KU LEUVEN

Branes in symplectic manifolds

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Branes

Let (M, ω_M) be symplectic.

Definition

A brane (Y, F) consists of

- Y a coisotropic submanifold (i.e. $TY^{\omega_M} \subset TY$)
- $F \in \Omega^2_{cl}(Y)$

such that

- $$\ker(F) = \ker(\omega) =: E$$
  where $\omega := \iota^* \omega_M$
- on TY/E have: $I := \omega^{-1} F$ satisfies
  $$I^2 = -1.$$

Remark

If smooth, $Y/\sim$ is a complex manifold.
$F + i\omega$ induces a holomorphic symplectic form there.

Branes

Remark

a) The definition is natural in terms of the generalized complex structure [Gualtieri 2003]

$$\begin{pmatrix} 0 & -\omega_M^{-1} \\ \omega_M & 0 \end{pmatrix} : TM \oplus T^*M \to TM \oplus T^*M.$$

b) Branes arose in the study of the A-model [Kapustin-Orlov 2001]

Aim

Given a brane (Y, F), understand the space of nearby branes:

- To produce new examples
- Is this space smooth? Is the moduli space smooth?
- Are all nearby coisotropic submanifolds also branes?

Notation

Given a 2-form σ, we denote the bundle map $TM \to T^*M, v \mapsto \iota_v \sigma$ by the same symbol.

Spacefilling branes: $Y = M$

Let (M, ω) be symplectic.

Remark
(M, F) is a brane
$\Leftrightarrow$ F is symplectic and $I := \omega^{-1}F$ satisfies $I^2 = -1$
$\Leftrightarrow$ $F + i\omega$ is holomorphic symplectic. (The imaginary part of fixed)

Example
Let $M = \mathbb{C}^2$ (or $M = \mathbb{T}^4$), with complex coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Take

$$\omega := Im(dz_1 \wedge dz_2) = dx_1 \wedge dy_2 + dy_1 \wedge dx_2$$
$$F := Re(dz_1 \wedge dz_2) = dx_1 \wedge dx_2 - dy_1 \wedge dy_2$$

Example
The $K3$ manifold is

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3.$$

It is a compact, simply connected, 4-dimensional smooth manifold.
(M, I_{can}) carries a holomorphic symplectic form.

A pencil of Poisson structures

Remark

Let F be a space-filling brane on (M, ω).

- The Poisson structures F^{-1} and ω^{-1} commute:

$$[F^{-1}, \omega^{-1}] = 0.$$

  That is, $F^{-1} + t\,\omega^{-1}$ is a Poisson structure for all $t \in \mathbb{R}$.
- (F^{-1}, I) is a Poisson-Nijenhuis structure.

Parametrizing nearby space-filling branes

Let F be a space-filling brane on (M, ω).

Lemma

Let $\{F_t\} \subset \Omega^2(M)$ *a curve with* $F_0 = F$ *and* $(\omega^{-1}F_t)^2 = -1$. *Then*

$$\dot{F} := \frac{d}{dt}|_0 F_t \in \Omega^{1,1}(M, \mathbb{R}).$$

I.e.: $\dot{F}(Iv, Iw) = \dot{F}(v, w)$.

Thus $\Omega^{1,1}(M, \mathbb{R})$ is the formal tangent space at F to

$$\{\widetilde{F} \in \Omega^2(M) : (\omega^{-1}\widetilde{F})^2 = -1\}.$$

Aim

Given a spacefilling brane F, find a good parametrization of nearby branes.

An idea from Dirac geometry

To parametrize space-filling branes near F:

- Encode F by
$$gr(F) := \{(v, \iota_v F) : v \in TM\}.$$
- Describe nearby branes $\widetilde{F}$ as graphs of linear maps
$$\psi\colon gr(F) \to \text{a complement of } gr(F)$$
- As complement take
$$D \oplus D^\circ$$
where $D \subset TM$ is an involutive distribution s.t.
  - D is lagrangian for ω
  - D is symplectic for F.

Proposition

There is a bijection between

- $\beta \in \Omega^{1,1}(M, \mathbb{R})$ *s.t.* $d\beta + \frac{1}{2}[\beta, \beta] = 0$
- *space-filling branes* $\widetilde{F} \in \Omega^2(M)$ *(for which* D *is symplectic)*

Here $[\ ,\ ]$ is a Lie bracket on $\Gamma(T^*M)$ determined by D and F.

Remark

The map $\beta \mapsto \widetilde{F}$ is non-linear:
Write $\beta = A' - FA$ for some $(A, A')\colon TM \to D \oplus D^\circ$.
Then $\widetilde{F} = (F + A')(Id + A)^{-1}$.

Remark

a) There is a DGLA (differential graded Lie algebra) whose Maurer-Cartan elements are the β above.

b) This is useful to answer questions like:
Given $B \in \Omega^{1,1}(M, \mathbb{R})$ s.t. $dB = 0$, is there a path $\{F_t\}$ of branes with $\frac{d}{dt}|_0 F_t = B$?

Moduli spaces in dimension 4

Claim

For any symplectic form on $M = K3$ (resp. $M = \mathbb{T}^4$),

$$\{\text{Spacefilling branes}\}/Sympl(M,\omega)_*$$

is smooth, of dimension 20 (resp. 4), non-compact.

Here $Sympl(M,\omega)_*$ denotes the symplectomorphisms inducing $Id_{H^\bullet(M,\mathbb{R})}$.

Remark

The argument relies on the Local Torelli Theorem.

An example of codimension 1 brane

Non-Example

$S^5 \subset (\mathbb{R}^6, \omega_{can})$ does not admit a brane structure.
Reason: $S^5/\sim = \mathbb{CP}^2$ doesn't admit a holomorphic sympl form: $H^2(\mathbb{CP}^2) = \mathbb{R}$.

Example

Let (N, ω_N) be symplectic. Then

$$Y := N \times S^1 \times \{0\}$$

is coisotropic in the symplectic manifold $(N \times S^1 \times \mathbb{R}\ ,\ \omega_N \times \omega_{T^*S^1})$.

Let $F_N \in \Omega^2(N)$ symplectic s.t. $I := \omega_N^{-1} F_N$ satisfies $I^2 = -1$.
Then

$$(Y, F)$$

is a brane, where $F :=$ trivial extension of F_N.

Branes nearby $Y = N \times S^1$

Question:
Let $f \in C^\infty(Y)$. When does $gr(f) := \{(y, f(y))\}$ admit a brane structure?

Proposition

There is a bijection between

a) *brane structures on $gr(f)$,*
b) *symplectic forms $\widetilde{F}_N$ on N, preserved by Φ_f, so that $\omega_N^{-1}\widetilde{F}_N$ squares to $-Id_{TN}$.*

Here:

$$\Phi_f := \text{time 1 flow of the time-dependent vector field } \{X^{\omega_N}_{f|_{N\times\{q\}}}\}_{q\in[0,1]}.$$

Using $gr(f) \cong Y$, the bijection reads

$$\widetilde{F} \mapsto \widetilde{F}|_{N\times\{0\}}$$
$$\widetilde{F}_N \mapsto \text{the unique extension invariant along } \ker(i^*_{gr(f)}\omega_M)$$

Stability of branes

Let (Y, F) be a brane in (M, ω_M).

Question:
Is the forgetful map

$$\begin{aligned}\{\text{Branes}\} &\to \{\text{Coisotropic submanifolds}\}\\ (\widetilde{Y}, \widetilde{F}) &\mapsto \widetilde{Y}\end{aligned}$$

surjective near Y?

Answer: No

As earlier, consider $(N, I, F_N + i\omega_N)$ holomorphic symplectic $\rightsquigarrow$

$$(\underbrace{N \times S^1 \times \{0\}}_{Y}, F)$$

brane in $(N \times S^1 \times \mathbb{R}, \omega_N \times \omega_{T^*S^1})$.

A) Answer by example

Proposition

For N the K3 manifold:
Let $f \in C^\infty(Y)$ s.t.

$$\Phi_f \neq Id_N.$$

Then $gr(f)$ is not a brane.

Here:

$$\Phi_f := \text{time } 1 \text{ flow of } \{X^{\omega_N}_{f|_{N\times\{q\}}}\}_{q\in[0,1]}.$$

Proof:
Suppose there is a brane structure F on $gr(f)$.

$$\text{Last Prop} \Rightarrow \Phi_f \text{ preserves } F_N \Rightarrow \Phi_f \text{ preserves } I.$$

The only automorphism of (N, I) inducing $Id_{H^2(N,\mathbb{R})}$ is Id_N. ↯

B) Answer by comparing first order deformations

Remark

Let Y be a codimension 1 submanifold. Then

$$\{\text{First order deformations of } Y\} = \Gamma(NY) \cong \Gamma(E^*).$$

Remark

Let (Y, F) be a codimension 1 brane. Choose G involutive s.t. $G \oplus E = TY$.
The first order deformations of Y as a brane are

$$(r, B) \in \Gamma(E^*) \oplus \Omega^2(Y)$$

s.t.

- $dB = 0$
- for all $v \in E$, on TY/E we have

  $$[\iota_v B] = -[\iota_v d\bar{r}] \circ I,$$

  where $\bar{r} \in \Omega^1(Y)$ is the extension of r annihilating G
- $B|_G$ is of type $(1, 1)$.

Proposition

For a brane $Y = N \times S^1$ as above, consider

$$\Upsilon\colon \{\text{First order def. of } Y \text{ as a brane}\} \to \Gamma(E^*),$$
$$(r, B) \mapsto r.$$

Its image consists of $r \in \Gamma(E^*)$ s.t.

$$d_N I^* d_N \left(\int_{S^1} r \right) = 0.$$

In particular, Υ in not surjective in general.

Here $\int_{S^1} r \in C^\infty(N)$ is obtained integrating r along the fibers of $Y \to N$.

TO DO
For an arbitrary brane (Y, F), parametrize its deformations as a brane.
Is the moduli space smooth?

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Thank you for your attention