Self-Bäcklund curves via Lame equation: explicit construction and applications

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joint work with
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Motivation

S.Ulam's problem in flotation theory, problem 19 in the Scottish book:

Is a solid of uniform density which will float in water in every position a sphere?

In $dim \geq 3$–recent progress by D. Ryabogin.

In $dim = 2$:

1. Auerbach, Zindler curves–solutions for density $\rho = 1/2$.
2. Bracho, Montejano, Oliveros first solutions for $\rho \neq 1/2$
3. Wegner curves:

Let $\gamma(s)$ be arc-length parametrization of a closed curve. Wegner constructed by means of elliptic functions the *Explicit* solutions to the floating condition:

$$\boxed{|\gamma(s+\alpha) - \gamma(s)| = 2l = const}$$

Wegner animations 1; and more complicated curves Wegner animations 2

Discovery by Franz Wegner

Wegner found on a physical level, not rigorously, a differential equation for γ, treating α as infinitesimal. Then solved and analyzed the solutions in elliptic functions, and then verified the floating property for the solution.

The solutions found by Wegner solve other interesting problems:

1. Motion of the electron in a radial magnetic field of the magnitude depending quadratically on r.

2. Bicycle curves: Sherlock Holmes and Dr. Watson discuss in view of the two tire tracks of a bicycle, which way the bicycle went. The problem is: Is it possible that tire tracks other then circles or straight lines are created by bicyclists going in both directions?

3. Buckled rings (pressurized elastica) (G. Bor, M. Levi, R. Perline, and S. Tabachnikov) Variational problem of relative extrema of the bending energy $\int k^2 ds$ under the length and area constraints.

4. Magnetic Gutkin billiards (Bialy, Mironov, Shalom)

<u>The goal today</u>: to find an analog of Wegner curves in centro-affine geometry

Centro-affine problem

Let $\gamma(t)$ be a parametrized smooth curve in the affine plane with a fixed area form. The curve is *centro-affine* if the Wronski determinant is constant:

$$[\gamma(t), \gamma'(t)] = 1$$

for all $t \in \mathbb{R}$. We assume that the curves are π-anti-periodic: $\gamma(t+\pi) = -\gamma(t)$ for all t. That is, the curve is closed, centrally symmetric and 2π-periodic. Then

$$\gamma''(t) = p(t)\gamma(t),$$

where $p(t)$ is $\pi-$periodic, called centro-affine curvature (for circle $p = -1$).

Pinkall' (Hamiltonian) flow on the space of centro-affine curves (dot is time derivative)

$$\dot{\gamma} = p\gamma' - \frac{p'}{2}\gamma.$$

Then p evolves according to Korteweg-de Vries equation:

$$\dot{p} = -\frac{1}{2}p''' + 3p'p$$

Tabachnikov: the Bäcklund transformation of the KdV equation can be interpreted as a geometric relation between centroaffine curves.

c-related curves

Centro-affine curves γ, δ are called *c-related* if $[\gamma(t), \delta(t)] = c$;
γ is called *Self-Bäcklund* if

$$\boxed{[\gamma(t), \gamma(t+\alpha)] = c, \quad \alpha \in (0, \pi) \text{ is called rotation number.}}$$

Construct a new centroaffine curve $\delta(t) = f(t)\gamma(t) + g(t)\gamma'(t)$, where $f(t)$ and $g(t)$ are π-periodic functions.

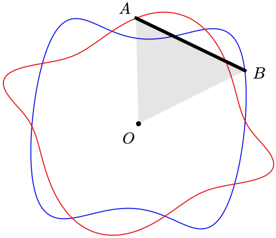


Figure: Area of the shaded triangle OAB remains constant.

One can immediately check that γ and δ are c-related iff $g(t) = c$ and the Riccati eq. holds $\boxed{cf'(t) - f^2(t) + c^2 p(t) + 1 = 0. \ (R)}$

Hill equation

The Riccati equation (R) is intimately related to the Hill equation

$$\boxed{y'' + (\lambda - p(t))y = 0. \ (H)}$$

Proposition

The Riccati equation (R) with a π-periodic $p(t)$ admits a π-periodic solution $f(t)$ for a parameter value $c \neq 0$ if and only if the Hill equation (H) admits a positive π-quasi-periodic solution $y(t)$ for $\lambda = -1/c^2$ and $f = -cy'/y$.

Recall that a solution $y(t)$ of (H) is called *π-quasi-periodic* if $y(t+\pi) = \mu\, y(t)$ for all t and some $\mu \neq 0$, called the *Floquet multiplier* of $y(t)$. If $\mu = 1$ then the solution is π-periodic and if $\mu = -1$ it is *π-anti-periodic.*

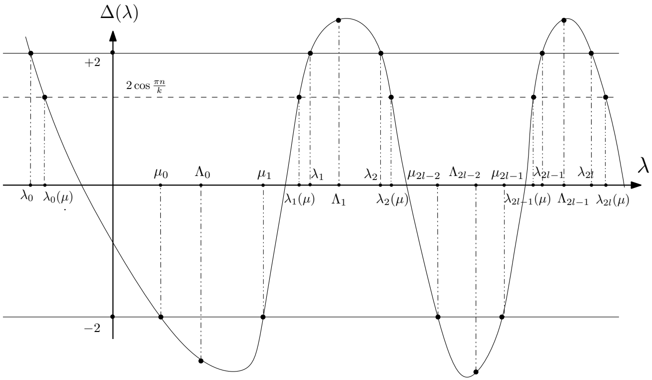
Theorem 1 [Liapunoff, Haupt] (Spectrum of the Hill operator).
$\exists \lambda_k, \mu_k \to +\infty, k = 0, 1, ...$

$$\boxed{\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \dots}$$

such that a non-trivial π-periodic solution exists iff $\lambda = \lambda_k$, and a π-anti-periodic non-trivial solution if $\lambda = \mu_k,\ k = 0, 1, \dots$. The number of zeros on $[0, \pi)$ for λ_{2k-1} or λ_{2k} is $2k$. In particular, no zeros means $\lambda = \lambda_0$. The number of zeros on $[0, \pi)$ for μ_{2k} or μ_{2k+1} is $2k+1$. Moreover, a solution to equation is unstable (that is, unbounded) if and only if λ belongs to one of the intervals $(-\infty, \lambda_0),\ (\mu_0, \mu_1),\ (\lambda_1, \lambda_2), \dots$ (instability intervals, or 'gaps').

Spectra

Graph of the function $\Delta(\lambda) := y_1(\lambda, 2\omega) + y_2'(\lambda, 2\omega)$, where $y_1(\lambda, t), y_2(\lambda, t)$ are the basic solutions of equation with
$y_1(\lambda, 0) = y_2'(\lambda, 0) = 1,\ y_1'(\lambda, 0) = y_2(\lambda, 0) = 0$; the positions of the periodic (λ_n), anti-periodic (μ_n), Dirichlet (Λ_n),
and Floquet $(\lambda_n(\mu)$ where $\mu = e^{i\pi n/k})$ eigenvalues are indicated.



Range of c

Given a centroaffine closed π-anti-periodic curve $\gamma(t)$, what is the range of the parameter c for which γ admits a closed centroaffine c-related curve $(\delta(t) = f(t)\gamma(t) + c\gamma'(t))$?
Thus one needs to find a π-periodic solution $f(t)$ to the Riccati equation (R) $cf' - f^2 + c^2 p(t) + 1 = 0$. By Proposition this happens if and only if the Hill equation (H) admits a positive π-quasi-periodic solution $y(t)$. We have:

Theorem 2 *Let γ be a centroaffine π-anti-periodic curve. Then $\lambda_0 < 0$. Moreover γ admits a c-related closed curve if and only if $|c| \leq 1/\sqrt{-\lambda_0}$.*

This is a centroaffine analog of Menzin's conjecture for hatchet planimeters (equivalently, bicycle monodromy) [Levi-Tabachnikov].

Moreover, $\lambda_0 \leq -P := \frac{1}{\pi}\int_0^\pi p(t)dt$ (Borg theorem). If γ is locally convex, so that $p(t)$ is strictly negative, then $P > 0$ and we have $\lambda_0 \leq -P < 0$. The geometric meaning of P is the area bounded by the dual curve γ^*.

Corollary

Suppose $P > 0$ (for example γ is locally convex) and γ admits a c-related π-anti-periodic closed curve. Then $|c| \leq 1/\sqrt{P}$.

Proof of Theorem 2.

Each component of γ is a non-trivial π-anti- periodic solution of equation (H) for $\lambda = 0$. This implies that $\mu_k = 0, k \geq 1$, hence $\lambda_0 < 0$.

We claim (H) admits a π-quasi-periodic positive solution $\Leftrightarrow$ $\lambda \leq \lambda_0$.

($\Leftarrow$.) If $\lambda = \lambda_0$ then equation (H) has a positive periodic solution, hence quasi-periodic. So we shall assume now that $\lambda < \lambda_0$. In this case equation (H) has no conjugate points, that is, a non-trivial solutions vanishing at two distinct points t_1, t_2 because, by the Sturm Comparison Theorem, any solution for every larger λ must have a zero between t_1, t_2. But for λ_0 there is a positive periodic solution. We complete the proof by the following
Lemma [after E. Hopf.] $y'' + q(t)y = 0$, *where* $q(t+\pi) = q(t)$, *has no conjugate points if and only if it admits a positive* π*-quasi-periodic solution.*

($\Leftarrow$.) Let us show that (H) admits no positive π-quasi-periodic solution for $\lambda > \lambda_0$. Indeed if $y(t)$, $y(t+\pi) = \mu\, y(t)$, then $\mu > 0$. Two possibilities:

1. If $\mu = 1$ then $y(t)$ is a positive periodic solution. But this is possible only for $\lambda = \lambda_0$, a contradiction.
2. If $\mu \neq 1$ then the solution $y(t)$ is unbounded, and hence λ belongs to one of the instability zones. In particular, $\lambda > \mu_0$. Then, by the Sturm Comparison Thm, $y(t)$ cannot be positive since solutions for μ_0 have zeroes.

$\square$

Self Bäcklund infinitesimal deformations of conics

Centroaffine ellipse is Self Bäcklund for any $\alpha \in (0, \pi)$.
An *infinitesimal deformation* of γ is a formal expression $\tilde{\gamma} = \gamma(t) + \epsilon\gamma_1(t)$, satisfying self Bäcklund condition modulo ϵ^2, for some $\tilde{\alpha} = \alpha + \epsilon\alpha_1$, $\tilde{c} = c + \epsilon c_1$.

Theorem 3 *Let $\gamma(t) = (\cos t, \sin t)$. Then*
1. A non-trivial infinitesimal deformation of γ within the class of self-Bäcklund π-anti-periodic centroaffine curves exists if and only if $\tilde{\alpha} = \alpha + \epsilon\alpha_1$ where $\alpha = \pi/2$, or $\alpha \neq \pi/2$ and α satisfies for some integer $k \geq 4$ the equation

$$\boxed{\tan(k\alpha) = k \tan \alpha}.$$

2. For $k \geq 2$, there are exactly $k - 2$ solutions of equation in the interval $(0, \pi)$, counting also $\alpha = \pi/2$ as a solution for k odd.

Remark Equation appeared in the context of bicycle kinematics in [Tabachnikov06]; [Bor-Levi-Perline-Tabachnikov]; in the papers by Wegner, summarized in [Wegner07]. It also appeared in [Gutkin] in the context of billiards and flotation problems, and in [Bialy-Mironov-Shalom 20,21], [Bialy-Mironov-Tabachnikov] for magnetic, outer and wire billiards. This ubiquitous equation has a countable number of solutions but, except for $\pi/2$, there are no π-rational solutions [Cyr].

Values of $\frac{\pi}{3}, \frac{\pi}{4}$

Theorem 4 *Let $\gamma(t)$ be a π-anti-periodic self-Bäcklund centroaffine curve, that is, $[\gamma(t), \gamma(t+\alpha)] = c \neq 0$. If $\alpha = \pi/3$ or $\alpha = \pi/4$ then γ is a centroaffine ellipse.*

Proof.

For the case $\alpha = \frac{\pi}{3}$ set $\gamma(t) = \gamma_0, \quad \gamma\left(t + \frac{\pi}{3}\right) = \gamma_1, \quad \gamma\left(t + \frac{2\pi}{3}\right) = \gamma_2$. Then

$$[\gamma_0, \gamma_1] = [\gamma_1, \gamma_2] = [\gamma_2, -\gamma_0] = c,$$

hence $[\gamma_0, \gamma_2] = [\gamma_0, \gamma_1]$, and the vector $\gamma_1 - \gamma_2$ is collinear with γ_0. Likewise, $\gamma_2 + \gamma_0$ is collinear with γ_1, and $\gamma_1 - \gamma_0$ with γ_2. We write

$$\gamma_1 - \gamma_2 = k_0\gamma_0, \quad \gamma_2 + \gamma_0 = k_1\gamma_1, \quad \gamma_1 - \gamma_0 = k_2\gamma_2.$$

It is easy to see that $k_0 = k_1 = k_2 = 1$. Thus $\gamma_2 = \gamma_1 - \gamma_0$.

It follows that $\gamma_2' = \gamma_1' - \gamma_0'$, and hence

$$1 = [\gamma_2, \gamma_2'] = [\gamma_1 - \gamma_0, \gamma_1' - \gamma_0'] = 2 - [\gamma_0, \gamma_1'] + [\gamma_0', \gamma_1].$$

Since $[\gamma_0, \gamma_1] = c$, one has $[\gamma_0', \gamma_1] + [\gamma_0, \gamma_1'] = 0.$ This implies that

$$[\gamma_0, \gamma_1'] = \frac{1}{2},\ [\gamma_0', \gamma_1] = -\frac{1}{2},$$

and hence $\gamma_1 = (1/2)\gamma_0 + c\gamma_0'$.
It follows that in the Riccati equation (R) one has $f = 1/2$, and hence, $c^2 p = -3/4$. That is, p is constant, which implies $p = -1$ and $c = \sqrt{3}/2$, and therefore the curve is a centroaffine conic.

Remark An analogous result, rigidity for periods 3 and 4, holds for bicycle curves, see [Tab06].

Radon curves, $\alpha = \frac{\pi}{2}$

There is a *functional* freedom for $\alpha = \frac{\pi}{2}$. For Ulam's floating problem this is analogous to the density $1/2$ (was known to Auerbach and Zindler).

Let Γ be a smooth closed convex centrally symmetric curve. Let $x, y \in \Gamma$. One says that x is Birkhoff orthogonal to y, ($x \perp_B y$), if y is parallel to the tangent line to Γ at x. Γ is called a *Radon curve*, if this relation is symmetric. Radon curves comprise a functional space, with ellipses providing a trivial example (conjugate diameters), see [Martini-Swanepoel] for a modern treatment.

Let Γ be a Radon curve, $x \in \Gamma$ be a point, and $y \in \Gamma$ be its Birkhoff orthogonal. Then the tangent lines at points $x, y, -x, -y$ form a parallelogram circumscribed about Γ. Area of the parallelogram is constant–<u>equiframe property</u>. As x traverses Γ, the vertices of the parallelogram describe a curve γ. Then γ is an invariant curve of the outer billiard transformation about Γ.

The relation of self-Bäcklund curves with Radon curves is as follows. Let γ be a self-Bäcklund curve with rotation number $\pi/2$, then the points $\gamma(t), \gamma(t+\pi/2), \gamma(t+\pi), \gamma(t+3\pi/2)$ form a parallelogram of constant area $2c$. Therefore the middle curve Γ is a Radon curve.

Thus we have for $\alpha = \pi/2$:

$$\boxed{\text{Self Backlund} \quad \Rightarrow \quad \text{Radon} \quad \Leftrightarrow \quad \text{Equiframe property.}}$$

A Picture of Radon curve

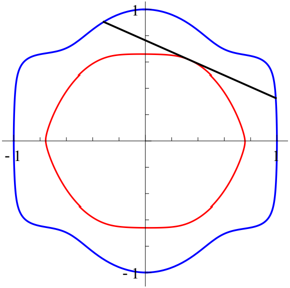


Figure: A self-Bäcklund curve with rotation angle $\alpha = \pi/2$ and $c = 1$.

We shall construct below analytic families of Radon curves.

Pictures of Self Bäcklund curves

Self-Bäcklund curves

Wegner ansatz

Emulating Wegner's approach, fix a small ϵ and consider the curves $\Gamma_\pm = \gamma \pm \epsilon\gamma'$. These curves are 2ϵ-related. We want them to be obtained from the same curve, Γ, by rotating it through small angles $\pm\delta$. The assumption is that δ is of order ϵ^3; all the calculations are mod ϵ^4...

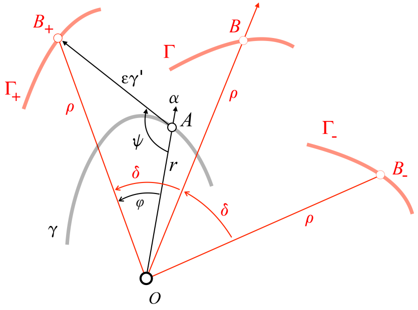


Figure: $r = |OA|,\ \rho = |OB_-| = |OB| = |OB_+|,\ \varphi = \angle AOB_+,\ \psi = \angle OAB_+,\ \delta = \angle BOB_+ = \angle B_-OB$. Curves γ and Γ are given in polar coordinates by $r(\alpha)$ and $\rho(\beta)$.

This ansatz gives...

$$\Gamma(t) = (R(t)^{1/2} \cos \alpha(t), R(t)^{1/2} \sin \alpha(t)) \tag{1}$$

where:

$$R'^2 = aR^3 + bR^2 + cR - 4. \tag{2}$$

Thus $R(t) = r^2(t)$ is an elliptic function. The curve is given by a parametric equation
with R as in equation (2) and $\alpha' = R^{-1}$. (If the curve is a centroaffine ellipse, one has $a = 0$ in equation (2).)

Lemma

One has $p(t) = \frac{1}{2}aR(t) + \frac{1}{4}b$. *Renaming the constants again, we obtain from equation* (2)

$$\boxed{p'^2 = 2p^3 + ap^2 + bp + c.}$$

Usual curvature of the curve $k = \frac{-8p(t)}{\sqrt{aR^2+bR+c}^3} = -\frac{4aR+2b}{(aR^2+bR+c)^{\frac{3}{2}}}$.
Thus the curvature is a function of the distance from the origin. This is a special class of curves. One can think of these curves as the trajectories of a charge in a rotationally symmetric magnetic field whose strength is a function of the distance from the origin. Note that Wegner's curves also have this property: their curvature satisfies $k = ar^2 + b$, where a, b are constants.

Comparing the equation on p to the equation satisfied by the Weierstrass $\wp$ function,

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \qquad \wp(z) := \frac{1}{z^2} + \sum_{\lambda\in\Lambda'}\left[\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right].$$

we conclude that $p(t)$ is given in terms of $\wp$ by

$$p(t) = 2\wp(t+\omega') + C. \tag{3}$$

Here $\wp$ is the Weierstrass function with half periods ω, ω', where the first one is real and the second one is pure imaginary. Since $p(t)$ needs to be periodic, we are in the case of three real roots $e_1 > e_2 > e_3$. Write $C = \wp(a)$ for some $a \in \mathbb{C}$. Thus

$$p(t) = 2\wp(t+\omega') + \wp(a). \tag{4}$$

Write our curve in complex form $X(t) = x(t) + iy(t)$, satisfying

$$\boxed{X'' + (-\wp(a) - 2\wp(t+\omega'))X = 0,} \tag{5}$$

which is precisely the Lamé equation [Akhiezer] (more general case–$n(n+1)$ instead of 2).

To construct a centroaffine π-anti-symmetric curve, we shall impose the Requirements:

1. The Wronskian $[X, X'] = 1$. This can be achieved by rescaling.
2. $\omega = \pi/2k$ for some integer $k \geq 2$, so that p is π/k-periodic.
3. The solution X is rotated over the period 2ω by $\pi n/k$, where $0 < n < k$ is odd and co-prime to k, so that after k periods we have $X(t+\pi) = -X(t)$. In other words, we require $X(t)$ to be a complex 2ω-quasiperiodic solution of equation (5), with Floquet multiplier $\mu = e^{i\pi n/k}$:

$$X(t + 2\omega) = X(t)e^{i\pi n/k}.$$

A basis X_+, X_- for the solutions of the Lamé equation (5) can be written in the following form (see [Akhiezer]):

$$\boxed{X_\pm(t) = e^{-t\zeta(\pm a)} \frac{\sigma(\pm a + t + \omega')\sigma(\omega')}{\sigma(\pm a + \omega')\sigma(t + \omega')},} \tag{6}$$

here and below ζ, σ are the Weierstrass zeta and sigma functions.

$\zeta'(z) = -\wp(z),\ \zeta$ has simple poles at Λ,
$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i,$ quasi-periodic, where $\eta_i := \zeta(\omega_i),\ i = 1, 2, 3.$
$\frac{\sigma'}{\sigma} = \zeta, \sigma$ is entire quasi-periodic, $\sigma(z + 2\omega_i) = -e^{2\eta_i(z+\omega_i)}\sigma(z)$, where
$\omega_1 = \omega,\ \omega_2 = -(\omega + \omega'),\ \omega_3 = \omega', \eta_i = \zeta(\omega_i).$

The construction of the self-Bäcklund curves is this section requires a careful choice of the parameter a in equation (5).

Proposition For every $a \in (0, \omega') \cup (\omega, \omega + \omega')$,

1. $\wp(a)$ is real, hence also the potential $2\wp(t + \omega') + \wp(a)$ in the Lamé eq.
2. $X_+(t)$ is regular: $X'_+(t) \neq 0$; $X_+(0) = 1$ and $X'_+(0) = ib, b \in \mathbb{R}, \ b > 0$.
3. $X_+(t)$ is locally star-shaped and positively oriented:

$$[X_+(t), X'_+(t)] = const > 0.$$

4. $X_+(t + 2\omega) = X_+(t)e^{2f(a)}$, where

$$\boxed{f(a) := a\zeta(\omega) - \omega\zeta(a).}$$

   So, $X_+(t)$ is a 2ω-quasiperiodic , with a Floquet multiplier $\boxed{\mu = e^{2f(a)}.}$

5. The function f of the previous item satisfies the identities

$$f(-a) = -f(a), \ f(a + 2\omega) = f(a), \ f(a + 2\omega') = f(a) + i\pi.$$

Thus, due to requirement 3 and Proposition (item 4), we need to solve $2f(a) \equiv i\pi n/k$ (mod $2\pi i$), or $\boxed{f(a) = \frac{i\pi n}{2k} + i\pi m,}$ for some $m, n \in \mathbb{Z}$, where n is odd, relatively prime to k, and $0 < n < k$.

Theorem 5 *Consider the equation for fixed integers k, n, where $k \geq 2$ and n is odd, relative prime to k, and $0 < n < k$. Then*

1. For every $m \geq 0$ there is a unique solution a_m of $\boxed{f(a) = \frac{i\pi n}{2k} + i\pi m,}$ such that $\underline{a_m \in (0, \omega') \text{ for } m > 0}$, and $\underline{a_0 \in (\omega, \omega + \omega'), \text{ for } m = 0}$.
2. The sequence $\lambda_m(\mu) := -\wp(a_m)$ is strictly monotone increasing and, in particular, the value $\lambda_0(\mu) = -\wp(a_0)$ is the smallest one.

Theorem 6 *For each k, m, n as in above, consider the curve X_+ determined by the value a_m.*

1. X_+ is locally star-shaped π-antisymmetric curve, with winding number
$$\mathrm{w} = 2k \left\lceil \frac{m}{2} \right\rceil + n.$$
2. X_+ is embedded (simple) if and only if $m = 0, n = 1$.
3. The curve X_+ satisfies the self-Bäcklund property $[X_+(t), X_+(t + \alpha)] = const$ for a value of the parameter $\alpha \in (0, \pi)$ if and only if
$$\boxed{\sigma(a + \alpha) = e^{2\alpha\zeta(a)}\sigma(a - \alpha).} \quad (7)$$

Proof of self-Bäcklund property

Set $\beta = \alpha/2$. Then self-Bäcklund property reads

$$Im\left(X_+(t+\beta)\overline{X_+(t-\beta)}\right) = c,$$

where overline denotes the complex conjugation. We can rewrite this equation as

$$X_+(t+\beta)X_-(t-\beta) - X_-(t+\beta)X_+(t-\beta) = 2c.$$

Next we substitute in the last equation the expressions for $X_\pm$ from equation (6):

$$\begin{aligned} 2c =& e^{-(t+\beta)\zeta(a)}\frac{\sigma(a+t+\beta+\omega')\sigma(\omega')}{\sigma(a+\omega')\sigma(t+\beta+\omega')}e^{(t-\beta)\zeta(a)}\frac{\sigma(-a+t-\beta+\omega')\sigma(\omega')}{\sigma(-a+\omega')\sigma(t-\beta+\omega')} - \\ &- e^{(t+\beta)\zeta(a)}\frac{\sigma(-a+t+\beta+\omega')\sigma(\omega')}{\sigma(-a+\omega')\sigma(t+\beta+\omega')}e^{-(t-\beta)\zeta(a)}\frac{\sigma(a+t-\beta+\omega')\sigma(\omega')}{\sigma(a+\omega')\sigma(t-\beta+\omega')}. \end{aligned}$$

Using the addition theorem $\wp(z) - \wp(w) = -\frac{\sigma(z-w)\sigma(z+w)}{\sigma^2(z)\sigma^2(w)}$ we get

$$\begin{aligned} 2c =& e^{-2\beta\zeta(a)}\frac{\left[\wp(t+\omega') - \wp(a+\beta)\right]\sigma^2(a+\beta)\sigma^2(\omega')}{\left[\wp(t+\omega') - \wp(\beta)\right]\sigma^2(\beta)\sigma(a+\omega')\sigma(-a+\omega')} - \\ &- e^{2\beta\zeta(a)}\frac{(\wp(t+\omega') - \wp(a-\beta))\sigma^2(a-\beta)\sigma^2(\omega')}{\left[\wp(t+\omega') - \wp(\beta)\right]\sigma^2(\beta)\sigma(a+\omega')\sigma(-a+\omega')}. \end{aligned}$$

Multiplying by the common denominator and renaming the constant,

$$\tilde{c} := 2c\sigma^2(\beta)\sigma(a+\omega')\sigma(-a+\omega')/\sigma^2(\omega'),$$

we get

$$\begin{aligned}\tilde{c}\left[\wp(t+\omega')-\wp(\beta)\right] =& e^{-2\beta\zeta(a)}\left[\wp(t+\omega')-\wp(a+\beta)\right]\sigma^2(a+\beta)- \\ & - e^{2\beta\zeta(a)}\left[\wp(t+\omega')-\wp(a-\beta)\right]\sigma^2(a-\beta).\end{aligned}$$

Thus we must have

$$\begin{aligned}&\tilde{c} = e^{-2\beta\zeta(a)}\sigma^2(a+\beta) - e^{2\beta\zeta(a)}\sigma^2(a-\beta)\\ &\wp(\beta)\tilde{c} = e^{-2\beta\zeta(a)}\wp(a+\beta)\sigma^2(a+\beta) - e^{2\beta\zeta(a)}\wp(a-\beta)\sigma^2(a-\beta).\end{aligned}$$

Substituting $\tilde{c}$ from the first identity into the second and simplifying, we get

$$\sigma^2(a+\beta)\left[\wp(a+\beta)-\wp(\beta)\right] = e^{4\beta\zeta(a)}\sigma^2(a-\beta)\left[\wp(a-\beta)-\wp(\beta)\right].$$

Using equation addition formula again, we obtain $\sigma(a+\alpha) = e^{2\alpha\zeta(a)}\sigma(a-\alpha)$, as needed. Moreover we have:

Theorem 7 *For integers k, m, n, where $k \geq 2$ and n is odd, relative prime to k, and $0 < n < k$, the associated curve X_+ satisfies the self-Bäcklund property $[X_+(t), X_+(t+\alpha)] = const$ for $k-2$ values of $\alpha \in (0,\pi)$ which are the solutions of* $\boxed{\sigma(a+\alpha) = e^{2\alpha\zeta(a)}\sigma(a-\alpha)}$ *on $(0,\pi)$.*

Important example

Important special case $m = 0, n = 1, \alpha = \pi/2$. We have an infinite family of self-Bäcklund simple closed curves with $\alpha = \pi/2$, but now we have an analytic examples.

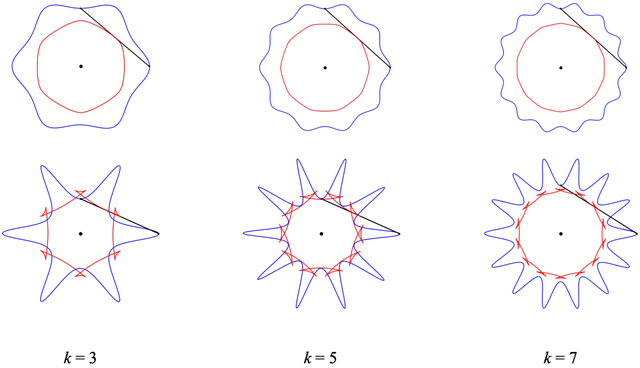


Figure: Self-Bäcklund simple curves $X_+(t)$ (blue) with $2k$-fold symmetry, $k = 3, 5, 7$, $\alpha = \pi/2$. The red curve is traced by the midpoint of the line segment $X_+(t)X_+(t + \pi/2)$ and is tangent to it. For large ω', the red curve is smooth and convex (is a Radon curve); for smaller ω'– cusps appear

Self-Bäcklund curves as deformations of conics

It turns out that the self-Bäcklund curves constructed above can be obtained as a genuine non-trivial deformations of a central conic. Recall, for every integer $k \geq 3$ and $\omega' \in i\mathbb{R}_+$ one considers the Weierstrass $\wp$-function with half periods $\omega = \pi/2k$, ω', the associated σ- and ζ-functions and the (unique) solution $a \in (\omega, \omega + \omega')$ to $a\zeta(\omega) - \omega\zeta(a) = i\omega$, then set

$$\boxed{Y(t) := X(t)/N,\ X(t) := \frac{\sigma(a+t+\omega')\sigma(\omega')}{\sigma(a+\omega')\sigma(t+\omega')} e^{-t\zeta(a)},\ N := \sqrt{|X'(0)|}.}$$

Remark

The normalization factor $N = \sqrt{|X'(0)|} = \sqrt{\frac{\wp'(a)}{2i(\wp(a)-e_3)}}$, *is introduced so as to render the normalized curve* Y *centroaffine and* π*-anti-periodic (enclosing area* π*).*

The deformation of the unit circle is obtained by fixing k and letting $\omega' \to \infty$ in the above construction. Namely we set $\boxed{\omega = \pi/2k,\ \omega' = i/s, s \in (0,1]}$, and use henceforth the subscript s to denote all associated objects $\wp_s, \sigma_s, \zeta_s, a_s, X_s, N_s$ and Y_s.

Theorem 8

For each integer $k \geq 3$,

1. *The family of curves* $Y_s(t)$, $s \in (0,1]$, *for* $\omega = \pi/2k,\ \omega' = i/s$, *extends smoothly to* $s \in [0,1]$ *by setting* $Y_0(t) := e^{it}$.
2. *Each curve* $Y_s(t)$ *is a centroaffine* π*-anti-periodic simple curve with* $2k$*-fold symmetry,* $Y_s(t+\pi/k) = Y_s(t)e^{i\pi/k}$, *self-Bácklund for* $s > 0$ *with respect to* $k-2$ *rotation numbers* $\alpha \in (0,\pi)$, *varying smoothly in* $s \in [0,1]$ *and converging as* $s \to 0$ *to the* $k-2$ *solutions of equation* $\tan(k\alpha) = k\tan\alpha$.
3. *The deformation* Y_s, $s \in [0,1]$, *is analytic away from* $s = 0$ *but not at* $s = 0$. *In fact, one has* $(\partial_s)^n|_{s=0} Y_s(t) = 0$, $n \geq 1$, *so the associated infinitesimal deformation of the unit circle vanishes to all orders, yet the deformation itself is non-trivial.*
4. *The change of parameter,* $\epsilon := \begin{cases} e^{-2k/s}, & s > 0, \\ 0, & s = 0, \end{cases}$ *gives a deformation* Y_ϵ *of the unit circle* Y_0, *analytic in* $\epsilon \in [0, e^{-2k}]$.
5. *The infinitesimal deformation associated with the analytic deformation* Y_ϵ *is non-trivial. That is,*

$$Y_\epsilon(t) = e^{it} + Y_1(t)\epsilon + O(\epsilon^2),$$

where Y_1 *is non-vanishing.*

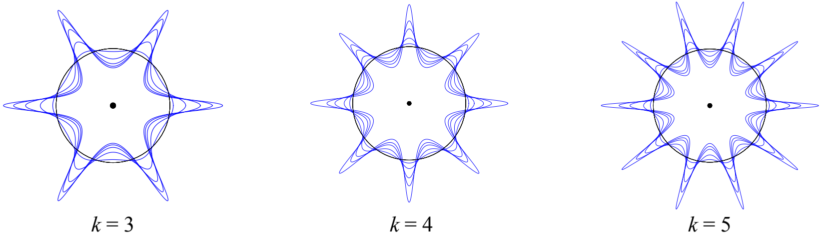


Figure: Three families of deformations of the circle (black) through a 1-parameter family of centroaffine self-Bäcklund curves Y_s (blue) with $2k$-fold symmetry, $k = 3, 4, 5$.

Main idea: functions $X_s, s \in [0, 1]$, are suitably normalized Floquet eigenfunctions of a Hill operator depending smoothly on s.
Then use a general argument of smooth dependence of the eigenfunctions of a Hill operator depending on the smooth parameter. Similarly, when replacing s with ϵ the Hill operator depends analytically on ϵ and so do its eigenfunctions.

In more detail, we recall that X_s, $s \in (0,1]$, is precisely the eigenfunction corresponding to the smallest eigenvalue $\lambda_{0,s}$ for the Floquet problem

$$X'' + (\lambda - 2q_s(t))X = 0,\ X(t + \pi/k) = \mu X(t),\ \mu = e^{i\pi/k}, \qquad (8)$$

where $q_s(t) = \wp(i/s + t)$ and X_s satisfy the normalization condition $X_s(0) = 1$. Moreover, we showed that $\lambda_{0,s} = -\wp_s(a_s)$, where $a_s \in (\omega, \omega + \omega')$ is uniquely defined.

Lemma

The function $\boxed{q_s(t) := \wp_s(t + i/s), s \neq 0,\ \text{and}\ q_0(t) := -k^2/3,}$ *depends smoothly on* $(s,t) \in [0,1] \times \mathbb{R}$*. Moreover, The change of parameter* $s \mapsto \epsilon$ *of equation* (4) *transforms the deformation* q_s *to* q_ϵ *which is real analytic in* $\epsilon \in [0, e^{-2k}]$*, with Taylor series* $q_\epsilon = -\frac{k^2}{3} - 8k^2\cos(2kt)\epsilon + O(\epsilon^2)$.

Lemma

The eigenfunctions $X_s(t)$*,* $s \in [0,1]$*, corresponding to the first eigenvalue* $\lambda_{0,s}$ *of the Floquet problem* (8)*, are uniquely determined by the condition* $X_s(0) = 1$ *and are smooth (analytic) in* s *if the potential* q_s *is smooth (analytic) in* s*.*

Remarks

Lemma

For every $s \in [0,1]$ the curves Y_s are self-Bäcklund for $k-2$ values of $\alpha_s \in (0,\pi)$, satisfying

$$\frac{\sigma_s(a_s+\alpha_s)}{\sigma_s(a_s-\alpha_s)} = e^{2\alpha_s \zeta_s(a_s)}. \tag{9}$$

All $k-2$ solutions α_s depend smoothly on $s \in [0,1]$. For $s=0$ this equation reduces to $k\tan(\alpha) = \tan(k\alpha)$. Moreover, the $k-2$ families α_ϵ are analytic in $\epsilon \in [0, e^{-2k}]$.

Lemma

X_ϵ has a Taylor series in ϵ: $X_\epsilon(t) = e^{it} + X_1(t)\epsilon + O(\epsilon^2)$, where X_1 is non-vanishing.

Question

It would be interesting to construct self Bäcklund curves via other finite zone potentials.

Thank you.