3)$ ( $R(3)$ with $a=0$ )

$$[A,B]=C, \quad [A,C]=\alpha A^2+\gamma\{A,B\}+\delta A+\epsilon B+\zeta$$
$$[B,C]=aA^2-\gamma B^2-\alpha\{A,B\}+dA-\delta B+z \quad .$$

- Granovskii, Zhedanov and Lutzenko (1991, 1992), Daskaloyannis (2001)

- Relation with special functions/orthogonal polynomials, Askey-Scheme, Inonu-Wigner type, Bocher contraction

$$K = C^2 - \alpha\{A^2, B\} - \gamma\{A, B^2\} + (\alpha, \gamma - \delta)\{A, B\}$$
$$+(\gamma^2 - \epsilon)B^2 + (\gamma\delta - 2\zeta)B + \frac{2a}{3}A^3 + (d + \frac{a\gamma}{3} + \alpha^2)A^2 + (\frac{a\epsilon}{3} + \alpha\delta + 2z)A$$

- deformation to cubic (Post and Ritter (2020), quartic and N order polynomial algebras from integrals of order 2 and N
- This Casimir operator can be written in terms of the Hamiltonian or other central element

- This provides constraints on the spectrum
- Realization of the quadratic algebra as deformed oscillator algebra $\{N, I, b, b^\dagger\}$ of the form

$$[N, b] = -b, \quad [N, b^\dagger] = b^\dagger, \quad bb^\dagger = \Phi(N+1) \quad b^\dagger b = \Phi(N)$$
$$A = A(N), \quad B = b(N) + b^\dagger \rho(N) + \rho(N) b$$

- $\Phi(N)$ is the structure function which is a polynomial in terms of the number operator $N$, parameter $u$
- Case $\gamma \neq 0$

$$\Phi = f_0 + f_1(N+u) + ... f_{10}(N+u)^{10}$$

- Case $\gamma = 0$

$$\Phi = g_0 + g_1(N+u) + ... g_4(N+u)^4$$

- the $f_i$ and $g_i$ depend explicitly on structure constants and Casimir
- representation via Fock basis $|k,n\rangle$, $n = 0, 1, 2, 3, ...$ and explicit relation between Casimir and Hamiltonian $K = a_0 + a_1 H + a_2 H^2 + a_3 H^3$

- The Smorodinsky-Winternitz model in 2D
- The integrals of motion of second order $A$ and $B$

$$H = \frac{1}{2}(P_x^2 + P_y^2) + \omega^2 r^2 + \frac{\mu_1}{x^2} + \frac{\mu_2}{y^2}$$

$$A = P_x^2 + \omega^2 x^2 + \frac{\mu_1}{x^2}, \quad B = (xP_y - yP_x)^2 + r^2(\frac{\mu_1}{x^2} + \frac{\mu_2}{y^2})$$

- The structure constant of the corresponding quadratic algebra

$$[A, B] = C,$$

$$[A, C] = 8\hbar^2 A^2 - 16\hbar^2 HA + 16\hbar^2\omega^2 B - 16\hbar^2(\mu_1 + \mu_2)\omega^2 + 8\hbar^4\omega^2$$

$$[B, C] = -8\hbar^2\{A, B\} + 16\hbar^4 A - 16\hbar^2\omega^2 B - 16\hbar^2(\mu_2 - \mu_1)\omega^2 - 16\hbar^4 H \quad .$$

- The cubic Casimir then take the form

$$K = 16\hbar^2((\mu_2-\mu_1)^2\omega^2+4\mu_1 H^2)-16\hbar^4(3H^2+2\hbar^2\omega^2-2(\mu_1+\mu_2))$$

- Using structure function and Casimir

$$\Phi(n,u,E) = 16\hbar^2(n+u-\frac{1}{2}-\frac{k_1}{2})(n+u-\frac{1}{2}+\frac{k_1}{2})$$
$$(n+u-\frac{1}{2}-\frac{k_2}{2}-\frac{E}{2\hbar\omega})(n+u-\frac{1}{2}+\frac{k_2}{2}-\frac{E}{2\hbar\omega})$$

- lead to the degenerate spectrum $E = 2\hbar\omega(p+1+\frac{\epsilon_1 k_1+\epsilon_2 k_2}{2})$

- Beyond: Higher order integrals, Painlevé and exceptional orthogonal polynomials and work with cubic of higher degree polynomial algebras

# Example of higher rank quadratic algebra

- M. A. Rodriguez and P. Winternitz (2001), Kalnins, Miller (2001) : n-dimensional superintegrable system
- Liao, Marquette, Zhang (2018), substructures and algebraic derivation
- Latini (2019), generalization to model on curved spaces and co algebra

$$H = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial {x_i}^2} - \frac{\gamma}{r} + \sum_{i=1}^{n-1}\frac{\beta_i}{x_i^2}$$

- where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$ and $\beta_i, \gamma$ are real parameters

- the corresponding Schrodinger/Hamilton-Jacobi equations are multiseparable
- wavefunctions can be described in terms of hypergeometric equation

- Ballesteros, Herranz, Musso, and Ragnisco (2004), Ballesteros, Enciso, Herranz, and Ragnisco (2009)
- Danilo Latini, Ian Marquette, Yao-Zhong Zhang, Embedding of the Racah algebra R (n) and superintegrability, Annals of Physics (2021)
- Danilo Latini, Ian Marquette, Yao-Zhong Zhang, Racah algebra R (n) from coalgebraic structures and chains of R (3) substructures, JPA (2021)

$$J_+^{[n]} = \frac{1}{2}(\boldsymbol{p}^2 + \sum_{j=1}^{n} \frac{a_j}{x_j^2}) \qquad J_-^{[n]} = \frac{1}{2}\boldsymbol{x}^2 \qquad J_3^{[n]} = \frac{1}{2}\boldsymbol{x}\cdot\boldsymbol{p}\,,$$

$$\{J_-^{[n]}, J_+^{[n]}\} = 2J_3^{[n]} \quad \{J_3^{[n]}, J_\pm^{[n]}\} = \pm J_\pm^{[n]}\,,$$

- The total Casimir, in the given realisation

$$C^{[n]} = (J_3^{[n]})^2 - J_+^{[n]}J_-^{[n]} = -\frac{1}{4}\left(\sum_{1\leq i<j}^{n}(L_{ij}^2 + a_i\frac{x_j^2}{x_i^2} + a_j\frac{x_i^2}{x_j^2}) + \sum_{i=1}^{n} a_i\right)$$

$$C_{ij} := -\frac{1}{4}\left(L_{ij}^2 + a_i\frac{x_j^2}{x_i^2} + a_j\frac{x_i^2}{x_j^2} + a_i + a_j\right)$$

$$C_i := -\frac{a_i}{4}, \quad P_{ij} := C_{ij} - C_i - C_j$$

- we can construct the so-called left and right Casimirs

$$C^{[m]} = \sum_{1\le i<j}^{m} P_{ij} + \sum_{i=1}^{m} C_i \qquad C_{[m]} = \sum_{n-m+1\le i<j}^{n} P_{ij} + \sum_{i=n-m+1}^{n} C_i$$

- $m = 1, \ldots, n$ , $C^{[1]} = C_1$, $C_{[1]} = C_n$ are just constants and $C^{[n]} = C_{[n]}$ is the total Casimir

- a consequence of the underlying coalgebra symmetry the set $U = U_1 \cup U_2$ with $U_1 := \{C_{[m]}\}$, $U_2 := \{C^{[m]}\}$ for $m = 1, \ldots, n$ provides $2n - 3$ integrals

- Moreover, since the elements in each subset $U_i$ $(i = 1, 2)$ Poisson commute each other, they define two abelian Poisson subalgebras composed by $n - 1$ elements in involution

- $C_{[1]}$ and $C^{[1]}$ are just constants and $C_{[n]} = C^{[n]})$

- the two sets $V_i := U_i \cup \{H\}$ $(i = 1, 2)$ is composed by $n$ functionally independent involutive constants of motion, thus leading to multi-integrability

- generic $n$D spherically symmetric curved space, under the influence of a central potental $V = V(r)$ and with additional non-central terms breaking the radial symmetry

$$H = J_+^{[n]} + V\left( \left(2J_-^{[n]}\right)^{1/2} \right) = \frac{1}{2}\left(\boldsymbol{p}^2 + \sum_{j=1}^{n} \frac{a_j}{x_j^2}\right) + V(|\boldsymbol{x}|)\,,$$

- connection with Racah algebra $R(n)$, $F_{ijk} := \frac{1}{2}\{P_{ij}, P_{jk}\}$

$$\{P_{ij}, H\} = \{F_{ijk}, H\} = 0, \{P_{ij}, P_{kl}\} = 0, \{P_{ij}, P_{ik} + P_{jk}\} = 0$$

- quadratic Poisson algebra, Racah algebra $R(n)$
- with $i, j, k, l, m, r \in \{1, \ldots, n\}$ all different

$$\begin{aligned}
\{P_{ij}, P_{jk}\} &= 2F_{ijk} \\
\{P_{jk}, F_{ijk}\} &= P_{ik}P_{jk} - P_{jk}P_{ij} + 2P_{ik}C_j - 2P_{ij}C_k \\
\{P_{kl}, F_{ijk}\} &= P_{ik}P_{jl} - P_{il}P_{jk} \\
\{F_{ijk}, F_{jkl}\} &= F_{jkl}P_{ij} - F_{ikl}(P_{jk} + 2C_j) - F_{ijk}P_{jl} \\
\{F_{ijk}, F_{klm}\} &= F_{ilm}P_{jk} - P_{ik}F_{jlm}
\end{aligned}$$

- Here some closure relations, high-order relations among the generators hold

$$F_{ijk}^2 - C_i P_{jk}^2 - C_j P_{ik}^2 - C_k P_{ij}^2 + P_{ij} P_{jk} P_{ik} + 4 C_i C_j C_k = 0$$

$$2F_{ijk}F_{klm} - P_{il}P_{jk}P_{km} - P_{ik}P_{jm}P_{kl} + P_{im}P_{jk}P_{kl} + P_{ik}P_{jl}P_{km} - 2C_k P_{im}P_{jl} + 2C_k P_{il}P_{jm} = 0$$

- For explicit choice of Hamiltonian with coalgebra symmetry we find larger symmetry algebras
- Quantum analog was also obtained, deformation terms with $\hbar$
- But still everything here is done with differential oeprators
- Problem with Jacobi/PBW basis, there is ideas such as special Racah R(n) algebras that was proposed

# Algebraic Hamiltonian

$$H = \sum_{a,b} c_{ab} T_a T_b + \sum_a c_a T_a + T_0$$

$$[T_a, T_b] = \sum_c c_{ab}^c T_c$$

- Common features to those algebraic approaches
- Dynamical algebra, hidden algebra, symmetry algebra : ( solvable, Casimir, Casimir + Cartan, partial Casimir ... )
- They still rely on explicit differential operator and often even acting on explicit wavefunctions

- Racah algebra $R(n)$ for the (generic) model on the (n-1)-sphere

- Racah/model of the sphere: Post (2010,2011), Gaboriaud, Vinet, Vinet and Zhedanov (2019), De Bie, Iliev, Van de Vijver and Vinet (2020) , Crampe, Gaboriaud, d'Andecy, and Vinet (2021)

- Hamiltonian on the (n-1)-sphere

$$C^{[n]} = -\frac{1}{4}\Big(\sum_{1\leq i<j}^{n} L_{ij}^2 + x^2 \sum_{j=1}^{n} \frac{a_j}{x_j^2}\Big)$$

- Connected with problems related to recoupling: Levy-Leblond and Levy-Nahas (1965), Granovskii and Zhedanov (1988)

- Campoamor-Stursberg, Latini, Marquette, Zhang (2023)
- Procedure based on the commutant of a subalgebra in the universal enveloping algebra of a given Lie algebra
- The case of $\mathfrak{sl}(n)$ ($\mathcal{A}_n$ ) is discussed, polynomial algebra of degree $n-1$
- Connection with the generic superintegrable model on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and the related Racah algebra $R(n)$
- 2-sphere and 3-sphere, the quadratic and cubic polynomial algebras ( $\mathcal{A}_3$ and $\mathcal{A}_4$)

- Starting from the conventional notion of a commutant in the enveloping algebra of a Lie algebra $\mathfrak{s}$

- We reformulate the problem by means of the Lie-Poisson structure of the corresponding symmetric algebra $S(\mathfrak{s})$, provides a computationally more adequate setting ( also calculation directly from noncommutative setting using NCalgebra, symmetrization map )

- We then define the notion of algebraic Hamiltonian with respect to a subalgebra $\mathfrak{a}$, and show that the commutant defines a polynomial algebra that can be identified with the symmetry algebra of the Hamiltonian

- With the constants of the motion obtained from the elements belonging to the centralizer $C_{S(\mathfrak{s})}(\mathfrak{a})$

- Let $\mathfrak{s}$ be an $n$-dimensional Lie algebra and $\mathcal{U}(\mathfrak{s})$ be its universal enveloping algebra.

- For positive integers $p$, we define $\mathcal{U}_{(p)}(\mathfrak{s})$ as the subspace generated by monomials $X_1^{a_1} \dots X_n^{a_n}$ subjected to the constraint $a_1 + a_2 + \cdots + a_n \leq p$, where $\{X_1, \dots, X_n\}$ is an arbitrary basis of $\mathfrak{s}$

- The symmetric algebra $S(\mathfrak{s})$, admits Poisson algebra through

$$\{P, Q\} = C_{ij}^k x_k \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial x_j}, \ P, Q \in S(\mathfrak{s})$$

- symmetrization map $(\Lambda : S(\mathfrak{s}) \to \mathcal{U}(\mathfrak{s}))$

$$\Lambda\left(x_{j_1} \dots x_{j_p}\right) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} X_{j_{\sigma(1)}} \dots X_{j_{\sigma(p)}},$$

- Commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ of a subalgebra $\mathfrak{a} \subset \mathfrak{s}$ is defined as the centralizer of $\mathfrak{a}$ in $\mathcal{U}(\mathfrak{s})$

$$C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) = \{Q \in \mathcal{U}(\mathfrak{s}) \mid [P, Q] = 0, \quad \forall P \in \mathfrak{a}\}$$

- The Lie-Poisson context (solution PDEs)

$$C_{S(\mathfrak{s})}(\mathfrak{a}) = \{Q \in S(\mathfrak{s}) \mid \{P, Q\} = 0, \ P \in \mathfrak{a}\}.$$

- An algebraic Hamiltonian with respect to $\mathfrak{a}$ is defined as

$$\mathcal{H}_a = \sum_{i,j} \alpha_{ij} X_i X_j + \sum_k \beta_k X_k + \sum_\ell \gamma_\ell \mathcal{C}_\ell,$$

- where $X_i, X_j, X_k \in \mathfrak{a}$, $\mathcal{C}_\ell$ is a Casimir operator of $\mathfrak{s}$

- The polynomials in $C_{S(\mathfrak{s})}(\mathfrak{a})$ may have all vanishing commutator/bracket, then Hamiltonian $\mathcal{H}_a$ is integrable and the symmetry algebra is Abelian

- Alternatively, there are non-commuting elements in $C_{S(\mathfrak{s})}(\mathfrak{a})$ then the algebraic Hamiltonian $\mathcal{H}_a$ possesses a non-commutative (super-)integrability property and the symmetry algebra is a non-Abelian (polynomial algebra)

- consider $\mathfrak{sl}(n)$ in its defining representation. A basis is given by the generators $E_{ij}$ with $1 \leq i, j \leq n$ subjected to the constraint $\sum_{i=1}^{n} E_{ii} = 0$.

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} \qquad (1 \leq i, j, k, l \leq n).$$

- The commutant $C_{\mathcal{U}(\mathfrak{sl}(n))}(\mathfrak{h})$ is finitely generated, monomials of the form $p_{i_1,\ldots,i_d} = e_{i_1,i_2} e_{i_2,i_3} \ldots e_{i_{d-1},i_d} e_{i_d,i_1}$

- linearly independent elements $\nu_d = \frac{n!}{d(n-d)!}$, monomial $P$ of order $d > n$ satisfying equation is decomposable.
- Summarizing, it has been shown that a basis of linearly independent elements in the centralizer $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ is given by the polynomials $h_\ell$, $p_{i_1,\ldots,i_d}$, $1 \leq \ell \leq n-1$ and $2 \leq d \leq n$

$$\dim_L C_{S(\mathfrak{sl}(n))}(\mathfrak{h}) = (n-1) + \sum_{d=2}^{n} \nu_d = \sum_{d=1}^{n} \frac{n!}{(n-d)!d} - 1$$

- $n^2 - n$ of these elements are functionally independent, the basis elements will satisfy several algebraic relations

- we have $\mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots \subset \mathcal{A}_n$, meaning that each $\mathcal{A}_k$ can be seen as a non-central extension of $\mathcal{A}_{k-1}$.

- $n = 3$, six functionally independent , dimension of $C_{S(\mathfrak{sl}(3))}(\mathfrak{h})$ is seven, a basis $\mathcal{B} = \{h_1, h_2, p_{1,2}, p_{1,3}, p_{2,3}, p_{1,2,3}, p_{1,3,2}\}$.
- The algebraic dependence of these elements is given by

$$p_{1,2}p_{1,3}p_{2,3} - p_{1,2,3}p_{1,3,2} = 0.$$

- Set of functionally independent polynomial solutions does not generate in general a polynomial algebra
- The elements in $\mathcal{B}$ generate a quadratic algebra with nontrivial brackets

$$
\begin{aligned}
\{p_{1,2}, p_{1,3}\} &= -\{p_{1,2}, p_{2,3}\} = \{p_{1,3}, p_{2,3}\} = p_{1,2,3} - p_{1,3,2},\\
\{p_{1,2}, p_{1,2,3}\} &= p_{1,2}(p_{1,3} - p_{2,3}) - h_1 p_{1,2,3},\\
\{p_{1,3}, p_{1,2,3}\} &= p_{1,3}(p_{2,3} - p_{1,2}) + (h_1 + h_2) p_{1,2,3},\\
\{p_{2,3}, p_{1,2,3}\} &= p_{2,3}(p_{1,2} - p_{1,3}) - h_2 p_{1,2,3},\\
\{p_{1,2}, p_{1,3,2}\} &= -p_{1,2}(p_{1,3} - p_{2,3}) + h_1 p_{1,3,2},\\
\{p_{1,3}, p_{1,2,3}\} &= -p_{1,3}(p_{2,3} - p_{1,2}) - (h_1 + h_2) p_{1,3,2},\\
\{p_{2,3}, p_{1,2,3}\} &= -p_{2,3}(p_{1,2} - p_{1,3}) + h_2 p_{1,3,2},\\
\{p_{1,2,3}, p_{1,3,2}\} &= h_1 p_{1,3} p_{2,3} + h_2 p_{1,2} p_{1,3} - (h_1 + h_2) p_{1,2} p_{2,3}.
\end{aligned}
$$

$$
\begin{aligned}
c^{[2]} =& p_{1,2} + p_{1,3} + p_{2,3} + \frac{1}{3}(h_1^2 + h_1 h_2 + h_2^2),\\
c^{[3]} =& p_{1,2,3} + p_{1,3,2} + \frac{1}{3}\left((h_1 + 2h_2)p_{1,2} + (h_1 - h_2)p_{1,3} - (2h_1 + h_2)p_{2,3}\right)\\
& + \frac{2}{27}(h_1^3 - h_2^3)).
\end{aligned}
$$

- also calculation via noncommutative setting, expression can be put in similar form with commutator instead of Poisson bracket

- Connection of the quadratic algebra associated to $C_{S(\mathfrak{sl}(3))}(\mathfrak{h})$ with a Racah-type quadratic algebra

$$c_1 := \frac{1}{3}(2h_1 + h_2)\,, \quad c_2 := \frac{1}{3}(h_2 - h_1)\,, \quad c_3 := -\frac{1}{3}(h_1 + 2h_2),$$

$$c_1 + c_2 + c_3 = 0,$$

- We take $c_{ij} := p_{i,j}$, and skew-symmetric and symmetric elements

$$f_{123} := \frac{1}{2}(p_{1,3,2} - p_{1,2,3}), \quad g_{123} := \frac{1}{2}(p_{1,3,2} + p_{1,2,3}).$$

$$c_{ij} = c_{ji} \qquad \text{(symmetric)}$$
$$f_{ijk} = -f_{jik} = f_{jki} \qquad \text{(skew-symmetric)}$$
$$g_{ijk} = g_{jik} = g_{jki} \qquad \text{(symmetric)}$$

- symmetry properties allow to write

$$\{c_{ij}, c_{jk}\} = 2f_{ijk}$$
$$\{c_{jk}, f_{ijk}\} = (c_{ik} - c_{ij})c_{jk} + (c_j - c_k)g_{ijk}$$
$$\{c_{jk}, g_{ijk}\} = (c_j - c_k)f_{ijk}$$
$$\{f_{ijk}, g_{ijk}\} = \frac{1}{2}((c_i - c_k)c_{ij}c_{jk} + (c_k - c_j)c_{ki}c_{ij} + (c_j - c_i)c_{jk}c_{ki})$$

$$f_{ijk}f_{kji} + g_{ijk}g_{kji} = c_{ij}c_{jk}c_{ki} \, .$$

- Algebra can be quantized via symmetrization map and correction terms obtained

- $\mathfrak{sl}(4)$, 12 functionally independent solutions, while the centralizer $C_{S(\mathfrak{sl}(4))}(\mathfrak{h})$ is of dimension 23, basis of $C_{\mathfrak{sl}(4)^*}(\mathfrak{h}^*)$, higher degree polynomials

$$f_{ijkl} := \frac{1}{2}(p_{i,l,k,j} - p_{i,j,k,l}), \qquad g_{ijkl} := \frac{1}{2}(p_{i,l,k,j} + p_{i,j,k,l}).$$

- other relations occurs, in particular we need to add

$$\{f_{iljk}, g_{ijkl}\} = \frac{1}{2}(((c_{jk} - c_{jl} - c_{kl})g_{ijk} + (c_{ij} + c_{ik} - c_{jk})g_{jkl})c_{il} + \ldots.$$
$$\{f_{iljk}, g_{ijlk}\} = \frac{1}{2}(((c_{jk} - c_{jl} + c_{kl})g_{ijl} + (c_{jl} - c_{ij} - c_{il})g_{jkl})c_{ik} + \ldots$$

- additional functional identities, cubic algebra $\mathcal{A}_4$
- quantization, correction terms for most relations, NCalgebra package

- using Marsden-Weinstein reduction, Del Olmo, Rodriguez, Winternitz 1993
- realization provide generic superintegrable systems on the three-sphere $\mathbb{S}^3$, $\mathcal{A}_4$ reduces to $R(4)$.
- Casimir reduce in following way

$$c^{[2]} = -\frac{1}{2}\left(H + \frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^2\right)$$
$$c^{[3]} = \frac{\mathrm{i}}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)c^{[2]} + \frac{\mathrm{i}}{16}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^3$$
$$c^{[4]} = -\frac{1}{16}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^2 c^{[2]} + \frac{1}{8}(\alpha_1 + \alpha_2 + \alpha_3)\ldots$$

- the element $c_{ij}$ becomes

$$c_{ij} = -\frac{1}{4}\left((s_i p_j - s_j p_i)^2 + \alpha_i^2 \frac{s_j^2}{s_i^2} + \alpha_j^2 \frac{s_i^2}{s_j^2} + 2\alpha_i \alpha_j\right)$$

- $g_{ijk}$, $f_{ijkl}$, $g_{ijkl}$ are all expressible in terms of elements with a lower number of indices after the reduction to canonical coordinates

- for any $i \neq j \neq k \neq l \in \{1, 2, 3, 4\}$

$$
\begin{aligned}
g_{ijk} =& \frac{\mathrm{i}}{2}(\alpha_k c_{ij} + \alpha_j c_{ik} + \alpha_i c_{jk} + \alpha_i \alpha_j \alpha_k), \\
g_{ijkl} =& \frac{1}{2}(c_{ij} c_{kl} + c_{il} c_{jk} - c_{ik} c_{jl} - \alpha_i \alpha_k c_{jl} - \alpha_j \alpha_l c_{ik} - \alpha_i \alpha_j \alpha_k \alpha_l), \\
f_{ijkl} =& \frac{\mathrm{i}}{2}(\alpha_l f_{ijk} + \alpha_k f_{ijl} + \alpha_j f_{ikl} + \alpha_i f_{jkl}).
\end{aligned}
$$

- This means specifically that, as a result of the canonical realization, the cubic algebra collapses to a quadratic one with basis elements

- Thus the connection with the Racah algebra $R(4)$, considering that several different bases are conceivable to give a presentation is obtained through the following identifications

$$\begin{aligned} C_i :=& -\alpha_i^2/4, \quad 1 \leq i \leq 4, \\ C_{ij} :=& c_{ij}, \quad 1 \leq i < j \leq 4, \\ F_{ijk} :=& f_{ijk}, \quad 1 \leq i < j < k \leq 4. \end{aligned}$$

- $sl(n)$ : the polynomial algebra $\mathcal{A}_n$ is of order $n-1$

- $C_{S(\mathfrak{s})}(\mathfrak{a})$ determines a finitely-generated polynomial algebra $\mathcal{A}$, can be identified with the symmetry algebra
- realization: $\mathcal{A}_n$ can reduce its polynomial order, the polynomial algebra reduces to the $(n-2)$-rank Racah algebra $R(n)$, , Hamiltonian $H$ (on sphere $\mathbb{S}^{n-1}$)

$$\dim_L C_{S(\mathfrak{sl}(5))}(\mathfrak{h}) = 88, \quad \dim_L C_{S(\mathfrak{sl}(6))}(\mathfrak{h}) = 414,$$
$$\dim_L C_{S(\mathfrak{sl}(7))}(\mathfrak{h}) = 2371,$$

- Notion of algebraic (super)integrability based on algebraic Hamiltonians and constants of the motion
- quantum case using symmetrization map or via Noncommutative NCalgebra package
- Generalisations to non semisimple algebras, more generally MASAs: Rodriguez, Winternitz, Zassenhaus (1990)
- Campoamor-Stursberg, Marquette, Junze Zhang, Yao-Zhong Zhang, the general problem of simple Lie algebras and commutant from Cartan subalgebra

- General proposal for the classification commutant of Lie algebras
- general framework, subalgebras chains and dynamical symmetries, missing label problems
- Campoamor-Stursberg, Latini, Marquette, Zhang, arXiv 2303.00975
- Polynomials even in the classical (bracket) setting can become quite large, 1000+ terms, not surprising when we know that Casimir can have large number of terms ( $G_2$ the degree 6 Casimir has more than 400 terms )
- Different approaches to terms with large number of terms