Billiard tables with rotational symmetry

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Joint work with Misha Bialy

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Motivation

The goal of this talk is to generalize, in various ways, the following simple claim:

Claim

Every centrally symmetric planar convex body of constant width is a disk.

And we will see the relation to the following fact:

Theorem (Cyr)

For $2 \leq n \in \mathbb{N}$, *the equation* $\tan(nx) = n\tan(x)$ *has no solutions for* $x \in \pi\mathbb{Q} \setminus \pi\mathbb{Z}$.

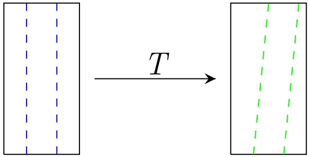
Outline

1. Introduction - billiards and twist maps
2. Formulation of the main results
3. Ideas of proofs

Introduction to twist maps of cylinders

$\mathcal{A} = S^1 \times \mathbb{R}$ - cylinder. $T : \mathcal{A} \to \mathcal{A}$ - area preserving diffeomorphism.

- $T(q,p) = (Q(q,p), P(q,p))$ is called a **twist map**, if $\frac{\partial Q}{\partial p} \neq 0$ (**twist condition**).
- $(\tilde{Q}(\tilde{q},p), P(\tilde{q},p)) = \tilde{T}(\tilde{q},p)$ - lift of T to $\mathbb{R}^2$.
- Alternative formulation of the twist condition: for every fixed $\tilde{q}_0 \in \mathbb{R}$, the function $p \mapsto \tilde{Q}(\tilde{q}_0, p)$ is monotone.



Generating functions of twist maps

- The fact that T is area preserving means that $dP \wedge dQ = dp \wedge dq \iff d\Big(PdQ - pdq\Big) = 0.$

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- We say that T is an **exact twist** map if there exists a function $S : \mathbb{R}^2 \to \mathbb{R}$ (**generating function**) for which $Pd\tilde{Q} - pd\tilde{q} = dS$, and $S(\tilde{q}+1, \tilde{Q}+1) = S(\tilde{q}, \tilde{Q})$.

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- Twist condition is equivalent to $\partial_1\partial_2 S$ having a constant sign. (∂_i denotes partial derivative w.r.t i^{th} variable)
- A twist map can be defined using a generating function, by setting

$$\tilde{T}(\tilde{q}, p) = (\tilde{Q}, P) \iff \begin{cases} p = -\partial_1 S(\tilde{q}, \tilde{Q}) \\ P = \partial_2 S(\tilde{q}, \tilde{Q}) \end{cases}$$

Variational setting

- A sequence $\{\tilde{q}_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ is called a **configuration** of $\tilde{T}$ if there exist $\{p_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ such that $(\tilde{q}_n, p_n) = \tilde{T}^n(\tilde{q}_0, p_0)$.

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- A sequence $\{\tilde{q}_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ is a configuration if and only if for all $M < N \in \mathbb{Z}$, the point $(\tilde{q}_{M+1}, ..., \tilde{q}_{N-1})$ is a critical point of the **action functional**

$$F_{M,N}(x_{M+1}, ..., x_{N-1}) = S(\tilde{q}_M, x_{M+1}) + \sum_{n=M+1}^{N-2} S(x_n, x_{n+1}) + S(x_{N-1}, \tilde{q}_N).$$

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- A configuration is called **minimizing** if every finite segment is a local minimum of the action functional.

Important theorems about twist maps

A **rotational invariant curve** α of a twist map T is a curve $\alpha \subseteq \mathcal{A}$ which is not contractible, and for which $T(\alpha) = \alpha$.

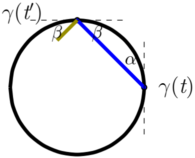
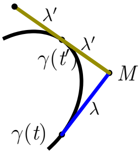
Theorem (Birkhoff's Theorem)

Every rotational invariant curve of an area preserving twist map is the graph of a Lipschitz function $f\colon S^1 \to \mathbb{R}$.

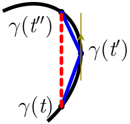
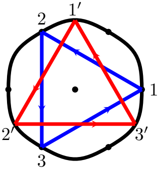
Theorem

Let $T\colon \mathcal{A} \to \mathcal{A}$ *be an exact twist map, and let* α *be an invariant curve of* T*. Then any orbit* $\{\tilde{q}_n\}_{n\in\mathbb{Z}}$ *that is associated to any point of* α *is minimizing.*

Billiards as twist maps

Billiard type	Phase cylinder and generating function	Picture
Birkhoff	$\{(t, \cos\alpha) \mid \alpha \in (0, \pi)\}$ t - arc length parameter on γ $S(t, t') = \lvert\gamma(t) - \gamma(t')\rvert$	
Outer	$\{(t, \lambda) \mid \lambda > 0\}$ $M = \gamma(t) + \lambda\dot{\gamma}(t) = \gamma(t') - \lambda'\dot{\gamma}(t')$ $S(t, t') = \text{area}(\text{conv}(\gamma \cup \{M\}))$ Partial derivatives of S are related to λ and λ'.	

Billiards as twist maps - continuation

Billiard type	Phase cylinder and generating function	Picture
Symplectic	$\{(t,s)\}$ $T(t,s)=(t',s') \Longleftrightarrow \begin{cases} s=-\det(\dot\gamma(t),\gamma(t')), \\ s'=\det(\gamma(t),\dot\gamma(t')) \end{cases}$ $S(t,t')=\det(\gamma(t),\gamma(t'))$	
Minkowski w.r.t norm N	$\{(t,s)\}$ $T(t,s)=(t',s') \Longleftrightarrow \begin{cases} s=-dN_{\gamma(t')-\gamma(t)}(-\dot\gamma(t)) \\ s'=dN_{\gamma(t')-\gamma(t)}(\dot\gamma(t')) \end{cases}$ $S(t,t')=N(\gamma(t')-\gamma(t))$	

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- We will consider higher order symmetry, and similar setting for other billiard systems.
- These results do not assume foliation by invariant curves, but rather the existence of one special invariant curve.

Formulation of the results

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Theorem 1

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Theorem 2

If γ is invariant under an order $k \geq 3$ element of $\mathrm{GL}(2, \mathbb{R})$, and either the Outer or Symplectic billiard map of γ have an invariant curve of k periodic orbits, then γ is an ellipse.

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Theorem 3

*Let γ be a curve invariant under an order $k \geq 3$ element of $\mathrm{GL}(2, \mathbb{R})$. The Minkowski billiard map of γ, **with the norm induced by γ**, has an invariant curve of k periodic orbits, if and only if γ is invariant under an element of $\mathrm{GL}(2, \mathbb{R})$ of order ak where*

$$\begin{cases} a = 1\,, \text{ if } k \equiv 2 \pmod 4\,, \\ a = 2\,, \text{ if } k \equiv 0 \pmod 4\,, \\ a = 4\,, \text{ if } k \equiv 1 \pmod 2\,. \end{cases}$$

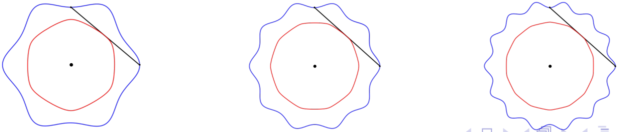
Corollaries

- Given a (symmetric) norm $\|\cdot\|$ on $\mathbb{R}^2$, and two unit vectors x, y, say that y is *Birkhoff orthogonal* to x if $t = 0$ is the local minimum of the function $t \mapsto \|x + ty\|$.
- The norm is called *Radon* norm if the Birkhoff orthogonality relation is symmetric.
- It can be shown that the unit circle of a Radon norm has an invariant curve of 4 periodic billiard orbits for the Outer billiard map.

Corollary

A Radon norm in $\mathbb{R}^2$ which is invariant under a linear map of order four is the Euclidean norm.

Remarkably, non-Euclidean analytic Radon norms having symmetries of order $2k$ for $k \geq 3$ odd have been constructed by Bialy, Bor, and Tabachnikov.



Corollaries

A twist map is said to be *totally integrable in an open set* U of the phase cylinder, if U has C^1 foliation by invariant curves of the twist map. Iterating Theorem 3 gives:

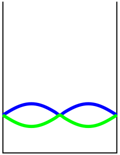
Corollary

Let γ *be a* C^2*-smooth, planar, strictly convex curve which is invariant under a linear map of order* $k \geq 3$. *Consider the Minkowski billiard system in* γ, *with the norm induced by* γ. *If the Minkowski billiard map of* γ *is totally integrable on the entire phase cylinder, then* γ *is a (Euclidean) ellipse.*

Twist map argument used in all proofs

Proposition

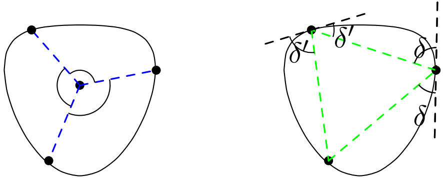
Let T be an exact twist map. Assume that α_1, α_2 are two rotational invariant curves of T, of the same rational rotation number, and that all points of those curves are periodic points of T. Then $\alpha_1 = \alpha_2$.



It is possible to have several invariant curves of the same rotation number (billiard in an ellipse), but not if all points of those curves are periodic.

Conclusion from this argument

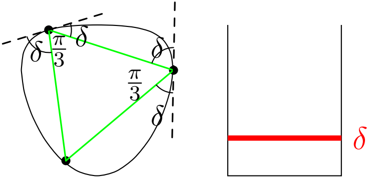
In all four systems, the symmetry of the table transforms the given invariant curve to another invariant curve of k periodic orbits. By the Proposition it must coincide with the original one. Eventually, this means that each k periodic billiard orbit coincides with an orbit of the symmetry.



After that, each type of billiard system is handled differently.

Birkhoff Case - Gutkin Condition

In the case of Birkhoff billiards, we conclude that the billiard orbit forms a regular polygon, and hence the incidence angles of the orbit are constant. As a result the curve satisfies the Gutkin condition with an angle which is a rational multiple of π.



Assuming γ is not a circle, the incidence angle must then satisfy $\tan(n\delta) = n\tan(\delta)$ for some $n \in \mathbb{N}$, but according to a theorem by Cyr, this is impossible for $\delta \in \pi\mathbb{Q} \setminus \pi\mathbb{Z}$.

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- One can choose the parameter t on γ in such a sophisticated way, so that there will be a constant $\lambda > 0$ for which: (Outer billiard law + Proposition)

$$\gamma(t) + \lambda\dot{\gamma}(t) = A\gamma(t) - \lambda A\dot{\gamma}(t),$$

and in addition,

$$A\gamma(t) = \gamma(t + \frac{2\pi}{k}).$$

Outer Billiard argument

- The Outer billiard law then becomes:

$$\lambda(\dot{\gamma}(t) + \dot{\gamma}(t + \frac{2\pi}{k})) = \gamma(t + \frac{2\pi}{k}) - \gamma(t)\,.$$

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- Write Fourier expansion of γ: $\gamma(t)=\sum_{n\in\mathbb{Z}} c_n e^{int}$.
- Compare coefficients in the above formula to get:

$$c_n(e^{i\frac{2\pi n}{k}}-1) = i\lambda n c_n(e^{i\frac{2\pi n}{k}}+1)\,.$$

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- According to Sturm-Hurwitz-Kellogg theorem, it follows that c_1 or c_{-1} are not zero, so $\lambda = \tan\frac{\pi}{k}$, so as a result we obtain that if $c_n \neq 0$ then we have a π-rational solution to Gutkin's equation:

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- As mentioned before, this is impossible.
- As a result, we must have $c_n = 0$ for all $|n| > 1$ which means that γ is an ellipse.

Sketch of argument for Minkowski case

- Let A be the symmetry of the table, so that the Minkowski billiard orbit that starts at $\gamma(t)$ is $\gamma(t), A\gamma(t), ..., A^{k-1}\gamma(t)$.
- Since all orbits that start from the same invariant curve are minimizers, it follows that

$$\sum_{i=1}^{k} g(A^i\gamma(t) - A^{i-1}\gamma(t))$$

  is constant (g is the norm with respect to which γ is the unit circle).
- Since γ is invariant under A it follows that $g(A\gamma(t) - \gamma(t))$ is constant.
- Therefore $A\gamma(t) - \gamma(t)$ is always on a fixed homothetic copy of γ.
- Analysis of this condition leads to the additional symmetry of γ.

Thank you!

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