Hamiltonian group actions and fixed points

Mantra: act globally, compute locally

Poisson 2026
Summer School
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Outline - Lectures 1 & 2

- Symplectic geometry (with lots of examples)
- Group actions & fixed points (with lots of examples)
- Localization (with lots of examples)
- Symplectic reduction (how to take a quotient)
  (with lots of examples)

The momentum map

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}$$

DEF: Combining these for all $\xi \in \mathfrak{g}$, we define the momentum map

$$\begin{aligned} \Phi : M &\to \mathfrak{g}^* \\ p &\mapsto \left(\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \phi^\xi(p) \end{array} \right). \end{aligned}$$

<u>Convexity Theorem</u> [Atiyah, Guillemin-Sternberg]:

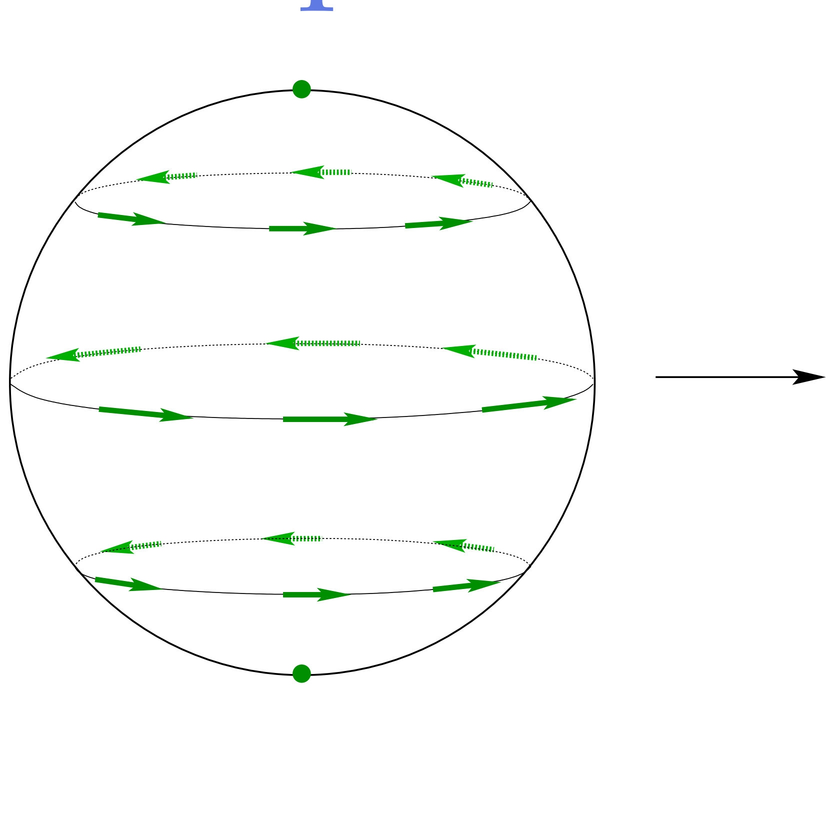
If $T = (S^1)^d \circlearrowright (M, \omega)$ is Hamiltonian, $\Phi(M)$ is a convex polytope.

compact

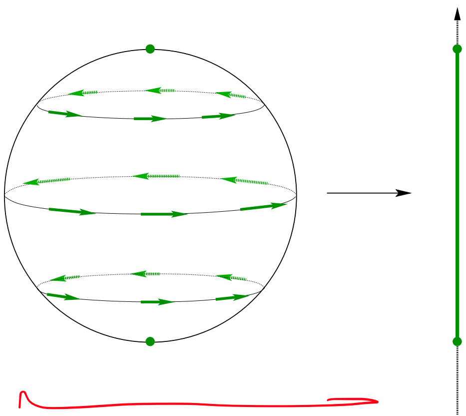
$$\Phi(M) = \mathrm{Conv}(\Phi(M^T)).$$

or ...

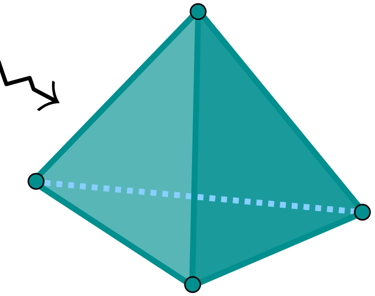
Examples



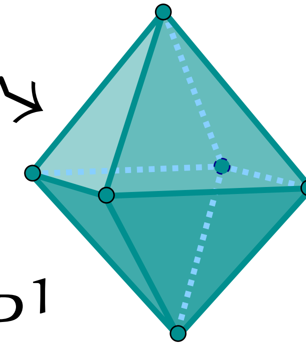
Examples



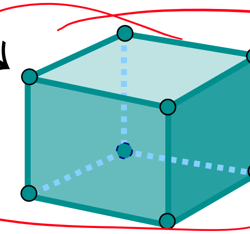
$T^3 \circlearrowright \mathbb{CP}^3$



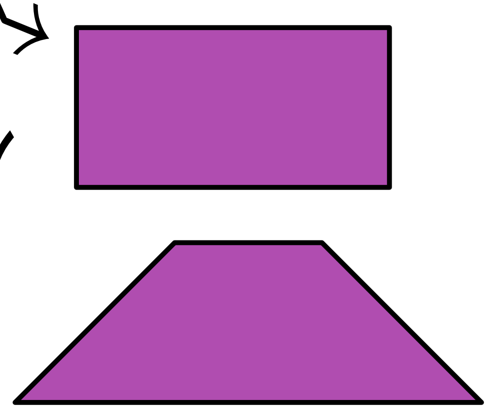
$T^3 \circlearrowright \mathcal{G}r(2, \mathbb{C}^4)$



$\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$

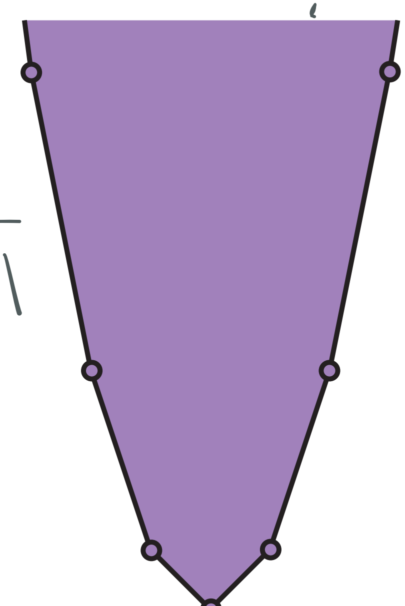


$\mathcal{P}ol_3(a_1, \ldots, a_5)$

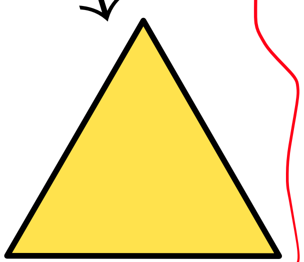


$\Omega SU(2)$

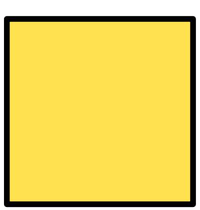
Pressley–Segal

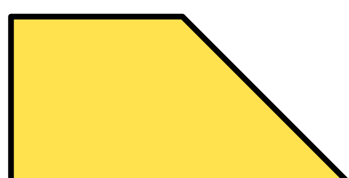


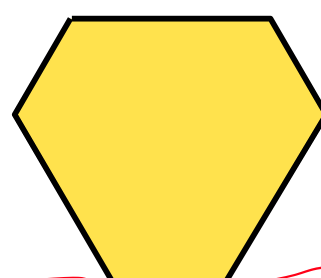
$\mathbb{CP}^2$



$\mathbb{CP}^1 \times \mathbb{CP}^1$







Localization

A **localization phenomenon** is a global feature of $T \circlearrowright M$ that can be described by the evidence of that feature at the T-fixed points.

<u>Convexity Theorem</u> [Atiyah,Guillemin-Sternberg]:

If $T = (S^1)^d \circlearrowright (M, \omega)$ is Hamiltonian, $\Phi(M)$ is a convex polytope.

$$\Phi(M) = \mathrm{Conv}(\Phi(M^T)).$$

There are non-abelian vesions...

In terms of topology, we use localization to make global equivariant computations in terms of local computations at fixed points.

Morse theory and cohomology

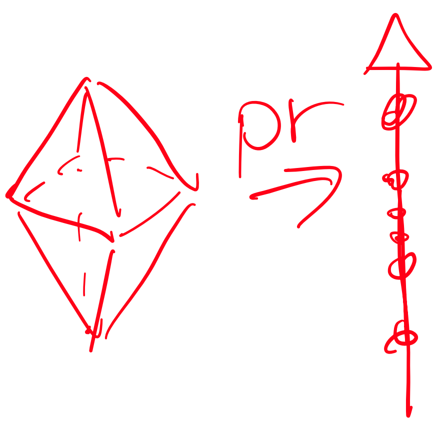
DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}$$

<u>Morse theory of the moment map</u> [Atiyah]:

ϕ^ξ is a Morse fcn in the sense of Bott

$$\mathrm{Crit}(\phi^\xi) = M^T$$



For a critical submfld $X \subset M^T$,

Index(X): $\nu(X \subset M)$ is always even!

Morse theory and cohomology

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$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}$$

Morse theory of the moment map [Atiyah]:

$\phi^\xi : M \to \mathbb{R}$ Morse–Bott

crit sets always even index

ϕ^ξ is perfect:

$$H^i(M;\mathbb{R}) = \sum_{X \subset M^T} H^{i-\lambda_X}(X;\mathbb{R})$$

λ_X = even index of X $\Rightarrow$ compute Betti #s,

Morse theory and cohomology

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

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Morse theory of the moment map [Atiyah]:

Examples

These have cohomology only in even deg.

$\mathbb{C}P^3$

index=0

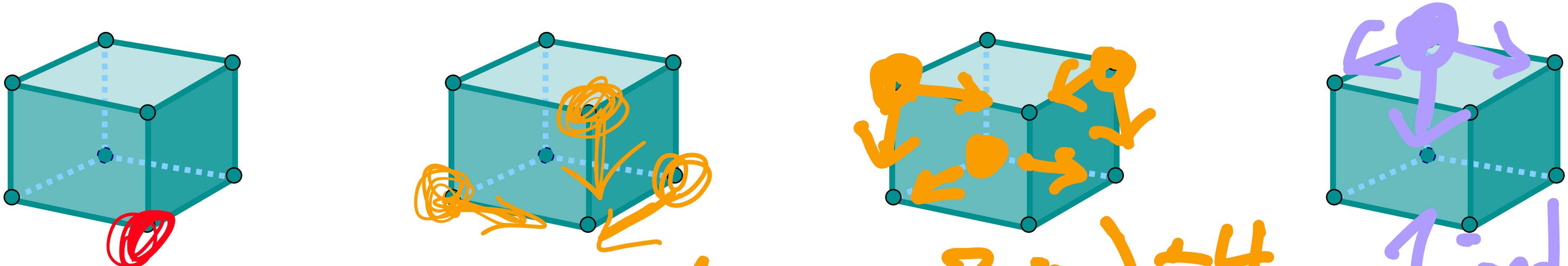
index 2

index 4

index 6

$$\beta_i = \begin{cases} 1 & i=0,2,4,6 \\ 0 & \text{o/w.} \end{cases}$$

$\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$



index 0

3 index 2

3 ind=4

1 ind 6

$\beta_{2i} = 1, 3, 3, 1$

Equivariant cohomology

Equivariant cohomology is a generalized cohomology theory in the equivariant category.

For $G = T = (S^1 \times \cdots \times S^1)^n$

$$T \circlearrowright \boxed{S^{2\infty-1} \times \cdots \times \underbrace{S^{2\infty-1}}_{\text{This sphere is contractible!}}} \text{ freely}$$

$$ET = S^{2\infty-1} \times \cdots \times S^{2\infty-1}, \qquad S^{2\infty-1} \subset \mathbb{C}^\infty$$

$$H_T^*(X) := H^*(X \times ET / T)$$

Equivariant cohomology

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- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$

Equivariant cohomology

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G-

- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$

$G \circlearrowright M \rightsquigarrow H^*_G(M;\mathbb{Z})$ or $H^*_G(M;\mathbb{R})$

$f : M \to N$ equivt. $\implies f^* : H^*_G(N) \to H^*_G(M)$

Mayer-Vietoris for invariant subsets

Et cetera

Equivariant cohomology

Equivariant cohomology is a generalized cohomology theory in the equivariant category.

- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$
- Equivariant cohomology of a point is not $\mathbb{Z}$

$$H_T^*(pt) = H^*\left(pt \times ET/T\right)$$

$$= H^*(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty)$$

$$\deg x_i = 2 \qquad = \mathbb{Z}[x_1, \ldots, x_n]$$

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$$T = T^d = S^1 \times \cdots \times S^1 \rightsquigarrow$$

$$H^*_T(pt; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_d]$$

$$\deg(x_i) = 2$$

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- Spaces, maps should be equivariant

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$$G \circlearrowright M \rightsquigarrow H^*_G(M;\mathbb{Z}) \text{ or } H^*_G(M;R)$$

$$f : M \to N \implies f^* : H^*_G(N) \to H^*_G(M)$$

Mayer-Vietoris

Et cetera

Equivariant cohomology

Equivariant cohomology is a generalized cohomology theory in the equivariant category.

- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$
- Equivariant cohomology of a point is not $\mathbb{Z}$
- Spaces, maps should be equivariant
- If G $\circlearrowright$ M is a free action, then $H^*_G(M) = H^*(M/G)$

$$H^*_G(M) = H^*(M \times EG/G)$$

$$= H^*(M/G \times EG)$$

Equivariant cohomology

Equivariant cohomology is a generalized cohomology theory in the equivariant category.

- Functor $\mathcal{S}paces \longrightarrow \mathcal{R}ings$
- Equivariant cohomology of a point is not $\mathbb{Z}$
- Spaces, maps should be equivariant
- If $G \circlearrowright M$ is a free action, then $H^*_G(M) = H^*(M/G)$
- For Hamiltonian actions on compact manifolds,

$$H^*_T(M;\mathcal{R}) \text{ is a free } \underbrace{H^*_T(pt;\mathcal{R})}_{\mathcal{R}[x_1,\dots,x_n]}\text{-module}$$

careful about $\mathcal{R}$...

Cohomological Localization

$$M^T \hookrightarrow M \quad \rightsquigarrow \quad H^*_T(M;R) \longrightarrow H^*_T(M^T;R)$$

if isolated

$\bigoplus R[x_1, \dots, x_n]$

Cohomological Localization

$$M^T \hookrightarrow M \quad \rightsquigarrow \quad H_T^*(M;R) \longrightarrow H_T^*(M^T;R)$$

<u>Theorem</u> [Frankel; Atiyah; Kirwan]:

If $T \circlearrowright M$ is a compact Hamiltonian T-manifold, then

$$\underline{H_T^*(M;\mathbb{Q})} \longrightarrow \underline{H_T^*(M^T;\mathbb{Q})}$$

is an injection.

$\bigoplus \mathbb{Q}[x_1, \ldots, x_n]$

$p \in M^T$

$\Rightarrow$ Can find $H_T^*(M)$ as a subring

Cohomological Localization

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<u>Theorem</u> [Frankel; Atiyah; Kirwan]:
If $T \circlearrowright M$ is a compact Hamiltonian T-manifold, then

$$H_T^*(M;\mathbb{Q}) \longrightarrow H_T^*(M^T;\mathbb{Q})$$

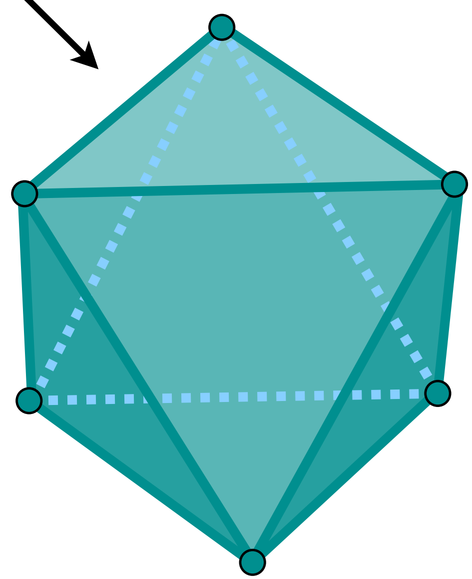
is an injection. (The statement sometimes holds over $\mathbb{Z}$.)

Key step: understand equivariant Euler classes

Example

$$T^3 \circlearrowright \mathcal{G}r(2,\mathbb{C}^4) \rightsquigarrow H^*_T(\mathcal{G}r(2,\mathbb{C}^4);\mathbb{Z}) \subseteq \bigoplus_{i=1}^{6} \mathbb{Z}[x,y,z]$$

Φ



Example

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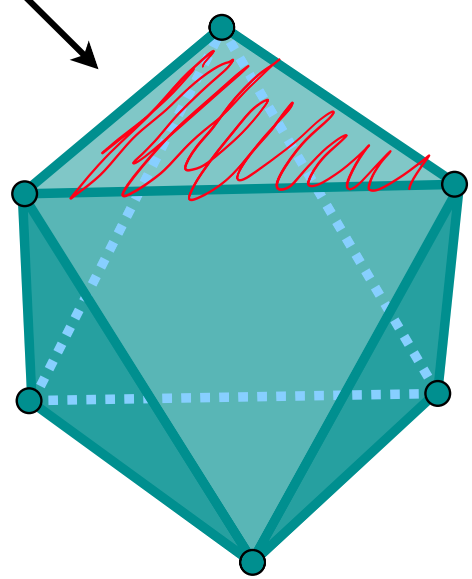
$$\begin{array}{c} H^*_T(\mathcal{G}r(2,\mathbb{C}^4);\mathbb{Z}) \\ \Uparrow\!\!\!\!\cup \\ \alpha = [\mathbb{P}^2_{e_1}] \end{array}$$

$= V$ in $\mathbb{C}^4$ that contain e_1

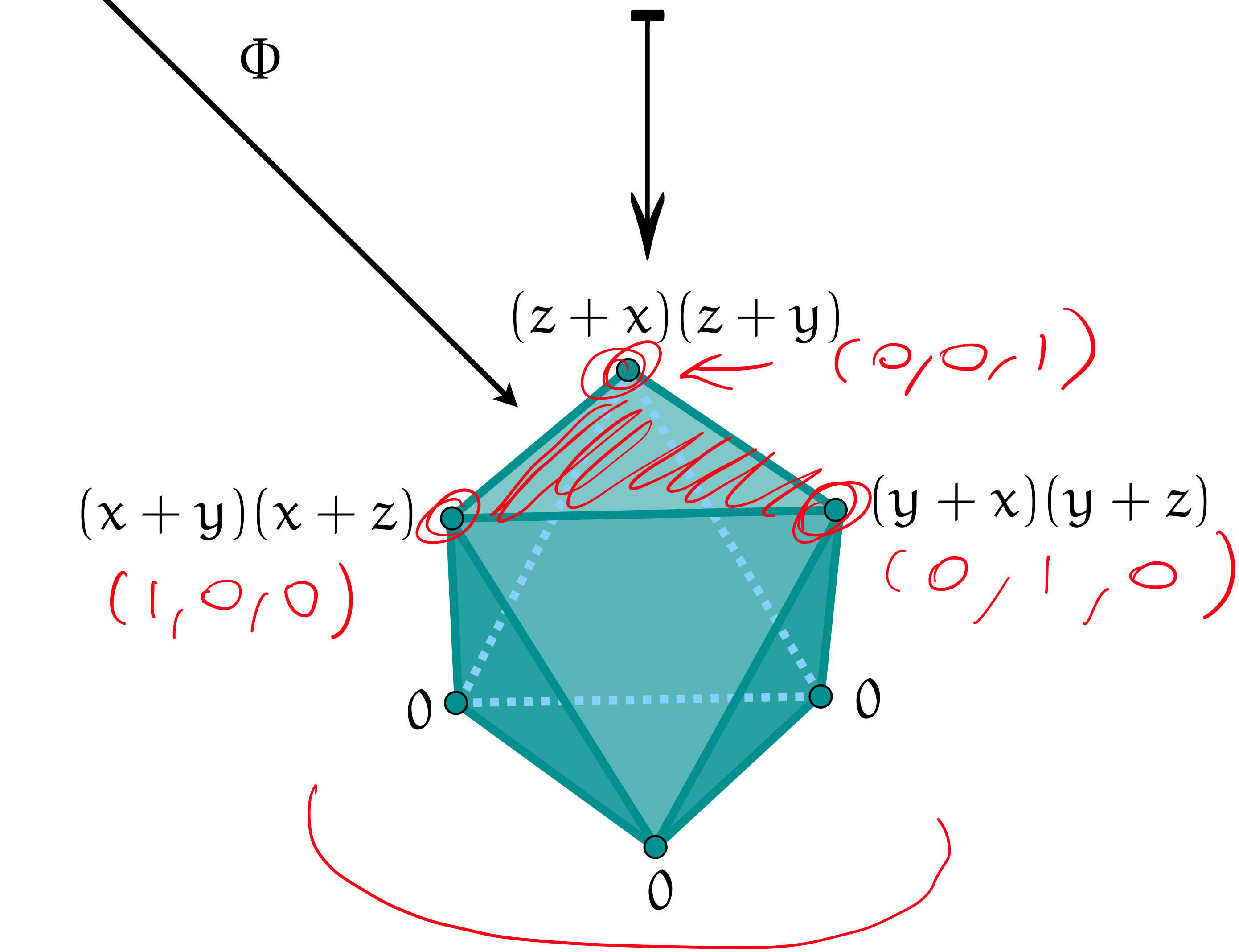
Φ



Example

$$T^3 \circlearrowright \mathcal{G}r(2, \mathbb{C}^4) \rightsquigarrow H_T^*(\mathcal{G}r(2,\mathbb{C}^4);\mathbb{Z}) \subseteq \bigoplus_{i=1}^{6} \mathbb{Z}[x,y,z]$$

$$\begin{array}{c} \Cup \\ \alpha = [\mathbb{P}^2] \end{array}$$



Equivariant cohomology

$$M^T \hookrightarrow M \quad \rightsquigarrow \quad H^*_T(M; R) \longrightarrow H^*_T(M^T; R)$$

1. M. Goresky, R. Kottwitz, and R. MacPherson, "Equivariant cohomology, Koszul duality, and the localization theorem." *Invent. Math.* **131** (1998), no. 1, 25–83.

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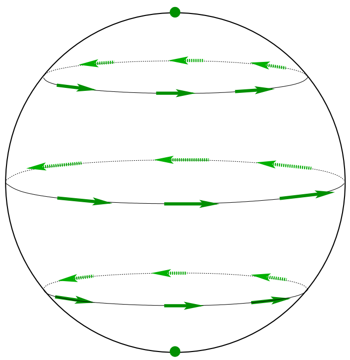

$$\alpha \in H^*_T(M;R) \Longrightarrow (\alpha|_N, \alpha|_S) \in \mathbb{R}[x] \oplus \mathbb{R}[x]$$

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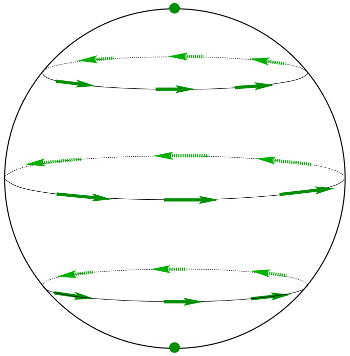

$$\alpha \in H_T^*(M;R) \Longrightarrow (\alpha|_N, \alpha|_S) \in \mathbb{R}[x] \oplus \mathbb{R}[x]$$

Fact: $(\alpha|_N, \alpha|_S) \in \mathrm{Im} \iff x \;\Big|\; (\alpha|_N - \alpha|_S)$

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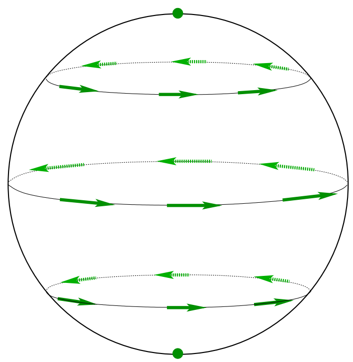

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<u>Fact:</u> $(\alpha|_N, \alpha|_S) \in \mathrm{Im} \iff x \,\Big|\, (\alpha|_N - \alpha|_S)$

<u>GKM Theorem</u>: Suppose that

(a) M^T consists of isolated points; and

(b) M^S consists of isolated points and S^2s, for each $S \subset T$ of codimension 1.

Weights are lin. indep. pairwise

HOLDS FOR: TVs, Coadj. orbits, ...

$$\text{Then } H_T^*(M;\mathbb{Q}) \cong \left\{ (f_v) \in \bigoplus_{v \in M^T} \mathbb{Q}[x_1, \ldots, x_d] \;\middle|\; \boxed{\alpha_e} \,|\, f_v - f_w \text{ for each } S_e^2 \right\}.$$

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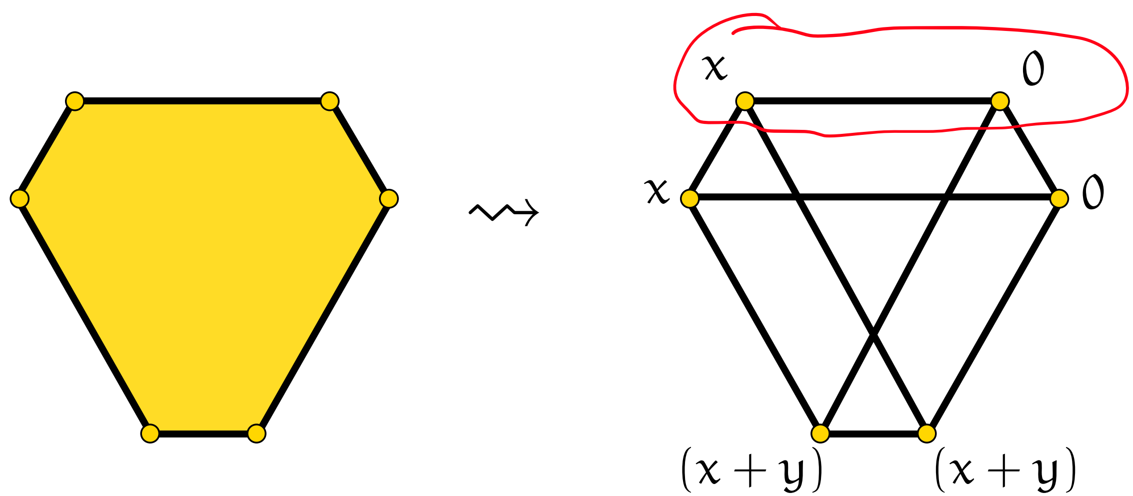
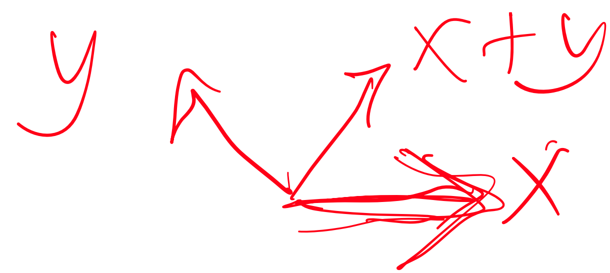
$$T^3 \circlearrowright \mathbb{C}^3 \rightsquigarrow T^3 \circlearrowright \mathrm{Flags}(\mathbb{C}^3)$$



$$\mathbb{R}^2 = \mathbb{R}^3 / \mathrm{Span}\begin{pmatrix}1\\1\\1\end{pmatrix}$$



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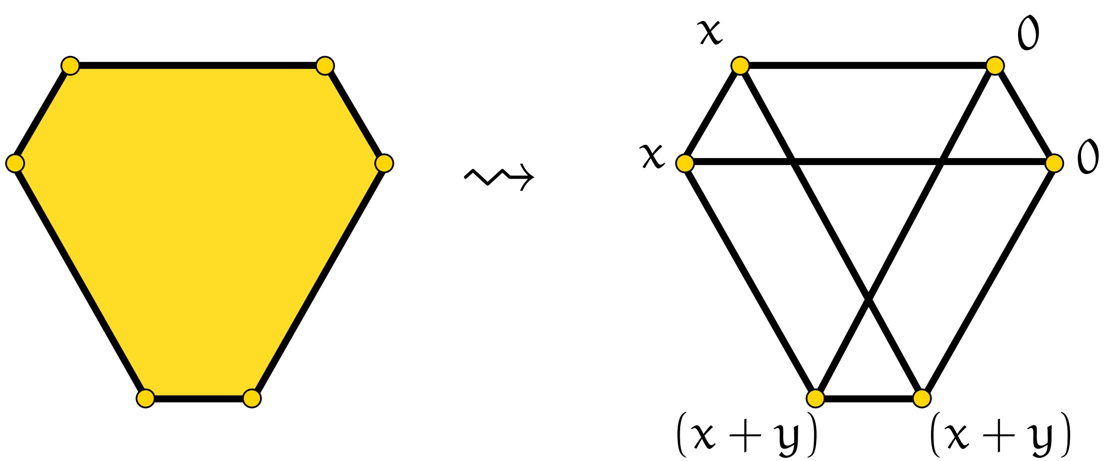

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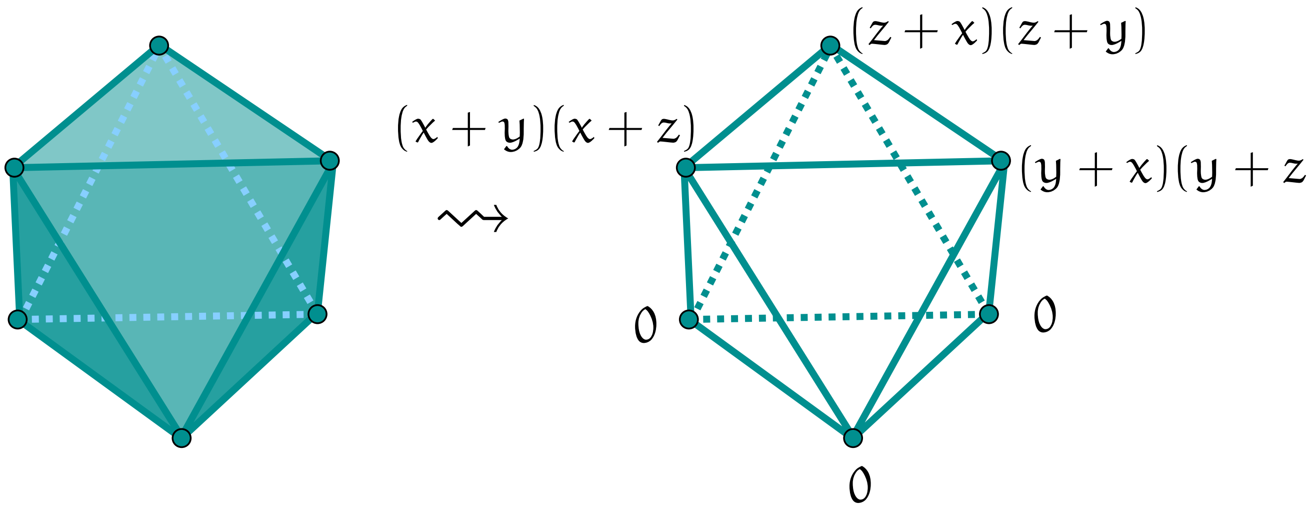

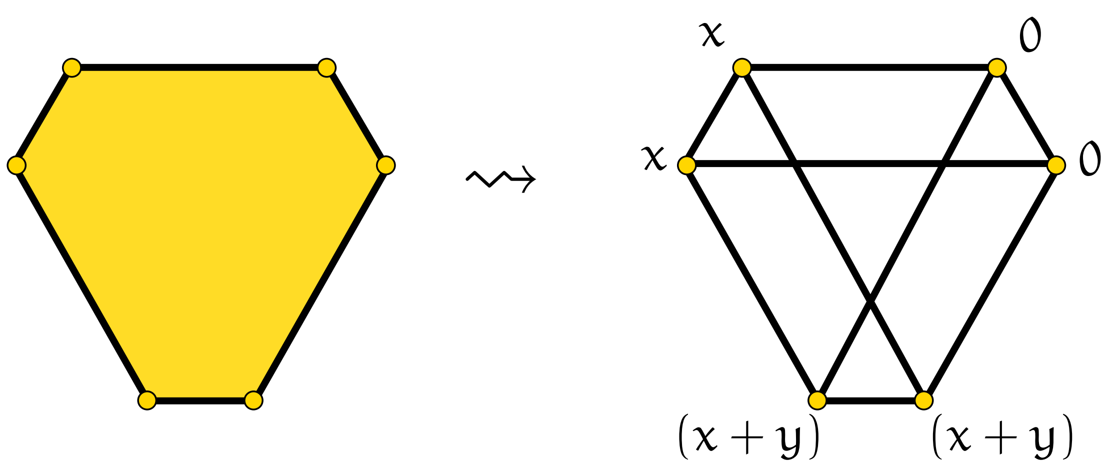

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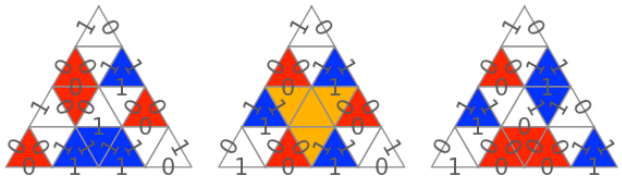
Further applications & generalizations

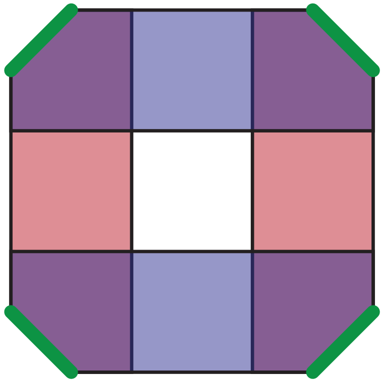
- Schubert calculus



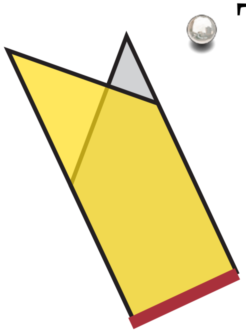
  <u>Questions:</u>
  - Does this lead to better/easier combinatorics?
  - How do you program “subrings” rather than “quotients”?

- Quantum invariants and Gromov-Witten theory

  <u>Question:</u>
  - How do you see quantum corrections in M^T?



- Torus manifolds (Masuda, Panov, Park)
- Toric origami manifolds (H-Pires)



  <u>Questions:</u>
  - What happens in the non-simply connected case?
  - Can you determine manifolds up to cobordism?