MKdV-Related Flows for Legendrian Curves in S^3

joint work with

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[check arXiv next week]

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Preamble / Motivation

Integrable evolution equations

"Integrable" $\leftrightarrow$ inducing integrable PDE on their geometric invariants

Integrable curve flows arise naturally:

from physics, e.g. vortex filament equation

from geometry: e.g. pseudo-spherical surfaces, Pinkall's flow, invariant flows in different geometries

curves

- geometric & topological features
- construct examples

integrability

Lax pairs, conservation laws, special solutions, Bäcklund transformation

3-sphere and Hopf fibration

Consider $\mathbb{C}^2$ with the Hermitian inner product

$$\langle z, w \rangle := z_1\overline{w_1} + z_2\overline{w_2} = \overline{\langle w, z \rangle}$$

Note: $\mathrm{Re}\langle \ , \ \rangle$ is the Euclidean inner product identified with $\mathbb{C}^2 \simeq \mathbb{R}^4$

$$S^3 = \{ z \in \mathbb{C}^2 \mid \langle z, z \rangle = 1 \}$$

Complex projectification : $\mathbb{C}^2 \setminus \{0\} \xrightarrow{\pi} \mathbb{C}P^1$

identify $(z, w) \sim (\lambda z, \lambda w)$ $\lambda \in \mathbb{C}, \lambda \neq 0$

Restricting π to S^3 gives the Hopf fibration

$$\pi_H : S^3 \to \mathbb{C}P^1 \qquad |\lambda| = 1$$

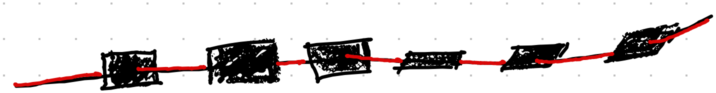
Contact distribution on S^3

Hopf map $\pi_H : S^3 \to \mathbb{C}P^1$

- pts on fiber differ by $p \to e^{i\theta} p$ (circles in S^3)
- define a contact distribution on S^3: the contact planes are (Euclidean) orthogonal to the Hopf fibers
- locally, contact planes are defined as tangent vectors in the kernel of a differential 1-form α, with $\alpha \wedge d\alpha \neq 0$ (α not unique, e.g. $\alpha(\cdot) = \mathrm{Re}\langle \cdot, v\rangle$ v tangent to fiber)
- The plane distribution has integral curves (tangent to a plane at each point), but no integral surface: $\alpha \wedge d\alpha \neq 0$

Integral curves are called

Legendrian curves
(L-curves)



Description of L-curves in S^3

$\gamma : \mathbb{R} \to S^3 \subseteq \mathbb{C}^2$ regular, smooth parametrized curve

γ is Legendrian iff

$$\langle \gamma_x , \gamma \rangle = 0$$

Why: - $\langle \gamma, \gamma \rangle = 1 \Rightarrow \mathrm{Re} \langle \gamma_x , \gamma \rangle = 0.$

- $i\gamma(x) \parallel$ Hopf fiber at $p = \gamma(x) \in S^3$
- γ L-curve iff $\gamma_x \perp_{Eucl.} i\gamma \Leftrightarrow \mathrm{Re} \langle \gamma_x , i\gamma \rangle = \mathrm{Im} \langle \gamma_x , \gamma \rangle = 0$

Group of Symmetries: $U(2) = \{ g \in GL(2, \mathbb{C}) \mid \bar{g}^T = g^{-1} \}$
(metric - preserving)

1. $U(2)$ preserves S^3
2. $U(2)$ preserves contact structure

Congruence of L-curves

Def. $\gamma, \tilde{\gamma}$ param. L-curves are congruent if $\tilde{\gamma} = g\gamma$, $g \in U(2)$

How to test congruence:

① $|\tilde{\gamma}_x| = \langle \tilde{\gamma}_x, \tilde{\gamma}_x \rangle^{\frac{1}{2}} = \langle g\gamma_x, g\gamma_x \rangle^{\frac{1}{2}} = |\gamma_x|$ speeds must be the same

② Assume unit speed (for simplicity)

$$\begin{tikzcd} & U(2) \arrow[d, "g \to g\cdot\binom{1}{0}"] \\ \mathbb{R} \arrow[ur, "\Gamma"] \arrow[r, "\gamma"] & S^3 \end{tikzcd}$$

Construct a lift Γ of the curve γ into the group [U(2)-value moving frame]

Check: $\Gamma(x) = \left(\gamma \,\vdots\, \gamma_x \right) \in U(2)$

Test congruence of $\gamma, \tilde{\gamma}$: do their lifts differ by left-multipl.

Curvature of L-curve

$\Gamma = (\gamma \,\vdots\, \gamma_x)$ U(2)-valued moving frame

$\Rightarrow$ ⊛ $\Gamma_x = \Gamma \begin{pmatrix} 0 & -1 \\ 1 & ik \end{pmatrix}$ $k(x) :=$ "curvature" of γ

Thm Two unit speed L-curves $\gamma, \tilde{\gamma}$ are congruent iff $k(x) = \tilde{k}(x)$ $\forall x$ (equal curvature functions)

⊛ $\Longrightarrow$ $\gamma_{xx} = -\gamma + ik\gamma_x$ $\Rightarrow$ $k = \mathrm{Im}\langle \gamma_{xx}, \gamma_x \rangle$

A complete set of differential invariants for L-curves <u>not</u> of unit speed are:

speed $\beta = |\gamma_x|$

curvature $k = \dfrac{\mathrm{Im}\langle \gamma_{xx}, \gamma_x \rangle}{|\gamma_x|^3}$

Constructing examples of L-curves

Use the curvature κ and Hopf map to reduce the reconstruction of γ to <u>computing an antiderivative</u>

- Identify $S^3 \simeq SU(2) = \{ g \in U(2) \mid \det g = 1 \}$

$$S^3 \longleftrightarrow SU(2)$$

$$\mathfrak{z} = (z_1, z_2) \longleftrightarrow \mathfrak{z}^* = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

- also identify $\mathfrak{su}(2) \simeq \mathbb{R}^3$
- using the Adjoint repr. of $SU(2)$ and taking the standard inner product on $\mathbb{R}^3$, get

$$\sigma : S^3 \xrightarrow{2:1} SO(3) \quad \text{(spin covering)}$$

- Moreover $\Pi_H(\mathfrak{z}) = \sigma(\mathfrak{z}^*)\vec{e}_1 \in S^2 \qquad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$

Hopf map factors through the double cover

Reconstruction of L-curves

$$\sigma:\ S^3 \simeq SU(2) \xrightarrow{2:1} SO(3)$$

$\gamma : \mathbb{R} \to S^3$ unit speed L-curve with curvature k

$$\eta = \pi_H \circ \gamma \ : \mathbb{R} \longrightarrow S^2$$

Fact: η has curvature $\varkappa = \frac{k}{2}$

Now, if $F = (\eta, T, N) \in SO(3)$ is the Frenet frame of the spherical curve η and $\tilde{F}$ its lift into $SU(2) \cong S^3$ (i.e. $\sigma \circ \tilde{F} = F$)

$\Rightarrow$ $\gamma = e^{i\int^x \varkappa(x')\,dx'}\tilde{F}$ is a unit-speed L-curve

(unique up to multipl. by $e^{i\theta}$, θ const)

The lifts of parallels at rational heights are torus knots in the 3-sphere

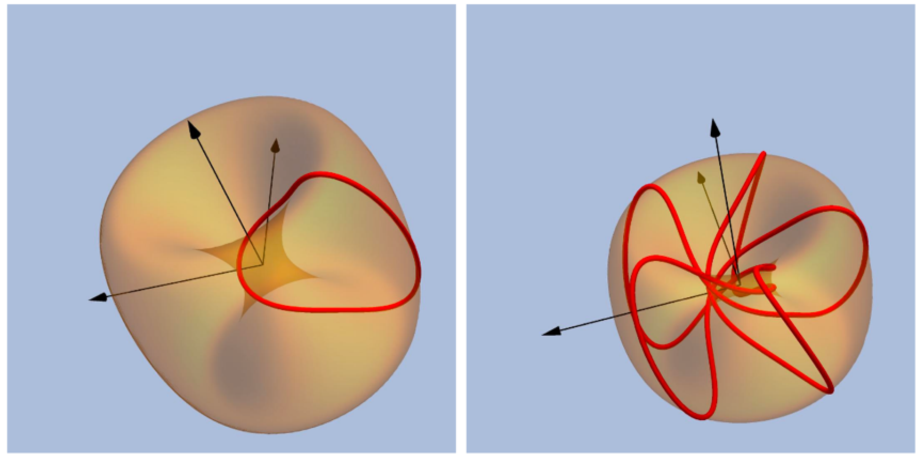


FIGURE 1. Left: the Heisenberg projection of the Legendrian knot $\gamma_{1,1}$, a topologically trivial knot with Maslov index 0 and Bennequin invariant -1. Right: the Heisenberg projection of the Legendrian knot $\gamma_{3,5}$, a torus knot of type $(-3,5)$ with Maslov index 2 and Bennequin invariant -15. The tori are the Heisenberg projections of $\mathcal{T}_{m,n} \subset S^3$, $m = n = 1$ (left) and $m = 3, n = 5$ (right).

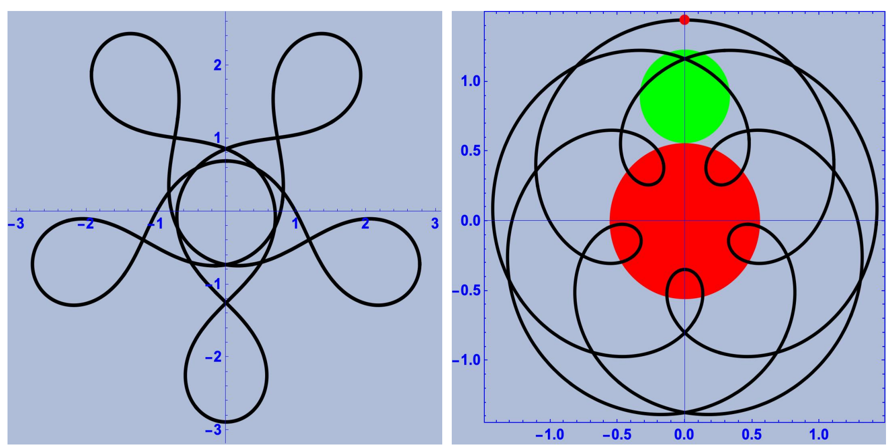


FIGURE 2. Left: the Lagrangian projection $\alpha_{3,5}$ of $\mathbf{p}_H \circ \gamma_{3,5}$. Its turning number is 2, and there are fifteen ordinary double points, each with intersection index -1. Right: the epicycloid obtained inverting $\alpha_{3,5}$ with respect to the origin.

Geometry of Legendrian Loop Space

$P_L = \{$regular param. L-curves of period $L\}$ (may not be unit-speed)

$\Rightarrow P_L$ has the structure of a Fréchet manifold

(e.g. Lerario & Mondino 2019, Haller & Vizman 2022)

Tangent Space: $\gamma \in P_L$, let $\hat{\gamma}(\cdot, t)$ be a variation of γ:

1. $\hat{\gamma}(\cdot, t) \in P_L$ $\forall t$, $|t|$ small enough, $\hat{\gamma}$ smooth
2. $\hat{\gamma}(x, 0) = \gamma(x)$

Variation Vector field $\quad \left.\dfrac{\partial \hat{\gamma}}{\partial t}\right|_{t=0} := V = p\gamma_x + q(i\gamma_x) + r(i\gamma)$

p, q, r periodic of period L

Require $\left.\dfrac{\partial}{\partial t}\right|_{t=0} \langle \gamma_x, \gamma \rangle = \langle V_x, \gamma \rangle + \langle \gamma_x, V \rangle = 0 \Rightarrow r_x = 2q|\gamma_x|^2$

$V = p\gamma_x + q(i\gamma_x) + r(i\gamma)$ $\qquad r_x = 2q|\gamma_x|^2$

Remark: $T_{\gamma(x)}S^3$ is spanned by $\underbrace{\gamma_x, i\gamma_x}_{\text{span contact plane at } \gamma}$ and $i\gamma$
tg to H-fiber

$p = q = 0, \; r = 1 \Rightarrow H = i\gamma$ generator of constant speed rotation along the fiber

$q = r = 0, \; p = f(x)$ periodic $\Rightarrow R_f = f(x)\gamma_x$ generator of reparam. in x (fixed period)

H and R_f form a group of transformations $\mathcal{G}$

Form the quotient $\mathcal{Q}_L = \mathcal{P}_L / \mathcal{G}$ of equivalence classes $[\gamma]$

Symplectic Structure on $\mathcal{Q}_L$

for $\mathcal{V}, \mathcal{W} \in T_{[\gamma]}\mathcal{Q}_L$ choose representatives $V, W \in T_\gamma \mathcal{P}_L$ and define the 2-form

$$\Omega_{[\gamma]}(\mathcal{V}, \mathcal{W}) := -\int_0^L \det_{\mathbb{R}}(\gamma, \gamma_x, V, W)\,dx$$

determinant in $\mathbb{R}^4$

- the value of Ω does not depend on the choice of representatives
- if $V = p_V \gamma_x + q_V (i\gamma_x) + r_V (i\gamma)$, $W = p_W \gamma_x + \dots$

$$\Omega_{[\gamma]}(\mathcal{V}, \mathcal{W}) = \frac{1}{2}\int_0^L (r_V r_W' - r_W r_V')\,dx = \int_0^L r_V r_W'\,dx$$

Symplectic Structure on Q_L

$$\Omega_{[\gamma]}(V, W) = -\int_0^L \det_{\mathbb{R}}(\gamma, \gamma_x, V, W)\,dx = \int_0^L r_V\, r_W'\, dx$$

(vector field $= p\gamma_x + q(i\gamma_x) + r(ij)$, $r^2 = 2g|\gamma_x|^2$)

Thm Ω is a symplectic form on Q_L

pf. • non-degeneracy

$\Omega_{[\gamma]}(V, W) = 0 \quad \forall V \Rightarrow r_W' = 0 \Leftrightarrow r_W = 0 \Leftrightarrow W = 0$

• closure

Nice application of Cornelia Vizman's "hat calculus"

Hat Calculus

Setting: S compact oriented k-manifold
M finite-dim manifold
ω p-form on M, α q-form on S

$$\Rightarrow \quad \widehat{\omega \bullet \alpha} := \int_S ev^*\omega \wedge pr^*\alpha \qquad \left(\int_S \text{ denotes fiber integr.}\right)$$

defines a $(p+q-k)$-form on $\mathcal{F}(S,M) = \{f : S \to M, \text{ smooth}\}$

Here: $ev : S \times \mathcal{F}(S,M) \to M$ $\qquad$ $pr : S \times \mathcal{F}(S,M) \to S$
$(x,f) \rightsquigarrow f(x)$ $\qquad$ $(x,f) \rightsquigarrow x$

In our case $S = S^1$, $M = S^3$

Claim: $\boxed{\Omega = \hat{\nu} = \widehat{\nu \bullet 1}}$

Ω: 2-form on $\mathcal{F}(S^1, S^3)$

ν standard volume form on S^3
1 constant fn on S^1
$p+q-k = 3+0-1 = 2$

Proof of Closure

volume form on S^3: pull-back of $\iota_E \mu$, where μ = standard volume form on $\mathbb{R}^4$, $E = r\frac{\partial}{\partial r}$ (Euler vect. field)

If $\gamma : S^1 \to S^3$ is an embedding

$$\hat{\nu}(V,W) = \int_{S^1} \gamma^*(\iota_W \iota_V \nu) = \int_\gamma \iota_W \iota_V \nu = \int_0^L \nu(\gamma_x, V, W)\, dx = -\int_0^L \det(\gamma, \gamma_x, V, W)\, dx$$

Use:

Thm (Vizman) $d(\widehat{\omega \cdot \alpha}) = \widehat{(d\omega) \cdot \alpha} + (-1)^p \widehat{\omega \cdot d\alpha}$

$$d\hat{\nu} = \underbrace{\widehat{(d\nu) \cdot \alpha}}_{\substack{= \\ 0 \\ \text{(top form)}}} - \widehat{\nu \cdot \underbrace{dl}_{= 1}} = 0$$

Hamiltonian Vector fields

Ω gives the correspondence

$$\mathcal{C}^\infty(Q_L) \longrightarrow TQ_L$$

$$\underset{\text{(Hamiltonians)}}{\mathcal{H}} \rightsquigarrow d\mathcal{H}_{[\gamma]}(V) = \Omega_{[\gamma]}(V, \underset{\substack{\text{Hamiltonian}\\ \text{vector fields}}}{W_{\mathcal{H}}})$$

Equiv., for $\hat{\gamma}(\cdot, t)$ a variation of γ, $\left.\frac{d\hat{\gamma}}{dt}\right|_{t=0} = V$

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{H}(\hat{\gamma}(\cdot, t)) = \Omega_{[\gamma]}(V, W_{\mathcal{H}})$$

Prop. Given $\mathcal{A}([\gamma]) = \int_0^L \langle \gamma_x, \gamma_x \rangle^{\frac{1}{2}} dx$ (total length)

$$W_{4\mathcal{A}} = \frac{k_x}{|\gamma_x|^2}(i\gamma_x) + 2k(i\gamma)$$

focusing mKdV Equation

Prop. If $V = p\gamma_x + q(i\gamma_x) + r(i\gamma)$, the induced curvature evolution is

$$k_t = \frac{(\beta q_x)_x}{\beta^2} + \frac{q}{\beta}\left(\frac{\beta_x}{\beta}\right)_x + k_x p + \beta(k^2+4)q \quad (*)$$

$\beta = |\gamma_x|$

Rk: If $\beta = 1$ and V preserves $\langle \gamma_x, \gamma_x \rangle$, then (setting $x = s$) $\quad p_s = kq$

$\Rightarrow (*)$ gives $\quad k_t = q_{ss} + (kp)_s + 4q$

A representative of $\mathcal{W}_{4A}$ is $W = \frac{1}{2}k^2\gamma_s + k_s(i\gamma_s) + 2k(i\gamma)$

$$\Rightarrow \quad k_t = k_{sss} + \frac{1}{2}(k^3)_s + 4k_s$$ (mKdV up to Galilean transform)

mKdV hierarchy

- $\exists$ ∞ hierarchy of Hamiltonian vectorfield inducing the mKdV hierarchy at the level of the curvature

Key step curvature flow induced by

$$V = p\gamma_s + q(i\gamma_s) + r(i\gamma) \in T_\gamma P' \quad \text{(unit speed case)}$$

has the form

$$k_t = (R+4)q \qquad R = \partial^2 + \partial k \partial^{-1} u$$

mKdV recursion operator

Then build vector fields recursively.