# Nijenhuis Geometry: conservation laws, symmetries and geodesically equivalent metrics

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# Context: Nijenhuis operators in Riemannian geometry and in finite and $\infty$-dimensional integrable systems

Nijenhuis geometry: conservation laws and symmetries

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Geodesically equivalent metrics

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Infinite dimensional integrable systems, finite-dimensional reductions

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Separation of variables and integration in quadratures

# Context: Nijenhuis operators in Riemannian geometry and in finite and $\infty$-dimensional integrable systems

Link 1 Levi-Civita, Sinjukov, BM, BKM (Nijenhuis Geometry 1, Appl. of Nijenhuis Geometry 5)

Link 2 Magri, Lorenzoni, Fordy, Ferapontov, Mokhov, BKM (Appl. of Nijenhuis Geometry 3, 4) + lots of papers on $\infty$-dim Nijenhuis recursion operators

Link 3 Levi-Civita, Eisenhardt, Stäckel, Benenti, Matveev-Topalov, BM, Tabachnikov *et al*

Link 4 BKM (Appl. of Nijenhuis Geometry 2)

Link 5 BKM (applications of Appl. of Nijenhuis Geometry 3 to separating coordinates for constant curvature metrics of arbitrary signeture)

Link 6 Dubrovin, Krichever, Novikov, Ferapontov, Fordy, Blaszak, Marciniak, Sergyeyev, ... BKM (Nijenhuis Geometry 4, Appl. of Nijenhuis Geometry 5)

# Our goal and approach

We first study Nijenhuis operators on their own and then apply general results so obtained wherever Nijenhuis operators appear in geometry and mathematical physics.

In this talk:

- AB, A.Konyaev, V.Matveev Nijenhuis Geometry 4: *conservation laws, symmetries and integration of certain non-diagonalisable systems of hydrodynamic type in quadratures*, arXiv:2304.10626.
- AB, A.Konyaev, V.Matveev Applications of Nijenhuis Geometry V: *geodesically equivalent metrics and finite-dimensional reductions of certain integrable quasilinear systems*, arXiv:2306.13238.

# Basic definitions (for this talk)

## Definition (differential geometric)

A field of endomorphisms $L = (L^i_j)$ is called a *Nijenhuis operator*, if

$$\mathcal{N}_L(\xi, \eta) \stackrel{\mathrm{def}}{=} L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta] = 0$$

for all vector fields $\xi$, $\eta$.

## Definition (algebraic)

An operator $L : V \to V$, $\dim V = n$, is called $\mathrm{gl}$-*regular*, if either of the following conditions holds:

- there is a vector $\xi$ such that $\xi, L\xi, \dots, L^{n-1}\xi$ are linearly independent (such a vector is called *cyclic*);
- the operators $\mathrm{Id}, L, \dots, L^{n-1}$ form a basis of the centraliser of $L$;
- for each eigenvalue of $L$ there is only one eigenvector;
- $L$ can be reduced to the *first (or second) companion form*.

# Basic definitions (for this talk)

Let $A = (A^i_j)$ be an operator (not necessarily Nijenhuis).

## Definition

A function $f$ is a *conservation law* for $A$, if the form $A^*\mathrm{d}\, f$ is closed. (Today all the constructions are local so that this condition is equivalent to the existence of a function $g$ such that $\mathrm{d}\, g = A^*\mathrm{d}\, f$.)

## Definition

An operator $B = (B^i_j)$ is called a *strong symmetry* (resp. just *symmetry*) for the operator $A$, if

- $AB = BA$
- (i) *strong symmetry:*

$$\langle A, B\rangle(\xi, \eta) \overset{\mathrm{def}}{=} A[B\xi, \eta] + B[\xi, A\eta] - [A\xi, B\eta] - AB[\xi, \eta] = 0,$$

  (ii) *symmetry:*

$$\langle A, B\rangle(\xi, \xi) = A[B\xi, \xi] + B[\xi, A\xi] - [A\xi, B\xi] = 0.$$

# Folklore: symmetries and conservation laws for a diagonal Nijenhuis operator

If $L = \mathrm{diag}(u_1, \dots u_n)$ or more generally

$$L = \mathrm{diag}(\lambda_1(u_1), \dots \lambda_n(u_n)),$$

where $\lambda_i(\cdot)$ are some functions (perhaps constant), satisfying $\lambda_i(u_i) \neq \lambda_j(u_j)$ almost everywhere, then the conservation laws and symmetries are very simple

$$f(u) = f_1(u_1) + f_2(u_2) + \dots + f_n(u_n)$$

and

$$M(u) = \begin{pmatrix} m_1(u_1) & & & \\ & m_2(u_2) & & \\ & & \ddots & \\ & & & m_n(u_n) \end{pmatrix}$$

Natural question. What about other types of Nijenhuis operators, e.g. $\mathfrak{gl}$-regular, which may admit Jordan blocks and collisions of eigenvalues?

# Jordan block in dimension 3 (example)

Let $L_{nc} = \begin{pmatrix} u_3 & u_2 & u_1 \\ 0 & u_3 & u_2 \\ 0 & 0 & u_3 \end{pmatrix}$ and $L_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Useful formula from Linear Algebra:

$$f(L_{nc}) = \begin{pmatrix} f(u_3) & f'(u_3)u_2 & f'(u_3)u_1 + \frac{f''(u_3)}{2}u_2^2 \\ 0 & f(u_3) & f'(u_3)u_2 \\ 0 & 0 & f(u_3) \end{pmatrix} = \begin{pmatrix} f & g & h \\ 0 & f & g \\ 0 & 0 & f \end{pmatrix},$$

with $f = f(u_3)$, $g = g(u_2,u_3) = f'(u_3)u_2$, $h = h(u_1,u_2,u_3) = f'(u_3)u_1 + \frac{f''(u_3)}{2}u_2^2$.

Symmetry of general type:

$$M = f_1(L_{nc})L_c^2 + f_2(L_{nc})L_c + f_3(L_{nc}) = \begin{pmatrix} f_3 & g_3 + f_2 & h_3 + g_2 + f_1 \\ 0 & f_3 & g_3 + f_2 \\ 0 & 0 & f_3 \end{pmatrix}$$

Conservation law of general type:

$$f(u_1, u_2, u_3) = h_3 + g_2 + f_1 = f_3'(u_3)u_1 + \frac{1}{2}f_3''(u_3)u_2^2 + f_2'(u_3)u_2 + f_1(u_3)$$

# Summary of results related to $\mathrm{gl}$-regular Nijenhuis operators

Symmetries and conservation laws of a $\mathrm{gl}$-regular Nijenhuis operator $L$ possess several remarkable properties:

P1. Each symmetry of $L$ is strong.

P3. Each symmetry of $L$ is Nijenhuis.

P2. If $M_1$ and $M_2$ are symmetries of $L$, then their product $M_1 M_2$ is a symmetry also.

P4. Symmetries $M_1$ and $M_2$ commute is the algebraic sense, i.e., $M_1 M_2 = M_2 M_1$, and are symmetries of each other.

P5. Every conservation law $f$ of the operator $L$ is a conservation law for each of its symmetry $M$, that is, $\mathrm{d}\,(M^* \mathrm{d}\, f) = 0$.

P6. Let $f$ be a regular conservation law of $L$. Then any other conservation law $h$ can be obtained from $\mathrm{d}\, g = M^* \mathrm{d}\, f$, where $M$ is a suitable symmetry of $L$.

P7. In the real analytic case, symmetries and conservation laws of $L$ are parametrised by $n$ arbitrary real analytic functions of one variable.

We have also obtained complete explicit description of symmetries and conservation laws at non-singular points and generic singular points.

# gl-regularity is essential: Example

Consider the constant operator $L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in $\mathbb{R}^3(x, y, z)$, which consists of two nilpotent Jordan blocks of size 2 and 1.

The symmetries of $L$ have the following form

$$M = \begin{pmatrix} f & xf_y + g & xf_z + a \\ 0 & f & 0 \\ 0 & b & c \end{pmatrix},$$

where the functions $f, g, a, b, c$ depend on $y$ and $z$ only.
Strong symmetries have a similar form with the additional condition that $f = f(y)$ (i.e., $f$ does not depend on $z$).

The conservation laws are $xu(y) + v(y, z)$.

None of the properties P1 – P7 are met.

# Context: Nijenhuis operators in Riemannian geometry and in finite and $\infty$-dimensional integrable systems

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{l}\text{Nijenhuis geometry:}\\ \text{conservation laws}\\ \text{and symmetries}\end{array} \arrow[r, leftrightarrow, "1"] \arrow[d, leftrightarrow, "2"'] \arrow[dr, leftrightarrow] & \begin{array}{c}\text{Geodesically equivalent}\\ \text{metrics}\end{array} \arrow[d, leftrightarrow, "3"] \arrow[dl, leftrightarrow] \\
\begin{array}{l}\text{Infinite dimensional}\\ \text{integrable systems,}\\ \text{finite-dimensional}\\ \text{reductions}\end{array} \arrow[r, leftrightarrow, "6"'] \arrow[ur, phantom, "4\qquad 5"] & \begin{array}{c}\text{Separation of variables}\\ \text{and integration}\\ \text{in quadratures}\end{array}
\end{tikzcd}
$$

# Link 1: Nijenhuis geometry and geodesically equivalent metrics

## Definition (Beltrami, Levi-Civita, ...)

Two (pseudo)-Riemannian metrics $g$ and $\bar{g}$ are called *geodesically equivalent* if they share the same geodesics viewed as unparameterized curves.

A manifold endowed with a pair of such metrics carries a natural Nijenhuis structure

$$L = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

In terms of $L$, the geodesic equivalence condition is given by the PDE equation

$$\nabla_\eta L = \frac{1}{2} \big( \eta \otimes \mathrm{d}\ \mathrm{tr}\, L + (\eta \otimes \mathrm{d}\ \mathrm{tr}\, L)^* \big), \tag{1}$$

where $\eta$ is an arbitrary vector field.

## Definition

If (1) holds, then the metric $g$ and Nijenhuis operator $L$ are said to be *geodesically compatible*.

# Singularities in the context of geodesically equivalent metrics

*Singular points* are those at which the algebraic type of $L$ changes, e.g., the eigenvalues of $L$ collide.

**Open problem.** What kind of singular points can appear in the context of geodesically equivalent metrics?

Riemannian case was understood by Matveev 2006 (dim 2: B-Matveev-Fomenko 1998), pseudo-Riemannian is still open.

**Example.** $\begin{pmatrix} 2x & y \\ y & 0 \end{pmatrix}$ is allowed, $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ is not.

If $L$ is a $\mathrm{gl}$-regular operator, then its eigenvalues can still collide without violating the $\mathrm{gl}$-regularity condition. In the Nijenhuis geometry, scenarios of such collisions can be very different. However, regardless of any particular scenario, we have the following general local result.

**Theorem**

*Let $L$ be a $\mathrm{gl}$-regular real analytic Nijenhuis operator. Then (locally) there exists a pseudo-Riemannian metric $g$ geodesically compatible with $L$. Moreover, such a metric $g$ can be defined explicitly in terms of the second companion form of $L$.*

# Magic formula

Fix second companion coordinates $u^1, \dots, u^n$ of $L$ so that

$$L = L_{\mathrm{comp2}} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

Let $p_1, \dots, p_n, u^1, \dots, u^n$ be the corresponding canonical coordinates on the cotangent bundle and consider the following algebraic identity

$$h_1 L^{n-1} + \dots + h_n \,\mathrm{Id} = \left( p_n L^{n-1} + \dots + p_1 \,\mathrm{Id} \right)^2. \qquad (2)$$

Since $L$ is $\mathfrak{gl}$-regular, the functions $h_1, \dots, h_n$ are uniquely defined. They are quadratic in $p_1, \dots, p_n$ and their coefficients are polynomials in $\sigma_i$'s.

## Theorem

*The function $h_1(u,p) = \sum h_1^{\alpha\beta}(u) p_\alpha p_\beta$ defines a non-degenerate (contravariant) metric which is geodesically compatible with $L$.*

# $\mathfrak{gl}$-regular Nijenhuis operators

## Theorem (Real analytic case)

*Let $L$ be a $\mathfrak{gl}$-regular Nijenhuis operator. Then there exist local coordinate systems $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in which $L$ reduces to the first and second companion forms:*

$$L(u) = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1} & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad L(v) = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

*where $\sigma_i$ are the coefficients of the characteristic polynomial of $L$ in the corresponding coordinate system.*

# Magic formula

Fix second companion coordinates $u^1, \ldots, u^n$ of $L$ so that

$$L = L_{\text{comp2}} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \ldots & \sigma_1 \end{pmatrix},$$

Let $p_1, \ldots, p_n, u^1, \ldots, u^n$ be the corresponding canonical coordinates on the cotangent bundle and consider the following algebraic identity

$$h_1 L^{n-1} + \cdots + h_n \,\mathrm{Id} = \left( p_n L^{n-1} + \cdots + p_1 \,\mathrm{Id} \right)^2. \tag{3}$$

Since $L$ is $\mathrm{gl}$-regular, the functions $h_1, \ldots, h_n$ are uniquely defined. They are quadratic in $p_1, \ldots, p_n$ and their coefficients are polynomials in $\sigma_i$'s.

## Theorem

*The function $h_1(u, p) = \sum h_1^{\alpha\beta}(u) p_\alpha p_\beta$ defines a non-degenerate (contravariant) metric which is geodesically compatible with $L$.*

# How to describe all geodesically compatible partners for $L$?

Let $L$ be an admissible Nijenhuis operator (in the context of geodesic equivalence), i.e. there is at least one (pseudo)-Riemannian metric $g$ geodesically compatible with $L$.

Natural problem. Describe all geodesically compatible partners for $L$.

Theorem

*Let $L$ and $g$ be geodesically compatible. Assume that $M$ is $g$-symmetric and is a strong symmetry of $L$, then $L$ and $gM := (g_{is}M^s_j)$ are geodesically compatible.*

*Moreover, if $L$ is $\mathrm{gl}$-regular, then every metric $\tilde{g}$ geodesically compatible with $L$ is of the form $\tilde{g} = gM$, where $M$ is a (strong) symmetry of $L$.*

Conclusion. Since we have complete description of symmetries for $\mathrm{gl}$-regular operators and our magic formula, this problem is solved in the $\mathrm{gl}$-regular case.

# Context: Nijenhuis operators in Riemannian geometry and in finite and $\infty$-dimensional integrable systems

| | | |
|---|---|---|
| Nijenhuis geometry: conservation laws and symmetries | $\xleftrightarrow{1}$ | Geodesically equivalent metrics |
| 2 $\updownarrow$ | 4 $\nwarrow\!\!\searrow$ $\nearrow\!\!\swarrow$ 5 | $\updownarrow$ 3 |
| Infinite dimensional integrable systems, finite-dimensional reductions | $\xleftrightarrow[6]{}$ | Separation of variables and integration in quadratures |

# Link 2: Nijenhuis operators and integrable PDEs

For a given Nijenhuis operator $L$, we define the operator fields $A_i$ by the following recursion relations

$$A_0 = \mathrm{Id}, \quad A_{i+1} = LA_i - \sigma_i \, \mathrm{Id}, \quad i = 0, \dots, n-1, \tag{4}$$

where functions $\sigma_i$ are coefficients of the characteristic polynomial of $L$ numerated as below:

$$\chi_L(\lambda) = \det(\lambda \, \mathrm{Id} - L) = \lambda^n - \sigma_1 \lambda^{n-1} - \dots - \sigma_n. \tag{5}$$

Equivalently, the operators $A_i$ can be defined from the matrix relation

$$\det(\lambda \, \mathrm{Id} - L) \cdot (\lambda \, \mathrm{Id} - L)^{-1} = \lambda^{n-1} A_0 + \lambda^{n-2} A_1 + \dots + \lambda A_{n-2} + A_{n-1}.$$

Consider the following system of quasilinear PDEs defined by these operators

$$\begin{aligned} u_{t_1} &= A_1 \, u_x, \\ &\dots \\ u_{t_{n-1}} &= A_{n-1} \, u_x, \end{aligned}$$

with $u^i = u^i(x, t_1, \dots, t_{n-1})$ being unknown functions in $n$ variables and $u = (u^1, \dots, u^n)^\top$.

# Integrability and other known properties of this system

$$\begin{aligned} u_{t_1} &= A_1\, u_x, \\ &\dots \\ u_{t_{n-1}} &= A_{n-1}\, u_x, \end{aligned} \tag{6}$$

Here $u = (u^1, u^2, \dots, u^n)^\top$ with $u^i = u^i(x, t_1, ..., t_{n-1})$ being $n$ unknown functions in $n$ variables. In total, we have a system of $n(n-1)$ quasilinear PDE equations. The system is compatible in the sense of Cartan-Kähler, i.e., solutions exists for any initial curve $\gamma(x) = u(x, 0, \dots, 0)$. Equivalently, we can say that evolutionary flows defined by individual equations $u_{t_i} = A_i\, u_x$ commute (Magri, Lorenzoni).

These equations are semi-hamiltonian in the sense of S. Tserev and hence can be integrated by means of the generalised hodograph method. In the diagonal case, each individual equation is weakly non-linear, leading to separation of variables and integration in quadratures (Ferapontov, Ferapontov-Fordy, Blaszak, Marciniak).

To extend these results to the non-diagonal case, one needs to describe the common symmetries and conservation laws of these equations.

# Symmetries and conservation laws of (6)

## Theorem

*If $L$ is $\mathrm{gl}$-regular, then*

1. *For any hierarchy of conservation laws $f_1, \dots, f_n$ of $L$, the operator*

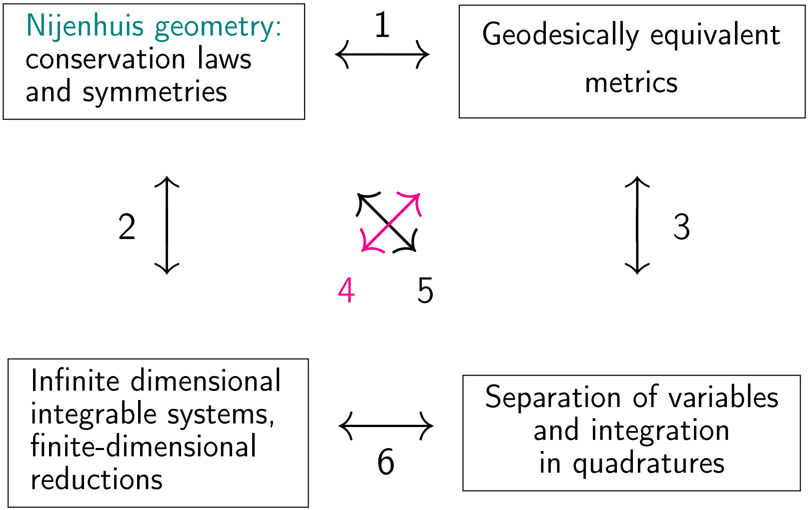
$$B = f_1 A_n + \dots + f_n A_1$$

   *is a common symmetry for $A_i$. Moreover, every common symmetry of $A_i$'s can be written in this way.*

2. *For any symmetry $M = g_1 L^{n-1} + \dots + g_n \,\mathrm{Id}$ of $L$, the first function $g_1$ is a common conservation law of $A_i$.*
   *Moreover, every common conservation law of $A_i$'s can be obtained in this way.*

This theorem leads us (via procedure described in "Nijenhuis Geometry 4") to Link 5 between Nijenhuis Geometry and integration in quadratures.

# Context: Nijenhuis operators in Riemannian geometry and in finite and $\infty$-dimensional integrable systems



# Link 4: geodesically equivalent metrics and finite dimensional reductions of integrable PDEs

Various types of finite-dimensional reductions of infinite-dimensional nonlinear integrable systems have been investigated since the middle of 70s (Antonowicz, Fordy, Bogoyavlenskij, Novikov, Hone, Marciniak, Blaszak, Veselov).

Informally, a finite-dimensional reduction of an integrable PDE system is a subsystem of it, which is finite-dimensional and still integrable.

It appears that such a reduction of (6) can be naturally obtained by fixing a metric $g$ geodesically compatible with $L$.

### Theorem

*Consider any metric $g$ geodesically compatible with $L$ and take any geodesic $\gamma(x)$ of this metric. Let $u(x, t_1, ..., t_{n-1})$ be the solution of* (6) *with the initial condition $u(x, 0, ..., 0) = \gamma(x)$. Then for any (sufficiently small) $t_1, ..., t_{n-1}$, the curve $x \mapsto u(x, t_1, ..., t_{n-1})$ is a geodesic of $g$.*

In other words, the evolutionary system corresponding to any of the equations from (6) sends geodesics of $g$ to geodesics.

# Explanation

The integrals of the geodesic flow of $g$ are closely related to the operators $A_i$ (Matveev-Topalov, Tabachnikov 1997).

Namely, if $g$ is geodesically compatible with $L$, then its geodesic flow (as a Hamiltonian system on $T^*\mathrm{M}$) admits $n$ commuting first integrals $F_0, \dots, F_{n-1}$ of the form

$$F_i(u, p) = \tfrac{1}{2}\, g^{-1}(A_i^* p, p). \tag{7}$$

# Finite dimensional reductions (continued...)

Let us consider the space $\mathfrak{G}$ of all $g$-geodesics (viewed as parameterised curves). Then system (6) defines a local action of $\mathbb{R}^n$ on $\mathfrak{G}$:

$$\Psi^{t_0, t_1, \dots, t_{n-1}} : \mathfrak{G} \to \mathfrak{G}, \qquad (t_0, t_1, \dots, t_{n-1}) \in \mathbb{R}^n.$$

More precisely, if $\gamma = \gamma(x) \in \mathfrak{G}$ is a $g$-geodesic, then we set $\Psi^{t_0, t_1, \dots, t_{n-1}}(\gamma)$ to be the geodesic $\tilde{\gamma}(x) = u(x + t_0, t_1, ..., t_{n-1})$, where $u(x, t_1, ..., t_{n-1})$ is the solution of (6) with the initial condition $u(x, 0, ..., 0) = \gamma(x)$.

**Theorem**

*The action $\Psi$ is conjugate to the Hamiltonian action of $\mathbb{R}^n$ on $T^*\mathrm{M}$ generated by the flows of the integrals $F_0, \dots, F_{n-1}$ defined by* (7). *The conjugacy is given by $\gamma \in \mathfrak{G} \mapsto (\gamma(0), g_{ij}\dot{\gamma}^i(0)) \in T^*\mathrm{M}$.*

**Remark.** Let $L$ be a $\mathrm{gl}$-regular real analytic Nijenhuis operator, then for every curve $\gamma$ with a cyclic velocity vector there exists a metric $g$ geodesically compatible with $L$ such that $\gamma$ is a $g$-geodesic. Thus, the above finite-dimensional reductions of (6) 'cover' almost all (local) solutions of the Cauchy problem.

# Conclusion

- General properties of conservation laws and symmetries of $\mathrm{gl}$-regular Nijenhuis operators are understood
- Explicit description of conservation laws and symmetries for various types of $\mathrm{gl}$-regular Nijenhuis operators
- Each $\mathrm{gl}$-regular Nijenhuis operator admits a geodesically equivalent partner $g$ (in other words, any kind of $\mathrm{gl}$-regular collisions of eigenvalues is allowed in the context of geodesically equivalent metrics).
- Complete description of all geodesically equivalent partners of $\mathrm{gl}$-regular Nijenhuis operators (in terms of their symmetries).
- To each $\mathrm{gl}$-regular Nijenhuis operator $L$, we assign an integrable system of hydrodynamic type (previously studied in the diagonal case by many authors). All the symmetries and conservation laws of this system are explicitly described in terms of the symmetries of conservation laws and symmetries of the Nijenhuis operator $L$, leading to integration in quadratures.
- Finite-dimensional reductions of this system are naturally isomorphic to integrable geodesic flows of metrics $g$ geodesically compatible with $L$.

# Thank you for your attention