Courant algebroids and their integrations

Poisson 2026

Plan

1 - The Lie functor

2 - Courant algebroids: Examples and Roytenberg thm

3 - Integration : - Quadratic Lie algebras
- Doubles of Lie bialgebroids
- Exact CA
- Cotangent type

Disclaimer

Everything was known by Ševera.

The Lie functor

Recall

1. A Lie gpd $G \rightrightarrows M$ is given by

2 manifolds $\{$ G and M

$+$

5 smoth maps $\begin{cases} s,t : G \rightrightarrows M & \text{surjective submersions} \\ u : M \to G & \text{unit} \\ i : G \to G & \text{inverse} \\ m : G \times_M G \to G & \text{multiplication} \end{cases}$

$+$

equations (associativity, unit and inverse)

2. A Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$ is a vector bundle with $\rho : A \to TM$ and $[\cdot,\cdot] : \Gamma A \times \Gamma A \to \Gamma A$

s.t

- $[a,b] = -[b,a]$
- $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ $\qquad a,b,c \in \Gamma A$
- $[a, fb] = \rho(a)(f)\, b + f[a,b]$ $\qquad f \in C^\infty(M)$

Thm (Pradines)

There is a functor

$$\mathrm{Lie} : \{\text{Lie gpds}\} \longrightarrow \{\text{Lie algds}\}$$

Classical proof

Given $G \rightrightarrows M$ define

- $\mathrm{Lie}(G) = A_G = \ker Ts|_{u(M)} \to M$
- $\rho = Tt|_{A_G}$
- $[a,b]^r = [a^r, b^r]$

under the identification $\Gamma A \simeq \mathfrak{X}(G)^R$ □

Towards a "Contravariant" proof

↳ Instead of defining the object we define its functions

① Functions on a gpd

Def

A simplicial manifold $X_\bullet = \{(X_\ell, d_i^\ell, s_j^\ell)\}$ consists of a collection of manifolds $\{X_\ell\}_0^\infty$ and smooth maps

$$d_i^\ell : X_\ell \to X_{\ell-1} \quad 0 \le i \le \ell \quad \text{face}$$

$$s_j^\ell : X_\ell \to X_{\ell+1} \quad 0 \le j \le \ell \quad \text{degeneracies}$$

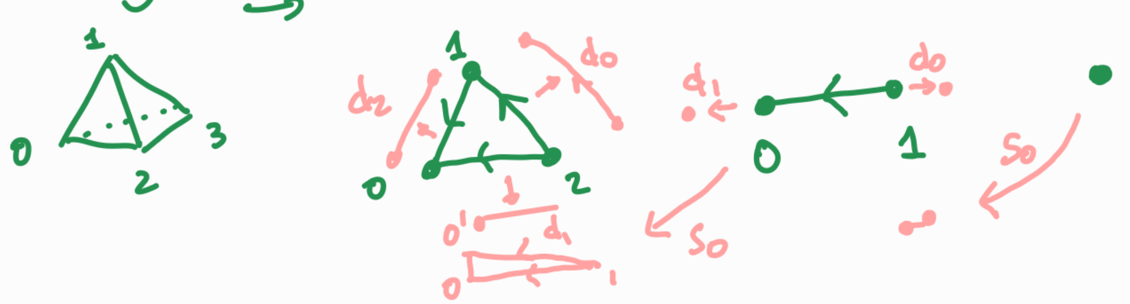
s.t

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ s_i s_j = s_{j+1} s_i & i \le j \\ d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases} \end{cases}$$

Simplicial identities

Idea

$$X_3 \;\substack{\to\\\to\\\to\\\to}\; X_2 \;\substack{\to\\\to\\\to}\; X_1 \;\substack{\to\\\to}\; X_0$$



Prop

$G \rightrightarrows M$ Lie gpd $\rightsquigarrow$ $N_\bullet G$ simplicial man given by

$N_0 G = M$

$N_\ell G = \{ \xi = (g_1, -, g_\ell) \mid s(g_i) = t(g_{i+1}) \}$

$d_0(g) = s(g), \ d_1(g) = t(g), \ s_0(x) = u(x)$

$d_0(\xi) = (g_2, -, g_\ell)$

$d_i(\xi) = (g_1, -, g_i g_{i+1}, -, g_\ell)$

$d_\ell(\xi) = (g_1, -, g_{\ell-1})$

$s_0(\xi) = (1_{t(g_1)}, g_1, -, g_\ell)$

$s_i(\xi) = (g_1, -, g_i, 1_{s(g_i)}, g_{i+1}, -, g_\ell)$

On the functions we get a differential graded algebra

$$\hat{C}^\infty(N_\bullet G) = \{ f : N_\bullet G \to \mathbb{R} \text{ smooth} \mid s_i^* f = 0 \}$$

$$\delta : \hat{C}^\infty(N_\ell G) \to \hat{C}^\infty(N_{\ell+1} G)$$

$$f \rightsquigarrow \delta(f) = \sum_{i=0}^{\ell+1} (-1)^i d_i^* f$$

$$\cup : \hat{C}^\infty(N_\ell G) \times \hat{C}^\infty(N_k(G)) \to \hat{C}^\infty(N_{\ell+k} G)$$

$$(f_1 \cup f_2)(g_1, -, g_{\ell+k}) = f_1(g_1, -, g_\ell) f_2(g_{\ell+1}, -, g_{\ell+k})$$

$(\hat{C}^\infty(N_\ell G), \delta, \cup)$ differential graded algebra.

② Functions on an algebroid

Given a Lie algebroid $(A \to M, \rho, [\cdot,\cdot])$ define the dga

$\rightsquigarrow (\Gamma \Lambda^\bullet A^*, d_{CE}, \wedge)$

$$(d_{CE}\eta)(a_0, \dots, a_k) = \sum_{i=0}^{k} (-1)^i \rho(a_i)\big(\eta(a_0, \dots, \hat{a}_i, \dots, a_k)\big) + \sum_{i<j} (-1)^i \eta(a_0, \dots, \hat{a}_i, \dots, [a_i, a_j], \dots, a_k)$$

Thm (Vaintrob)

For a vector bundle $A \to M$

$$\left\{ \begin{array}{c} \text{Lie algebroid structures} \\ (\rho, [\cdot,\cdot]) \end{array} \right\} \underset{}{\overset{1:1}{\rightleftarrows}} \left\{ Q : \Gamma \Lambda^\bullet A^* \to \Gamma \Lambda^{\bullet+1} A^* \right\}$$

s.t. $Q(\omega \wedge \eta) = Q(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge Q\eta$

and $Q^2 = 0$.

$\longleftrightarrow$

Contravariant proof (Cabrera - del Hoyo)

Given a Lie gpd $G \rightrightarrows M$

$$(\Gamma \Lambda^\bullet A^*, d_{CE}, \wedge) = \left(\frac{\hat{C}^\infty(N.G)}{J + \delta(J)}, [\delta], [\cup] \right)$$

↖ Higher vanishing ideal.

$$J_\ell = \sum_{i=0}^{\ell-1} I^2(s_i(N_{\ell-1}G))$$

Thm (Severa)

There is a Lie functor

$$\mathrm{Lie}: \left\{ \begin{array}{c}\text{simplicial manifolds} \\ X_\bullet \end{array} \right\} \longrightarrow \left\{ \begin{array}{c}\text{sheaves of} \\ \text{dga over } X_0 \end{array} \right\}$$

$$\begin{array}{ccc} \cup\!\shortmid & & \cup\!\shortmid \\ \{\ \text{Lie } n\text{-gpds}\ \} & \xrightarrow{\mathrm{Lie}} & \left\{ \begin{array}{c}\deg\ n \\ dg\text{-manifolds}\end{array} \right\} \end{array}$$

Def
A simplicial man $G_\bullet$ is a Lie n-gpd if the horn projections

$$P^i_j : G_i \to \Lambda^i_j(G) \equiv \left\{ \begin{array}{l}\text{we remove the interior of} \\ G_i \text{ and the } j\text{-face}\end{array} \right\}$$

are surjective submersions $\forall i,j$ and diffeomorphisms for all $i > n$

Def
Given a sheaf of dga $(M, \mathcal{A}_\bullet)$ we say that it is a deg n dg-manifold if $\forall p \in M \ \exists U$ s.t

$$\mathcal{A}_\bullet|_U \simeq C^\infty(U) \otimes \mathrm{Sym}(V_1 \oplus \_\_ \oplus V_n)$$

Notation $\quad (\mathcal{M}, Q) = (M, C^\infty_\bullet(\mathcal{M}), Q)$

Final Remarks

1) Interested on $n = 2$
2) Lie n-gpds form an iCFO (Rogers-Zhu)

$$\text{Lie } n\text{-gpds}[W^{-1}] \simeq \text{differentiable } n\text{-stacks}$$

3) Some care is need it when applying the Lie functor to the iCFO structure.