Symplectic Connections.

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Connections

Let M be a smooth manifold and let $\Gamma(TM)$ be the space of vector fields on M, i.e. the space of smooth sections of the tangent bundle TM. A linear connection on M, also called an affine connection on M or a linear connection on TM, is a bilinear map

$$\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) : (X, Y) \mapsto \nabla_X Y\,,$$

such that for all f in the set of smooth functions on M, written $C^\infty(M, \mathbb{R})$, and all vector fields X, Y on M:

$$\nabla_{fX} Y = f \nabla_X Y, \qquad \text{and} \qquad \nabla_X (fY) = (Xf) Y + f \nabla_X Y = df(X) Y + f \nabla_X Y.$$

The vector field $\nabla_X Y$ is called the covariant derivative of Y along X (with respect to the connection ∇).

Such a connection and covariant derivative can be defined more generally for sections of a vector bundle $E \xrightarrow{\pi} M$, as a bilinear map $\Gamma(TM) \times \Gamma(E) \to \Gamma(E) : (X, s) \mapsto \nabla_X s$, with the above conditions with respect to multiplication by a smooth function.

Equivalently, as a map $\Gamma(E) \to \Gamma(T^*M \otimes E) : s \to \nabla s$ so that $\nabla fs = f \nabla s + df \otimes s$.

Why linear connections?

Linear connections are tools to define parallel transport along a curve.

The holonomy at a point of a connection is a Lie group; it is defined using parallel transport along loops starting at that point.

They give a way to write in a coordinate free way mutidifferential operators.

They provide a way to define invariants on a manifold.

The group of diffeomorphisms preserving a connection is always a Lie group.

Content of the course:

Some definitions about connections on vector bundles.

General facts about Symplectic connections : existence, space of connections, properties of the curvature;

Symplectic connections and deformation quantisation on symplectic manifolds;

Symplectic connections of Ricci type, local, twistor spaces;

Some definitions about connections on principal bundles.

Global models for symplectic connections of Ricci type.

Generalization of the construction (by Cahen and Schwachhöfer) for special symplectic connections.

Some definitions about Connections on vector bundles

Local expression. Pullback connections.

Local expression: Let ∇ be a connection on the vector bundle $E \stackrel{\pi}{\to} M$. Let $s = \{s_1, \dots, s_k\}$ be a local frame for E over an open subset $U \subset M$. The connection defines a matrix valued 1-form Γ on U such that $\nabla_X s_i = \sum_k (\Gamma(X))_i^k \, s_k$. Hence for any section of E over U:

$$\nabla_X(\sum_i U^i s_i) = \sum_k \left(XU^k + (\Gamma(X))_j^k \, U^j \right) s_k.$$

Christoffel symbols: In the case of a tangent bundle, a coordinate chart (U, φ) with local coordinates $\{y^1, \dots y^m\}$ defines a local frame for TM :$\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m}\}$ and the $\Gamma_{ij}^k = \left(\Gamma(\frac{\partial}{\partial y^i})\right)_j^k$ so that $\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial y^k}$ are called the Christoffel symbols of the connection.

Pull back connection : If $f : M' \to M$ is smooth, on the pullback bundle $f^*E = \{(e, x') \in E \times M' \,|\, \pi(e) = f(x')\} \stackrel{pr_2}{\to} M'$ there is a pullback connection $f^*\nabla$: it is defined on $s' = \{s_1 \circ f, \dots, s_k \circ f\}$ which is a local frame for f^*E via the matrix valued 1-form $f^*\Gamma$. If u is a section of E and $u' = u \circ f$ then $(f^*\nabla)_X u' = (\nabla_{f_* X} u) \circ f$.

In particular, if $\gamma : I \to M$ is a smooth parametrized curve and if $Y : I \to TM$ is a vector field along the curve (i.e. $Y(t) \in T_{\gamma(t)}M, \forall t \in I$) one denotes $\nabla_{\dot\gamma(t)} Y = (\gamma^*\nabla)_{\frac{d}{dt}} Y$..

Parallel transport. Holonomy

Let ∇ be a connection on the vector bundle $E \xrightarrow{\pi} M$.
Let $\gamma : I \to M$ be a smooth parametrized curve.

Parallel sections along a curve : A section $Y : I \to E$ along γ –i.e. $Y(t) \in E_{\gamma(t)}$, i.e. Y is a section of $\gamma^* E$– is said to be parallel iff $\nabla_{\dot\gamma(t)} Y = ((\gamma^*\nabla)_{\frac{d}{dt}} Y) = 0 \; \forall t \in I$. This writes locally, using a local frame $s = \{s_1, \dots, s_k\}$ of E above U and writing $Y(t) = \sum_i Y^i(t) s_i(\gamma(t))$, as

$$\frac{d}{dt} Y^i(t) + \sum_j (\Gamma(\dot\gamma(t)))^i_j \, Y^j(t) = 0 \qquad \forall i \quad (\forall t \text{ such that } \gamma(t) \in U.)$$

Parallel transport along a curve : Given $t_0, t_1 \in I$ and given $Y(t_o) \in E_{\gamma(t_0)}$ there is one and only one parallel section $\bar Y$ along γ such that $\bar Y(t_0) = Y(t_o)$. The element $\bar Y(t_1)$ is called the image of $Y(t_o)$ under the parallel transport along γ from t_0 to t_1 denoted $T^\gamma_{t_0,t_1}$; from the equation above

$$T^\gamma_{t_0,t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)} \quad \text{is a linear map.}$$

Parallel transport around a loop γ at a point p yields a linear endomorphism T^γ_p of E_p. The group $H_p = \{ T^\gamma_p \,|\, \gamma \text{ is a loop at } p \}$ is called the holonomy group of ∇ at p. Its connected component H^0_p, called the restricted holonomy group, corresponds to parallel transport along contractible loops. They are Lie subgroups of $Gl(E_p)$ whose Lie algebra $\mathfrak{h}_p$ is called the holonomy Lie algebra at p. .

Curvature of a connection.

The curvature R^∇ of a connection ∇ on a vector bundle $E \xrightarrow{\pi} M$ is the 2-form R^∇ on M with values in $\mathrm{End}(E)$ defined by

$$R^\nabla(X,Y)u := \nabla_X(\nabla_Y u) - \nabla_Y(\nabla_X u) - \nabla_{[X,Y]}u \qquad \forall X, Y \in \Gamma(TM), \forall u \in \Gamma(E).$$

The curvature measures an infinitesimal parallel transport along an infinitesimal rectangle, so $R^\nabla_p(X,Y)$ is in the holonomy Lie algebra at p.

It is still a largely open problem to know which subgroups of $Gl(m,\mathbb{R})$ can be the holonomy subgroup of a torsion free linear connection on a manifold of dimension m. (It can be any subgroup without the torsion free condition.)

Let ∇ be a linear connection on M; its torsion $T^\nabla \in \Gamma(\Lambda^2 T^*M \otimes TM)$ is the tensor defined by

$$T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \qquad \forall X, Y \in \Gamma(TM).$$

We shall only deal here with linear connections which are torsion free, i.e. for which $T^\nabla = 0$.

General properties of symplectic connections

Connections adapted to the symplectic framework

(M, ω) symplectic of dimension $2n$ (i.e. $d\omega = 0$ and ω non degenerate)

A linear connection ∇ on M is said to be symplectic (P. Tondeur, A. Lichnerowicz) if
–its torsion T^∇ vanishes :

$$\Leftrightarrow T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0;$$

– the symplectic form ω is parallel:

$$\Leftrightarrow (\nabla_X \omega)(Y, Z) := X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0.$$

Existence : Canonical projection from torsion free connections to symplectic connections

Take ∇^0 torsion free, define N by $(\nabla^0_X \omega)(Y, Z) =: \omega(N(X, Y), Z)$.

One uses the fact that $\underset{XYZ}{\oplus}(\nabla^0_X \omega)(Y, Z) = d\omega(X, Y, Z) = 0$ because ∇^0 is torsion free.

Then $\nabla_X Y := \nabla^0_X Y + \frac{1}{3} N(X, Y) + \frac{1}{3} N(Y, X)$ is symplectic.

Indeed $(\nabla_X \omega)(Y, Z) = (\nabla^0_X \omega)(Y, Z) - \frac{1}{3}(\omega(N(X, Y), Z) + \omega(N(Y, X), Z) + \omega(Y, N(X, Z)) + \omega(Y, N(Z, X))) = 0$
because $\omega(Y, N(X, Z))) = -\omega(N(X, Z), Y) = \omega(N(X, Y), Z)$ and $\omega(N(Y, X), Z) + \omega(Y, N(Z, X)) = -\omega(N(X, Z), Y)$.

Non uniqueness :Take ∇ symplectic, then $\nabla'_X Y := \nabla_X Y + S(X, Y)$ is symplectic iff $\omega(S(X, Y), Z)$ is totally symmetric.

Indeed $(\nabla'_X \omega)(Y, Z) = (\nabla_X \omega)(Y, Z) - \omega(S(X, Y), Z) - \omega(Y, S(X, Z))$.

When is there a natural unique symplectic connection?

To get unicity, one needs extra structure on M.

- Symmetric symplectic space

(M, ω) symplectic with symmetries, i.e. $S : M \times M \to M$ C^∞, such that for each symmetry $s_x := S(x, \cdot)$, x is an isolated fixed point of s_x, $s_x^2 = \mathrm{Id}$, $s_x^* \omega = \omega$, and $s_x s_y s_x = s_{s_x y}$.

$$\Downarrow$$

there exists a unique symplectic connection for which each s_x is an affinity. It is given by

$$\omega_x(\nabla_X Y, Z) = \tfrac{1}{2} X_x \omega(Y + s_{x*} Y, Z).$$

Indeed $s_{x*_x} = -\mathrm{Id}$ and $s_{x*}(\nabla_X Y) = \nabla_{s_{x*}X} s_{x*} Y$; at x, we have $(\nabla_X Y)(x) = (\nabla_X s_{x*} Y)(x)$ and thus

$$\omega_x(\nabla_X Y, Z) = \tfrac{1}{2}\omega_x(\nabla_X Y + \nabla_X s_{x*} Y, Z) = \tfrac{1}{2} X_x \omega(Y + s_{x*} Y, Z) - \tfrac{1}{2}\omega_x(Y + s_{x*} Y, Z).$$

Any connection preserved by symmetries has vanishing torsion and is such that $\nabla R^\nabla = 0$.

Indeed $s_{x*} T^\nabla(X, Y) = T^\nabla(s_{x*}X, s_{x*}Y)$; computing at x : $-T_x^\nabla(X_x, Y_x) = T_x^\nabla(-X_x, -Y_x)$.

Levi Civita connection.

• pseudo-Kähler (resp. Kähler) manifold.

An almost complex structure J on a manifold M is a field of endomorphisms of TM which squares to $-\mathrm{Id}$. It is compatible with a symplectic structure ω if $\omega(JX, JY) = \omega(X, Y)$, hence iff g^J defined by $g^J(X, Y) := \omega(X, JY)$ is symmetric.
It is positive if g^J is positive definite.

A linear symplectic connection ∇ for which $\nabla J = 0$ for a compatible J is such that $\nabla g^J = 0$.
On a pseudo-Riemannian manifold (M, g) there is a unique torsion free connection such that $\nabla g = 0$ (where $(\nabla_X g)(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$); it is called the Levi Civita connection ∇^g and is given by

$$g(\nabla^g_X Y, Z) = \tfrac{1}{2}\left(Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([Z, X], Y) + g([Z, Y], X) + g([X, Y], Z)\right)$$

Indeed $g(\nabla^g_X Y, Z) = Xg(Y, Z) - g(Y, \nabla^g_X Z) = Xg(Y, Z) - g(Y, [X, Z]) - g(Y, \nabla^g_Z X) =$
$Xg(Y, Z) - Zg(Y, X) - g(Y, [X, Z]) + g(\nabla^g_Z Y, X) = Xg(Y, Z) - Zg(Y, X) - g(Y, [X, Z]) + g([Z, Y], X) + g(\nabla^g_Y Z, X) =$
$Xg(Y, Z) - Zg(Y, X) + Yg(Z, X) - g(Y, [X, Z]) + g([Z, Y], X) - g(Z, \nabla^g_Y X) =$
$Xg(Y, Z) - Zg(Y, X) + Yg(Z, X) - g(Y, [X, Z]) + g([Z, Y], X) - g(Z, [Y, X]) - g(Z, \nabla^g_X Y)$.

A manifold (M, ω, J) with a torsionfree connection preserving ω and J is called a pseudo-Kähler manifold and the connection in question is unique: it is the Levi Civita connection for g^J.

Curvature tensors associated to a symplectic connection.

The curvature tensorR^∇ associated to a linear connection ∇ is defined by

$$R^\nabla(X,Y) := \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}.$$

For a symplectic connection, the curvature at a point x in M, R^∇_x, is a 2-form on T_xM with values in symplectic Lie algebra $sp(T_xM,\omega_x)$: $\omega(R^\nabla(X,Y)Z,T)+\omega(Z,R^\nabla(X,Y)T)=0$.

Indeed $\omega(\nabla_X(\nabla_Y Z),T) = X\omega(\nabla_Y Z,T) - \omega(\nabla_Y Z,\nabla_X T) =$
$X(Y\omega(Z,T)) - X\omega(Z,\nabla_Y T) - Y\omega(Z,\nabla_X T) + \omega(Z,\nabla_Y(\nabla_X T))$. Hence $\omega(R^\nabla(X,Y)Z,T) =$
$X(Y\omega(Z,T)) - Y(X\omega(Z,T) - \omega(Z,\nabla_X(\nabla_Y T) - \nabla_Y(\nabla_X T)) - [X,Y]\omega(Z,T) + \omega(Z,\nabla_{[X,Y]}T)$.

The Ricci curvature, r^∇, is defined as $r^\nabla(X,Y) = \mathrm{Trace}\left(Z \mapsto R^\nabla(X,Z)Y\right)$.

The first Bianchi Identity ($\underset{X,Y,Z}{\oplus} R^\nabla(X,Y)Z) = 0$) and the fact that any element in $sp(T_xM,\omega_x)$ is traceless, imply that $r^\nabla(X,Y) = r^\nabla(Y,X)$.

A second "trace" of the curvature could be defined as $\omega(R(e_i,e^i)X,Y)$ where $(\omega(e_i,e^j) = \delta_i^j$. It is equal to $-2r^\nabla(X,Y)$ since $r^\nabla(X,Y) = \omega(R(X,e_i)Y,e^i) = \omega(R(X,e_i)e^i,Y)$.

There is no "scalar" curvature.

Moment map on the space $\mathcal{E}(M,\omega)$ of sympl. connections

$\mathcal{E}(M,\omega)$ is an affine space modelled on the space $\Gamma^\infty(S^3T^*M)$ of symmetric covariant 3–tensorfields on M; its tangent space at a point ∇ is identified with $\Gamma^\infty(S^3T^*M)$. We assume here that M is compact.

A symplectic 2-form Ω_∇ on $\mathcal{E}(M,\omega)$ is defined by

$$\Omega_\nabla(\underline{A},\underline{B}) := \int_M (\underline{A},\underline{B})\,\frac{\omega^n}{n!}$$

where $(\cdot,\cdot)$ denotes the pairing induced by ω in a chart, $(\underline{A},\underline{B})(x) = (\omega_x^{-1})^{i_1j_1}(\omega_x^{-1})^{i_2j_2}(\omega_x^{-1})^{i_3j_3}\underline{A}_{i_1i_2i_3}\underline{B}_{j_1j_2j_3}$.

Symplectic diffeomorphisms act on $\mathcal{E}(M,\omega)$: $(g\cdot\nabla)_X Y(x) := g_{*_{g^{-1}x}}\left(\nabla_{g_*^{-1}X}g_*^{-1}Y\right)$.

For $Y\in\chi(M)$ with $\mathcal{L}_Y\omega=0$, $Y^{*\mathcal{E}}_\nabla = \mathcal{L}_Y\nabla$.

The Lie algebra of hamiltonian vector fields is identified to $\mathcal{C}^\infty_{(0)}(M)$ (=smooth functions on M having 0 mean) via $i(X)\omega = df_X$.

A moment map will be a map $J:\mathcal{E}(M,\omega)\to\mathcal{C}^\infty(M)$, with $\mathcal{C}^\infty(M)$ viewed as a subspace of the dual of the algebra $\mathcal{C}^\infty_{(0)}(M)$, so that, for any hamiltonian vector field X

$< J(\nabla), f_X >:= \int_M J(\nabla) f_X \frac{\omega^n}{n!} = \phi_X(\nabla)$ where ϕ_X is a real function on $\mathcal{E}(M,\omega)$ such that $\frac{d}{dt}\phi_X(\nabla+tA)_{|0} = \Omega_\nabla(\underline{X^{*\mathcal{E}}},\underline{A}) = \int_M \left(\underline{\mathcal{L}_X\nabla},\underline{A}\right)\frac{\omega^n}{n!}$.

Such a map is given by $J(\nabla) = -\frac{1}{2}\, r^\nabla_{pq}r^{\nabla\,pq} + \frac{1}{4}\, R^\nabla_{pqrs}R^{\nabla\,pqrs} - (\nabla^2_{pq}r^\nabla)^{pq}$. (Cahen-G.)

Lecture 2: Deformation Quantization and symplectic connections

Why Quantization?

Quantum theory provides a description of nature which is more fundamental than classical theory. We shall consider here only non relativistic descriptions.

Why are we interested in quantization, nature being quantum?

Giving a quantum description a priori of a physical system is difficult, and the classical description is often easier to obtain; hence one often uses the classical description as a starting point to find a quantum description.

Classical vs Quantum Mechanics

Classical mechanics, in its Hamiltonian formulation on the motion space, has for framework a symplectic manifold (or more generally a Poisson manifold). The motion space is in general the quotient of the evolution space by the motion.

Observables are families of smooth functions on that manifold M.

The dynamics is defined in terms of a Hamiltonian $H \in C^\infty(M)$ and the time evolution of an observable $\{f_t\} \in C^\infty(M \times \mathbb{R})$ is governed by the equation :

$$\frac{d}{dt} f_t = -\{H, f_t\}.$$

Quantum mechanics, in its usual Heisenberg's formulation, has for framework a Hilbert space (states are rays in that space).

Observables are families of selfadjoint operators on the Hilbert space.

The dynamics is defined in terms of a Hamiltonian H, which is a selfadjoint operator, and the time evolution of an observable $\{A_t\}$ is governed by the equation :

$$\frac{dA_t}{dt} = \frac{i}{\hbar}[H, A_t].$$

Quantization

A natural suggestion for quantization is a correspondence $\mathcal{Q}\colon f \mapsto \mathcal{Q}(f)$ mapping a function f to a self adjoint operator $\mathcal{Q}(f)$ on a Hilbert space $\mathcal{H}$ in such a way that $\mathcal{Q}(1) = \mathrm{Id}$ and

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\}) + O(\hbar^2).$$

There is no correspondence defined on all smooth functions on M so that

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\}),$$

when one puts an irreducibility requirement which is necessary not to violate Heisenberg's principle. More precisely, Van Hove proved that there is no irreducible representation of the Heisenberg algebra, viewed as the algebra of constants and linear functions on $\mathbb{R}^{2n}$ endowed with the Poisson braket, which extends to a representation of the algebra of polynomials on $\mathbb{R}^{2n}$.

Flato, Lichnerowicz and Sternheimer introduced Deformation Quantization where they

" suggest that quantisation be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables."

Deformation quantisation

A natural star product on a Poisson manifold (M, P) is a bilinear map

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\nu]], \qquad (u, v) \mapsto u * v := \sum_{r \geq 0} \nu^r C_r(u, v),$$

which defines a formally associative product on $C^\infty(M)[[\nu]]$ when the map is extended $\mathbb{K}[[\nu]]$-linearly,i.e., for all $k > 0$, $\sum_{r+s=k, r,s \geq 0} (C_r(C_s(u, v), w) - C_r(u, C_s(v, w)) = 0$, such that :

- $C_0(u, v) = \mu(u, v) := uv$ and the skewsymmetric part of C_1 is the Poisson bracket;
- 1 is a unit : $1 * u = u * 1 = u$;
- each C_r is a bidifferential operator on M of order at most r in each argument.

Two star products $*_1$ and $*_2$ are equivalent if there exist a series of differential operator $B = \sum_{r \geq 1} \nu^r B_r$ such that

$$u *_2 v = \exp \nu B \left(\exp(-\nu B) u *_1 \exp(-\nu B) v \right).$$

Example: Moyal-Weyl $\star$-product and the Weyl algebra

Let P be a bivector on $V = \mathbb{R}^m$ with constant coefficients:

$$P = \sum_{i,j} P^{ij} \partial_i \wedge \partial_j, \quad P^{ij} \in \mathbb{R}$$

Then $P^{-ij} = \frac{1}{2}(P^{ij} - P^{ji})$ defines a Poisson bracket on M.

The Weyl-Moyal $\star$ product is

$$(u \star_M v)(z) = \exp\left(\nu P^{rs} \partial_{x^r} \partial_{y^s}\right)\left(u(x)v(y)\right)\big|_{x=y=z}.$$

Associativity follows from the fact that $\partial_{t^k}(u \star_M v)(t) = (\partial_{x^k} + \partial_{y^k}) \exp\left(\nu P^{rs} \partial_{x^r} \partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=t}$:

$$\begin{aligned} ((u \star_M v) \star_M w)(x') &= \exp\left(\nu P^{rs} \partial_{t^r} \partial_{z^s}\right)((u \star_M v)(t) w(z))\big|_{t=z=x'} \\ &= \exp\left(\nu P^{rs}(\partial_{x^r} + \partial_{y^r})\partial_{z^s}\right) \exp\left(\nu P^{r's'} \partial_{x^{r'}} \partial_{y^{s'}}\right)((u(x)v(y))w(z))\Big|_{x=y=z=x'} \\ &= \exp\left(\nu P^{rs}(\partial_{x^r}\partial_{z^s} + \partial_{y^r}\partial_{z^s} + \partial_{x^r}\partial_{y^s})\right)((u(x)v(y))w(z))\Big|_{x=y=z=x'} = (u \star_M (v \star_M w)(x'). \end{aligned}$$

When P is skewsymmetric and non degenerate, i.e; $V = \mathbb{R}^{2n}$, $P = \Omega^{-1}$ with $\Omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$, $(S(V^*)[\nu], \star_M)$ is called **the Weyl algebra** .

Relation to Weyl's quantization

For the usual quantization of $\mathbb{R}^{2n}$ with the canonical Poisson bracket $P = \Omega^{-1}$ as before [in coordinates $\{ q^i, p_i; 1 \le i \le n \}$ $\{u, v\} = \sum_{j=1}^n \left(\frac{\partial u}{\partial q^j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q^j} \right)$]

the *Weyl ordering* yields a bijection $\mathcal{Q}_{Weyl}$ between polynomials on $\mathbb{R}^{2n}$, $\mathbb{R}[p_i, q^j]$ and the space of differential operators with polynomial coefficients $D_{polyn}(\mathbb{R}^n)$:
$\mathcal{Q}_{Weyl}(1) = \mathrm{Id}$, , $\mathcal{Q}_{Weyl}(q^i) := Q^i := q^i \cdot$ is the multiplication by q^i, and
$\mathcal{Q}_{Weyl}(p_i) := P_i = i\hbar \frac{\partial}{\partial q^i}$ and to a polynomial in $p's$ and $q's$ the corresponding totally symmetrized polynomial in Q^i and P_j, e.g.

$$\mathcal{Q}_{Weyl}(q^1(p^1)^2) = \frac{1}{3}(Q^1(P^1)^2 + P^1 Q^1 P^1 + (P^1)^2 Q^1).$$

Then

$$\begin{aligned} f *_w g : &= \mathcal{Q}_{Weyl}^{-1}\left(\mathcal{Q}_{Weyl}(f) \circ \mathcal{Q}_{Weyl}(g)\right) \\ &= f.g + \frac{i\hbar}{2}\{f, g\} + O(\hbar^2) = f \star_M g|_{\nu = \frac{i\hbar}{2}}. \end{aligned} \tag{1}$$

Hence the Moyal star product corresponds (for $\nu = \frac{i\hbar}{2}$) to the composition of operators via Weyl's quantisation of polynomials on $\mathbb{R}^{2n}$.

Examples : Moyal-type star products on $\mathbb{R}^m$ and equivalence

We have seen that if $P(u,v) = P^{ij}\partial_{x_i} u \partial_{x_j} v$ with the P^{ij} constants then $P^-(u,v) = \frac{1}{2}(P(u,v) - P(v,u))$ is a Poisson bracket and

$$(u *_P v)(x) := \left(exp \nu \left(P^{ij} \partial_{y_i} \partial_{z_j} \right) u(y) v(z) \right)_{y=z=x}$$

defines a natural star product on (M, P^-). If P_1 and P_2 are 2 such constant bivectors with $P_1^- = P_2^-$, then $P_2^{ij} = P_1^{ij} + 2A^{ij}$ and

$$exp \nu \left(P_2^{ij} \partial_{y_i} \partial_{z_j} \right) = exp \nu A^{ij} (\partial_{y_i} + \partial_{z_i})(\partial_{y_j} + \partial_{z_j})\, exp \nu P_1^{ij} \partial_{y_i} \partial_{z_j}\, exp - \nu A^{ij} \partial_{y_i} \partial_{y_j}\, exp - \nu A^{ij} \partial_{z_i} \partial_{z_j};$$

defining $B(u) = A^{ij} \partial_{x_i} \partial_{x_j}(u)$, we have

$$u *_{P_2} v = \exp \nu B \left(\exp(-\nu B) u *_{P_1} \exp(-\nu B) v \right)$$

In particular all classical quantisations (Weyl, normal ordering, standard ordering...) on $\mathbb{R}^{2n}$ yield equivalent star products.

Existence and classification of star products, Kontsevich (1995)

The set of equivalence classes of differential star products on a Poisson manifold (M, P) coincides with the set of equivalence classes of Poisson deformations of P:

$$P_\nu = P\nu + P_2\nu^2 + \cdots \in \nu\Gamma(X, \Lambda^2 T_X)[[\nu]], \text{ such that } [P_\nu, P_\nu]_S = 0,$$

where equivalence of Poisson deformations is defined via the action of a formal vector field on M, $X = \sum_{r\geq 1} \nu^r X_r$, via $\{u, v\}' := e^X \left\{e^{-X} u, e^{-X} v\right\}$.

A general yoga sees any deformation theory encoded in a differential graded Lie algebra structure.

A **differential graded Lie algebra** (briefly DGLA) $(\mathfrak{g}, [\ ,\ ], d)$ is a $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}^i, [\ ,\ ]$ with $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$,together with a differential $d\colon \mathfrak{g} \to \mathfrak{g}$, i.e. a graded derivation of degree 1 ($d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}$, $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$) so that $d \circ d = 0$.

A **deformation** is a Maurer-Cartan element, i.e. a $C \in \nu\mathfrak{g}^1[[\nu]]$ so that $dC - \frac{1}{2}[C, C] = 0$.

To express star products in that framework, consider **the DGLA of polydifferential operators.**

To express formal Poisson structures in that framework, consider **the DGLA of polyvector fields.**

Star products and DGLA of polydifferential operators

Consider the **Hochschild complex** of multilinear maps from $C^\infty(M)$ to itself, with $\mathcal{C}^i := \mathrm{Hom}(C^\infty(M)^{\otimes(i+1)}, C^\infty(M))$; the degree is shifted by one; the degree $|A|$ of a $(p+1)$–linear map A is equal to p. For $A_1 \in \mathcal{C}^{m_1}$, $A_2 \in \mathcal{C}^{m_2}$, define:

$$(A_1 \circ A_2)(f_1, \ldots, f_{m_1+m_2+1}) := \sum_{j=1}^{m_1} (-1)^{(m_2)(j-1)} A_1(f_1, \ldots, f_{j-1}, A_2(f_j, \ldots, f_{j+m_2}), f_{j+m_2+1}, \ldots, f_{m_1+m_2+1})$$

The **Gerstenhaber bracket** is defined by $[A_1, A_2]_G := A_1 \circ A_2 - (-1)^{m_1 m_2} A_2 \circ A_1$.

An element $M \in \mathcal{C}^1$ defines an associative product iff $[M, M]_G = 0$.

Define $\mathcal{D}_{poly}(M) := \bigoplus \mathcal{D}^i_{poly}(M)$ with $\mathcal{D}^i_{poly}(M)$ the set of multi differential operators acting on $i+1$ smooth functions on M and vanishing on constants.

The Gerstenhaber bracket gives $\mathcal{D}_{poly}(M)$ the structure of a graded Lie algebra.

The **differential** d_μ is defined by $d_\mu A = -[\mu, A]_G$.

$$(\mathcal{D}_{poly}(M), [\ ,\ ]_G, d_{\mathcal{D}} := d_\mu|_{\mathcal{D}}) \quad \text{is a DGLA.}$$

A $\star$-product is given by a series of bidifferential operators $\star = \mu + C$ with $C \in \nu\mathcal{D}^1_{poly}(M)[[\nu]]$ a Maurer-Cartan element of the DGLA $(\mathcal{D}_{poly}(M), [\ ,\ ]_G, d_{\mathcal{D}})$; indeed the associativity $[\mu + C, \mu + C]_G = 0$ is equivalent to $d_D C - \frac{1}{2}[C, C]_G = 0$.

Parenthesis: writing a differential operator with a connections

If ∇ is a linear connection on M, we have seen that one can covariantly derive any tensor on M. In particular, for any function $u \in C^\infty(M)$ one has the following

∇u is a 1-form :$\nabla u(X) = X(u)$ so that $\nabla u = du$;

$\nabla^2 u = \nabla(du)$ is a 2-form : $\nabla^2 u(X, Y) = (\nabla_X du)(Y) = X(du(Y)) - du(\nabla_X Y)$ hence $\nabla^2 u(X, Y) = X(Yu)) - (\nabla_X Y)u$

$\nabla^{k+1} u := \nabla(\nabla^k u)$ is a $k+1$-form and

$$\nabla^{k+1} u(X_1, \ldots X_{k+1}) = X_1(\ldots(X_{k+1}u)..) + \text{lower order terms.}$$

In a local chart with coordinates $\{y^1, \ldots, y^m\}$

$$\nabla^k u(\frac{\partial}{\partial y^{i_1}}, \ldots \frac{\partial}{\partial y^{i_k}}) = \frac{\partial^k}{\partial y^{i_1} \ldots \partial y^{i_k}} u + \text{lower order terms.}$$

and any differential operator D of order k writes uniquely as

$$D = \sum_{r=0}^{k} D_r \nabla^r$$

where the D_r are symmetric r-contravariant tensors.

Connections and quantisation (Lichnerowicz, Fedosov)

A natural star product on a symplectic manifold determines a unique symplectic connection $\nabla = \nabla(*)$ such that $C_1 = \{.,.\} + d_\mu E_1$ i.e. $C_1(u,v) = \{u,v\} + E_1(uv) - uE_1(v) - vE_1(u)$ and

$$C_2 = \frac{1}{2}(\mathrm{ad}\, E_1)^2\,\mu + ((\mathrm{ad}\, E_1)\,\{.,.\}) + \frac{1}{2}P^2(\nabla^2\cdot, \nabla^2\cdot) + A_2$$

where A_2 is of order at most 1 in each argument, $((\mathrm{ad}\, E_1)\, C) = [E_1, C]_G$, and

$$P^2(\nabla^2 u, \nabla^2 v) := P^{ij}P^{i'j'}\,\nabla^2_{ii'}u\,\nabla^2_{jj'}v$$

in a chart.
Moreover the map $* \mapsto \nabla(*)$ is equivariant under the symplectomorphism group.(Rawnsley-G.)

In particular all classical quantisations on $\mathbb{R}^{2n}$ yield (equivalent) star products which all correspond to the standard flat connection.

Existence of a star product on any symplectic manifold

On any symplectic manifold (M, ω) there exists a differential star product (1983, De Wilde and Lecomte). In 1985 and 1994, Fedosov gave a recursive construction when one has chosen a symplectic connection and a sequence of closed 2-forms $\tilde{\Omega} = \sum_{k\geq 1} \nu^k \omega_k$ on M.

It is obtained by identifying $C^\infty(M)[[\nu]]$ with an algebra of flat sections of the Weyl bundle $\mathcal{W} = B(M,\omega) \times_{Sp(V,\Omega),\rho} W$ endowed with a flat connection D built from ∇ et $\tilde{\Omega}$.

$B(M,\omega)$ is the bundle of symplectic frames : a symplectic frame at $x \in M$ is a linear symplectic isomorphism $\xi_x : (V = \mathbb{R}^{2n}, \Omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}) \to (T_xM, \omega_x))$.
$B(M,\omega)$ is a principal $Sp(V,\Omega)$-bundle over M.

One puts a grading on the Weyl algebra $S(V^*)[\nu]$, assigning the degree 1 to the linear function y and the degree 2 to ν. Recall that the product is given by

$$(u \star_M v)(z) = \exp\left(\frac{\nu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right) (u(x)v(y))\Big|_{x=y=z},$$

with $P = \Omega^{-1}$. The **formal Weyl algebra** W is the completion in that grading.

ρ is the natural representation of the symplectic group $Sp(V,\Omega)$ on W (for any $B \in sp(V,\Omega)$, $\rho_*(B)a = \frac{-1}{\nu}[\overline{B}, a]$ where $[a,b] := (a \star_M b) - (b \star_M a)$ for any $a, b \in W$ and $\overline{B} = \frac{1}{2}\sum_{ijr} \Omega_{ri} B^r_j y^i y^j$); it acts by automorphisms of $\star_M$, so the Weyl bundle is a bundle of algebras, the fiber product being $\star_M$.

Fedosov's construction on any symplectic manifold

The symplectic connection induces a connection ∂ in $\mathcal{W}$: $\partial a = da - \frac{1}{\nu}[\frac{1}{2}\omega_{ki}\Gamma^k_{rj}y^iy^jdx^r, a]$.

Deform it into $Da = \partial a - \delta(a) - \frac{1}{\nu}[r,a]$ where $\delta(a) = \frac{1}{\nu}\left[-\omega_{ij}y^idx^j, a\right] = \sum_k dx^k \wedge \frac{\partial a}{\partial y^k}$, with $r \in \Gamma(\mathcal{W}\otimes\Lambda^1)$. Its curvature is given by $\frac{1}{\nu}\left[\overline{R} - \partial r + \delta r + \frac{1}{2\nu}[r,r], a\right]$;

for it to vanish, choose r so that $\delta r = -\overline{R} + \partial r - \frac{1}{\nu}r^2 + \tilde{\Omega}$; such a r is given inductively by $r = -\hat{\delta}\overline{R} + \hat{\delta}\partial r - \frac{1}{\nu}\hat{\delta}r^2 + \hat{\delta}\tilde{\Omega}$ where, writing $a = \sum_{p\geq 0, q\geq 0} a_{pq} = \sum_{2k+p\geq 0, q\geq 0}\nu^k a_{k,i_1,\dots,i_p,j_1,\dots,j_q}y^{i_1}\dots y^{i_p}\,dx^{j_1}\wedge\dots\wedge dx^{j_q}$, for any $a \in \Gamma(\mathcal{W}\otimes\Lambda^q)$,

$$\hat{\delta}(a_{pq}) = \frac{1}{p+q}\sum_k y^k i(\frac{\partial}{\partial x^k})a_{pq} \text{ if } \mathrm{p+q>0} \text{ and } 0 \text{ if } \mathrm{p+q=0}$$

A flat section of $\mathcal{W}$ is given inductively by $a = \hat{\delta}\left(\partial a - \frac{1}{\nu}[r,a]\right) + a_{00}$.
It is determined by $a_{00} \in \mathbb{C}^\infty(M)[[\nu]]$ and is denoted $Q(a_{00})$.

The Fedosov's star product $*^{Fed}_{\nabla,\Omega}$ is then obtained by $u \star^{Fed}_{\nabla,\Omega} v := (Q(u) \star_M Q(v))_{00}$.

Links between the moment map and quantization

A star product is said to be closed if the integral on the manifold defines a trace on the deformed algebra $(C^\infty(M)[[\nu]], \star)$.
The star product constructed by Fedosov from the data of a symplectic connection ∇ and a series of closed 2-forms A is denoted $*^{Fed}_{\nabla, A=\nu^k A_k}$.

Laurent La Fuente-Gravy showed that the Fedosov star product $*^{Fed}_{\nabla,0}$ is closed at order 3 in the deformation parameter ν iff $J(\nabla)$ is a constant function on M.

With Akito Futaki, he defined further obstructions to the closedness of a star product,

considering the normalized trace of Fedosov star products.

Lecture 3 : Selecting symplectic connections

Curvature tensor of a symplectic connection

We have defined a symplectic connection on a symplectic manifold (M, ω) as a torsionfree linear connection ∇ on TM such that $\nabla\omega = 0$ and we have seen that the space of those connections is an affine space modelled on the space of symmetric covariant 3-tensor fields on M. The curvature tensor is $R^\nabla(X, Y) := \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}$. For a symplectic connection, it satisfies the following pointwize algebraic identities at each point $x \in M$:

$$R^\nabla_x(X, Y)Z = -R^\nabla(Y, X)Z \qquad \bigoplus_{X,Y,Z} R^\nabla_x(X, Y)Z = 0$$

$$\underline{R}_x(X, Y, Z, T) := \omega_x(R^\nabla_x(X, Y)Z, T) = \omega_x(R^\nabla_x(X, Y)T, Z),$$

hence $\underline{R}_x$ is an element in $\Lambda^2 T^*_x M \otimes S^2 T^*_x M$ which is in the kernel of

$$a : \Lambda^2 T^*_x M \otimes S^2 T^*_x M \to \Lambda^3 T^*_x M \otimes T^*_x M$$

where a is the skexsymmetrization map.

The Koszul long exact sequence

Given any finite dimensional vector space V, the Koszul long exact sequence is

$$0 \longrightarrow S^q(V) \xrightarrow{a} V \otimes S^{q-1}(V) \xrightarrow{a} \Lambda^2 V \otimes S^{q-2}(V) \xrightarrow{a} \cdots \xrightarrow{a} \Lambda^{q-1}(V) \otimes V \xrightarrow{a} \Lambda^q(V) \longrightarrow 0$$

where a is the skewsymmetrisation operator:

$$a(v^1 \wedge \ldots \wedge v^q \otimes w^1 \cdots w^p) = \sum_{i=1}^{p} v^1 \wedge \ldots \wedge v^q \wedge w^i \otimes w^1 \cdots w^{i-1} w^{i+1} \cdots w^p.$$

Indeed, introducing the symmetrisation operator as:

$$s(v^1 \wedge \ldots \wedge v^q \otimes w^1 \cdots w^p) \sum_{i=1}^{q} (-1)^{q-i} v^1 \wedge \ldots v^{i-1} \wedge v^{i+1} \ldots \wedge v^q \otimes v^i \cdot w^1 \cdots w^p$$

the operators satisfy $a^2 = 0,\ s^2 = 0,\ (a_\circ s + s_\circ a)_{|\Lambda^q V \otimes S^p(V)} = (p+q)\mathrm{Id}.$

In particular one has the long exact sequence

$$0 \longrightarrow S^4(V) \xrightarrow{a} V \otimes S^3(V) \xrightarrow{a} \Lambda^2 V \otimes S^2(V) \xrightarrow{a} \Lambda^3(V) \otimes V \xrightarrow{a} \Lambda^4(V) \longrightarrow 0$$

and

$$\begin{aligned} \ker a|_{\Lambda^2 V \otimes S^2(V)} &= \operatorname{Im} a|_{V \otimes S^3(V)} \simeq (V \otimes S^3(V))/\ker a|_{V \otimes S^3(V)} \\ &= (V \otimes S^3(V))/\operatorname{Im} a|_{S^4(V)} = (V \otimes S^3(V))/S^4(V). \end{aligned}$$

Decomposition of the curvature R_x^∇ under $Sp(T_xM,\omega_x)$

The space $\mathcal{R}_x$ of 4-tensors satisfying the algebraic identities of a symplectic curvature tensor at x is:

$$\mathcal{R}_x := \ker a_{|_{\Lambda^2(V)\otimes S^2(V)}} \simeq \left(V\otimes S^3(V)\right)/S^4(V) \qquad \text{for} \quad V = T_x^*M.$$

Under the action of the group $Sp(V,\Omega)$ the space $V\otimes S^3(V)$, in dimension $2n\geq 4$, decomposes into three irreducible subspaces $S^4(V)\oplus S'^2(V)\oplus W$ where $S'^2(V) = a(s(\Omega_x\otimes S^2(V)))\sim S^2(V)$

$a(s(\Omega\otimes r))(X,Y,Z,T) = 2\Omega(X,Y)r(Z,T)+\Omega(X,T)r(Y,Z)+\Omega(X,Z)r(T,Y)-\Omega(Y,T)r(X,Z)-\Omega(Y,Z)r(T,X)$

so that the curvature of a symplectic connection splits into two irreducible components

$$R^\nabla = E^\nabla + W^\nabla \qquad dim M = 2n\geq 4 \qquad (I.\ Vaisman, M.\ De\ Visscher)$$

where W^∇ has no trace and

$$\begin{aligned} E^\nabla(X,Y)Z &= \tfrac{1}{2n+2}\Big(2\omega(X,Y)\rho^\nabla Z+\omega(X,Z)\rho^\nabla Y-\omega(Y,Z)\rho^\nabla X \\ &\quad +\omega(X,\rho^\nabla Z)Y-\omega(Y,\rho^\nabla Z)X\Big)\end{aligned}$$

with the Ricci tensor $r^\nabla(X,Z) = \mathrm{Trace}\big(Y\to R^\nabla(X,Y)Z\big)$ converted into an endomorphism

$$\omega(X,\rho^\nabla Y) = r^\nabla(X,Y).$$

A symplectic connection for which $W^\nabla = 0$ is said to be of **Ricci-type**.

Preferred connections?

To select symplectic connections through a variational principle one considers a Lagrangian $L(R^\nabla)$, which is a polynomial in the curvature of the connection ∇, invariant under the action of the symplectic group

$$\int_M L(R^\nabla)\omega^n.$$

and the easiest choice is a polynomial of degree 2 in R^∇
The first Pontryagin class is represented by the 4-form $P_1(\nabla)$ with

$$P_1(\nabla)\wedge\omega^{n-2} = \frac{1}{16\pi^2}[r^\nabla \cdot r^\nabla - \tfrac{1}{2}R^\nabla \cdot R^\nabla]\omega^n.$$

Since this combination will be constant under variations, all non-trivial Euler equations coming from a variational principle built from a second order invariant polynomial in the curvature are the same; they take the form:

$$\underset{X,Y,Z}{\oplus}(\nabla_X r^\nabla)(Y,Z) = 0. \quad (Bourgeois - Cahen)$$

A symplectic connection ∇ satisfying this equation is said to be **preferred** .

Ricci-type symplectic connections, local models

Construction of Ricci-type connections by reduction.

- $\mathbb{R}^{2n+2}, \Omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$ and $a \in sp(\mathbb{R}^{2n+2}, \Omega)$ i.e. ${}^t a\Omega + \Omega a = 0$.

- $\Sigma_a = \{x \in \mathbb{R}^{2n+2} | \Omega(x, ax) = 1\}$ (suppose it is not empty!)

$$M^{red} := {}_{\exp ta}\backslash\Sigma_a \qquad \pi : \Sigma_a \to M^{red}$$

This can always be locally defined as follows.
Since the vector field ax is nowhere 0 on Σ_a, for any $x_0 \in \Sigma_a$, there exist:
– a neighbourhood $U_{x_0} (\subset \Sigma_a)$,
– a ball $D^{red} \subset \mathbb{R}^{2n}$ of radius r_0, centred at the origin,
– a real interval $I = (-\epsilon, \epsilon)$
– and a diffeomorphism $\chi : D^{red} \times I \to U_{x_0}$ such that $\chi(0, 0) = x_0$ and $\chi(y, t) = \exp -ta(\chi(y, 0))$. We shall denote

$$\pi : U_{x_0} \to D^{red} \quad \pi = p_1 \otimes \chi^{-1}.$$

The space D^{red} is a local version of the Marsden–Weinstein reduction of Σ_a around the point x_0.

- $H_x := \{x, ax\}^{\perp_\Omega} \subset T_x\mathbb{R}^{2n+2}$; $\pi_* : H_x \simeq T_{\pi(x)}M^{red}$ $(\overline{X}_x \leftrightarrow X_{y=\pi(x)})$.

- $\omega^{red}_{y=\pi(x)}(X, Y) := \Omega(\overline{X}_x, \overline{Y}_x)$ then (M^{red}, ω^{red}) is symplectic.

- $(\nabla^{red}_X Y)_y := \pi_{*_x}(\nabla_{\overline{X}}\overline{Y} - \Omega(a\overline{X}, \overline{Y})x + \Omega(\overline{X}, \overline{Y})ax)$ where ∇ is the flat connection on $\mathbb{R}^{2n+2}$;

then ∇^{red} is symplectic and of Ricci-type (Baguis, Cahen).

A Ricci type connection is preferred.

For a Ricci type connection
$R^\nabla(Y,Z)T =$
$\frac{1}{2n+2}\left(2\omega(Y,Z)\rho^\nabla T + \omega(Y,T)\rho^\nabla Z - \omega(Z,T)\rho^\nabla Y + \omega(Y,\rho^\nabla T)Z - \omega(Z,\rho^\nabla T)Y\right),$
the second Bianchi identity $\underset{X,Y,Z}{\oplus}(\nabla_X R)(Y,Z) = 0$ yields

$0 = \underset{X,Y,Z}{\oplus}\big(2\omega(Y,Z)(\nabla_X\rho^\nabla)T + \omega(Y,T)(\nabla_X\rho^\nabla)Z - \omega(Z,T)(\nabla_X\rho^\nabla)Y +$
$\omega(Y,(\nabla_X\rho^\nabla)T)Z - \omega(Z,(\nabla_X\rho^\nabla)T)Y\big)\,.$

For $Y = e_i, Z = e^i$ with $\omega(e_i, e^j) = \delta_i^j$ it gives after summation
$(4n-2)(\nabla_X\rho^\nabla)T + 2\omega(T,(\nabla_{e_i}\rho^\nabla)e^i)X - 2\omega(X,T)(\nabla_{e_i}\rho^\nabla)e^i.$

Hence, there is a vector U^∇ such that

$$(\nabla_X\rho^\nabla)Y = \omega(U^\nabla, Y)X + \omega(X,Y)U^\nabla.$$

Since $r^\nabla(Y,Z) = \omega(Y,\rho^\nabla Z) = -\omega(\rho^\nabla Y, Z)$, this implies

$$(\nabla_X r^\nabla)(Y,Z) = -\omega(X,Z)\omega(U^\nabla,Y) - \omega(U^\nabla,Z)\omega(X,Y),$$

and $\underset{X,Y,Z}{\oplus}(\nabla_X r^\nabla)(Y,Z) = 0.$

Thus any Ricci-type connection is preferred.

Properties of the curvature of a Ricci-type connection

Computing $(\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})\rho^\nabla)$ from the formula $(\nabla_X \rho^\nabla) = \omega(U^\nabla, \cdot)X + \omega(X, \cdot)U^\nabla$ gives

$$\omega(\nabla_X U, \cdot)Y + \omega(Y, \cdot)\nabla_X U - \omega(\nabla_Y U, \cdot)X - \omega(X, \cdot)\nabla_Y U.$$

On the other hand, it is equal to $[R(X, Y), \rho^\nabla]$. Since

$R^\nabla(X, Y) = \frac{1}{2n+2}\left(2\omega(X, Y)\rho^\nabla + \omega(X, \cdot)\rho^\nabla Y - \omega(Y, \cdot)\rho^\nabla X + \omega(X, \rho^\nabla \cdot)Y - \omega(Y, \rho^\nabla \cdot)X\right),$

it is equal to

$$\frac{1}{2n+2}\left(\omega((\rho^\nabla)^2 X, \cdot)Y + \omega(Y, \cdot)(\rho^\nabla)^2 X - \omega((\rho^\nabla)^2 Y, \cdot)X - \omega(X, \cdot)(\rho^\nabla)^2 Y\right).$$

This implies, that there exists a functionf^∇ such that

$$\nabla_X U^\nabla = \frac{1}{2n+2}(\rho^\nabla)^2 X + f^\nabla X.$$

because the only field of endomorphisms A such that

$$\omega(AX, \cdot)Y + \omega(Y, \cdot)AX - \omega(AY, \cdot)X - \omega(X, \cdot)AY = O \qquad \forall X, Y$$

is $A = f\mathrm{Id}$.

More properties of the curvature of a Ricci-type connection

(M,ω,∇) Ricci-type $\Rightarrow \begin{cases} (\nabla_X\rho^\nabla)Y = X\omega(U^\nabla,Y)+U^\nabla\omega(X,Y), \\ \nabla_X U^\nabla = \frac{1}{2n+2}(\rho^\nabla)^2X + f^\nabla X. \end{cases}$

Introducing the 1-form u^∇ defined by $u^\nabla(Y)=\omega(U^\nabla,Y)$ the relation above yields

$$(\nabla_X u^\nabla)(Y) = \omega(\nabla_X U^\nabla, Y) = \frac{1}{2n+2}\omega((\rho^\nabla)^2X,Y) + f^\nabla\omega(X,Y)$$

and, by skewsymmetrization,

$$du^\nabla = \frac{1}{n+1}r^{2,\nabla} + 2f^\nabla\omega$$

with the 2-form $r^{2,\nabla}$ defined by $r^{2,\nabla}(X,Y)=\omega(X,(\rho^\nabla)^2Y)=\omega((\rho^\nabla)^2X,Y)$.
Observe that

$$\begin{aligned} dr^{2,\nabla}(X,Y,Z) &= \underset{XYZ}{\oplus}(\nabla_X r^{2,\nabla})(Y,Z) = -\underset{XYZ}{\oplus}\omega((\nabla_X\rho^\nabla)\rho^\nabla Y + \rho^\nabla(\nabla_X\rho^\nabla)Y,Z) \\ &= -2\underset{XYZ}{\oplus}\omega(X,Z)\omega(U^\nabla,\rho^\nabla Y). \end{aligned}$$

Hence $dr^{2,\nabla} = -2\left(\iota(\rho^\nabla U^\nabla)\omega\right)\wedge\omega = -2(n+1)df^\nabla\wedge\omega$.
This implies, in dimension ≥ 4:

$$df^\nabla = \frac{1}{n+1}\iota(\rho^\nabla U^\nabla)\omega,$$

Connection determined by its jet of order 3 at a point.

Remark that

$$d(\mathrm{Trace}(\rho^\nabla)^2)(X) = \nabla_X(\omega((\rho^\nabla)^2 e_i, e^i) = \omega((\nabla_X\rho^\nabla)\rho^\nabla e_i, e^i) - \omega((\nabla_X\rho^\nabla)e_i, \rho^\nabla e^i)$$

which is equal to $-4\omega(\rho^\nabla U^\nabla, X)$ since $(\nabla_X\rho^\nabla)Y = X\omega(U^\nabla, Y) + U^\nabla\omega(X, Y)$.

Since we have seen that $df^\nabla = \frac{1}{n+1}\iota(\rho^\nabla U^\nabla)\omega$, there exists a constant k^∇ such that

$$f^\nabla = \frac{-1}{4(n+1)}(\mathrm{Trace}(\rho^\nabla)^2) + k^\nabla$$

To summarize, (M, ω, ∇) Ricci-type $\Rightarrow \begin{cases} (\nabla_X\rho^\nabla)Y = X\omega(U^\nabla, Y) + U^\nabla\omega(X, Y), \\ \nabla_X U^\nabla = \frac{1}{2n+2}(\rho^\nabla)^2 X + f^\nabla X., \\ f^\nabla = \frac{-1}{4n+4}\,\mathrm{Trace}(\rho^\nabla)^2 + k^\nabla \quad k^\nabla \in \mathbb{R} \end{cases}$

Hence, $R, \nabla R, \nabla^2 R, \ldots$ are determined by $\rho^\nabla,\ U^\nabla, k^\nabla$ so that the germ of the connection at one point up to order 3 determines the whole germ.

Construction of the local model.

In the example $(M^{red} = \Sigma_a/_{\exp ta}, \omega^{red}, \nabla^{red})$, $\begin{cases} \overline{\rho^{\nabla^{red}} X}(x) = -2(n+1)\overline{a_x}\bar{X} \\ \bar{U}^{\nabla^{red}}(x) = -2(n+1)(2n+1)\overline{a_x^2}x \\ (\pi^* f^{\nabla^{red}})(x) = 2(n+1)(2n+1)\Omega'(a^2x, ax) \end{cases}$

with $\overline{a_x^k}(X) = a^kX + \Omega'(a^kX, x)Ax - \Omega'(a^kX, ax)x$.

Any symplectic manifold with a Ricci-type connection (M, ω, ∇) is locally diffeomorphic to a model $(M^{red}, \omega^{red}, \nabla^{red})$ (Cahen, G., Schwachhöffer)

Take $p \in M$, and ξ a symplectic frame at p [i.e. $\xi : (\mathbb{R}^{2n}, \Omega^{(2n)}) \to (T_pM, \omega_p)$ iso],

set $\tilde{u}(\xi) = (\xi)^{-1}\ U^{\nabla}(p) \in \mathbb{R}^{2n}$, $\quad \tilde{\rho}(\xi) = (\xi)^{-1}\ \rho^{\nabla}(p)\ \xi \in \mathrm{End}(\mathbb{R}^{2n})$, and:

$$\tilde{a}(\xi) = \begin{pmatrix} 0 & \frac{f(p)}{2(n+1)(2n+1)} & \frac{-\underline{\tilde{u}(\xi)}}{2(n+1)(2n+1)} \\ 1 & 0 & 0 \\ 0 & \frac{-\tilde{u}(\xi)}{2(n+1)(2n+1)} & \frac{-\tilde{\rho}(\xi)}{2(n+1)} \end{pmatrix} \qquad \text{with} \qquad \underline{u} := \Omega(u, \cdot).$$

Some definitions about Connections on principal bundles.

Principal connections and their curvature

Let $P \xrightarrow{p} M$ be a principal G-bundle. We denote by $\cdot$ or $r : P \times G \to P : (\xi, g) \mapsto \xi \cdot g = r_g \xi$ the right action of the Lie group G on M (with $p(\xi \cdot g) = p(\xi)$).

A (principal) connection on $P \xrightarrow{p} M$ is a 1-form α on P with values in the Lie algebra $\mathfrak{g}$ of G such that $r_g^* \alpha = \operatorname{Ad} g^{-1} \alpha \quad \forall g \in G$ and $\alpha(A^{*P}) = A \quad \forall A \in \mathfrak{g}$ where $A_\xi^{*P} = \frac{d}{dt} \xi \cdot \exp tA|_{t=0}$.

It gives a splitting of the tangent space to P :

$$T_\xi P = V_\xi \oplus H_\xi \qquad \text{with } V_\xi = \ker P_{*\xi} \quad \text{and} \quad H_\xi = \ker \alpha_\xi .$$

An element $X_x \in T_x M$ admits a unique horizontal lift $\bar{X}_\xi$ for $\xi \in p^{-1}(x)$ defined by $\alpha_\xi(\bar{X}_\xi) = 0$ and $p_*(\bar{X}_\xi) = X_x$.

The curvature of the principal connection α is the 2-form on P with values in $\mathfrak{g}$ given by

$$\Omega = d\alpha + \tfrac{1}{2}[\alpha \overset{\wedge}{,} \alpha] \quad \text{i.e. } \Omega(X, Y) = d\alpha(X, Y) + [\alpha(X), \alpha(Y)].$$

Link with linear Connections.

Any connection ∇ on a vector bunble $E \stackrel{\pi}{\to} M$ with fibers of dimension k gives rise to a principal connection α on the frame bundle $B(E) \to M$ which is a $Gl(\mathbb{R}, k)$-principal bundle, a frame at the point x being a linear isomorphism $\xi : \mathbb{R}^k \to E_x = \pi^{-1}x$.

Let $s = \{s_1, \ldots, s_k\}$ be a local frame for E over an open subset $U \subset M$. We can view s as a map from $\sigma : U \to B(E)$ with $s_i(x) = \sigma(x)(e_i)$ for $\{e_1, \ldots e_k\}$ the usual canonical basis of $\mathbb{R}^k$.

We have recalled that The connection ∇ on E defines a matrix valued 1-form Γ on U such that $\nabla_X s_i = \sum_k (\Gamma(X))_i^k \, s_k$.

The associated principal connection α on $B(E)$ is entirely determined by the fact that

$$\sigma^* \alpha = \Gamma.$$

Holonomy of a connection on a principal bundle.

Given a curve $\gamma : I \to M$, a curve $\bar\gamma : I \to P$ is a horizontal lift of γ if $p(\bar\gamma(t)) = \gamma(t)$ and $\dot{\bar\gamma}(t)$ is horizontal for all $t \in I$. Given an initial point in the fiber above $x = \gamma(t_0)$, this horizontal lift exists and is unique.

The horizontal lift through ξ of a loop γ in M based at $x = p(\xi)$ is not necessarily closed and there is an element $g_\xi^\gamma \in G$ such that the end point is $\xi \cdot g_\xi^\gamma$.

The set Hol_ξ^α of all group elements g_ξ^γ form a Lie subgroup of G called the holonomy at ξ of the principal connection α.

Let $\xi \in P$ be an arbitrary point of the principal bundle. Let $H(\xi)$ be the set of points in P which can be joined to ξ by a horizontal curve. Then $H(\xi)$ is a principal bundle over M with structure group Hol_ξ^α; it is called the holonomy bundle (through ξ) of the connection.

The Ambrose-Singer's theorem states that the Lie algebra of Hol_ξ^α is spanned by all the elements of $\mathfrak{g}$ of the form $\Omega_\eta(X, Y)$ as η ranges over all points which can be joined to ξ by a horizontal curve, and X and Y are horizontal tangent vectors at η.

Global construction of Ricci type connections.

Global construction.

Theorem : If (M, ω, ∇) is of Ricci type with M simply connected there exists (P, ω^P) symplectic of dimension 2 higher with a flat connection ∇^P so that (M, ω, ∇) is obtained from (P, ω^P, ∇^P) by reduction.

$P = N \times \mathbb{R}$ where N is the holonomy bundle over M corresponding to a (principal) connection defined on the $Sp(\mathbb{R}^{2n+2}, \Omega')$-principal bundle

$$B'(M, \omega) = B(M, \omega) \times_{Sp(\mathbb{R}^{2n}, \Omega)} Sp(\mathbb{R}^{2n+2}, \Omega')$$

where $B(M, \omega) \xrightarrow{\pi} M$ is the $Sp(\mathbb{R}^{2n}, \Omega)$-principal bundle of symplectic frames over M and where $Sp(\mathbb{R}^{2n}, \Omega)$ injects into $Sp(\mathbb{R}^{2n+2}, \Omega')$ as $\tilde{j}(A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}$.

The connection 1-form α' on $B'(M, \omega)$ is characterised by

$$\alpha'_{[\xi,1]}([\overline{X}^{hor}, 0]) = \begin{pmatrix} 0 & \frac{-\omega_x(U^\nabla, X)}{2(n+1)(2n+1)} & \frac{-\underline{\widetilde{\rho^\nabla(X)}(\xi)}}{2(n+1)} \\ 0 & 0 & -\underline{\tilde{X}(\xi)} \\ \tilde{X}(\xi) & \frac{-\widetilde{\rho^\nabla(X)}(\xi)}{2(n+1)} & 0 \end{pmatrix}$$

where $\overline{X}^{hor}$ is the horizontal lift in $T_\xi B(M, \omega)$ of $X \in T_x M$, with $x = \pi(\xi)$.

Holonomy bundle N.

The curvature 2-form $\Omega^{\alpha'}$ of α' is equal to $-2\tilde{a'}\pi'^*\omega$, where $\tilde{a'}$ is the unique $Sp(\mathbb{R}^{2n+2},\Omega')$-equivariant extension of $\tilde{a}$ to $B'(M,\omega)$.
It is invariant by parallel transport thus the holonomy algebra of α' is of dimension 1.

Assume M is simply connected. The holonomy bundle of α' is a circle or a line bundle over M, $N \xrightarrow{\pi'} M$. This bundle has a natural contact structure ν given by the restriction to $N \subset B(M,\omega)'$ of the 1-form $-\frac{1}{2}\alpha'$ (viewed as real valued since it is valued in a 1-dimensional algebra). One has $d\nu = \pi'^*\omega$.

The symplectic manifold with connection (P,ω^P,∇^P) is then obtained by an induction procedure.

Construction of symplectic manifolds by induction.

A **contact quadruple** (M, N, α, π) is a $2n$ dimensional smooth manifold M, a $2n+1$ dimensional smooth manifold N, a co-oriented contact structure α on N (i.e. α is a 1-form on N such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing), and a smooth submersion $\pi : N \to M$ with $d\alpha = \pi^*\omega$ where ω is a symplectic 2-form on M.

Given a contact quadruple (M, N, α, π) the **induced symplectic manifold** is the $2n+2$ dimensional manifold

$$P := N \times \mathbb{R}$$

endowed with the (exact) symplectic structure

$$\mu := 2e^{2s}\ ds \wedge p_1^*\alpha + e^{2s}\ dp_1^*\alpha = d(e^{2s}\ p_1^*\alpha)$$

where s denotes the variable along $\mathbb{R}$ and $p_1 : P \to N$ the projection on the first factor.

Notations: $p = \pi \circ p_1 : P \to M$; if $X \in \chi(M)M$, $\bar{\bar{X}} \in \chi(P)$ is defined by

$$(i)\ \ p_*\bar{\bar{X}} = X \qquad (ii)\ \ (p_1^*\alpha)(\bar{\bar{X}}) = 0 \qquad (iii)\ \ ds(\bar{\bar{X}}) = 0.$$

E is the vector field on P such that

$$(i)\ \ p_{1*}E = Z \qquad (ii)\ \ ds(E) = 0$$

where Z is the Reeb vector field on N (i.e. $i(Z)d\alpha = 0$ and $i(Z)\alpha = 1$).

Clearly the values at any point of P of $\bar{\bar{X}}, E, S = \partial_s$ span the tangent space to P at that point.

Construction of Ricci-flat connections by induction.

Starting from a symplectic connection ∇ on M, define ∇^P by:

$$\begin{aligned}
\nabla^P_{\bar{\bar{X}}}\bar{\bar{Y}} &= \overline{\overline{\nabla_X Y}} - \tfrac{1}{2}p^*(\omega(X,Y))E - p^*(\hat{s}(X,Y))\partial_s\\
\nabla^P_E\bar{\bar{X}} &= \nabla^P_{\bar{\bar{X}}}E = 2\overline{\overline{\sigma X}} + p^*(\omega(X,u))\partial_s \qquad \nabla^P_{\partial_s}\bar{\bar{X}} = \nabla^P_{\bar{\bar{X}}}\partial_s = \bar{\bar{X}}\\
\nabla^P_E E &= p^* f\,\partial_s - 2\bar{\bar{U}} \qquad \nabla^P_E\partial_s = \nabla^P_{\partial_s}E = E \qquad \nabla^P_{\partial_s}\partial_s = \partial_s
\end{aligned}$$

where f is a function on M, U is a vector field on M, $\hat{s}$ is a symmetric 2-tensor on M, and σ is the endomorphism of TM associated to s, hence $\hat{s}(X,Y) = \omega(X,\sigma Y)$.

∇^P is a symplectic connection on (P,μ).

Choosing

$$\begin{aligned}
\hat{s} &= \frac{-1}{2(n+1)}r^\nabla \qquad \underline{U} := \omega(U,\cdot) = \frac{2}{2n+1}\mathrm{Trace}[Y \to \nabla_Y\sigma]\\
f &= \frac{1}{2n(n+1)^2}\mathrm{Trace}(\rho^\nabla)^2 + \frac{1}{n}\mathrm{Trace}[X \to \nabla_X U],
\end{aligned}$$

the connection ∇^P on (P,μ) is Ricci-flat (i.e. has zero Ricci tensor);

if the symplectic connection ∇ is of Ricci-type, then the connection ∇^P on (P,μ) is flat.

Star products by reduction

The reduction of Moyal star product on $\mathbb{R}^{2n+2}$ yields a star product on the reduced space which corresponds to the Ricci-type connection.

Twistor bundle

Let (M, ω) be a symplectic manifold.
An almost complex structure J, i.e. a smooth section of $\mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$ on (M, ω), is *compatible* if $\omega(JX, JY) = \omega(X, Y)$, hence
iff g^J defined by $g^J(X, Y) := \omega(X, JY)$ is symmetric.
It is *positive* if g^J is positive definite.

On any symplectic manifold, there exist positive compatible almost complex structures, and the set of those is path connected.

Choose a metric g, write $\omega(X, Y) = g(AX, Y)$; set $J := \left(\sqrt{AA^{*g}}\right)^{-1} A$.

On the model vector space $V = \mathbb{R}^{2n}$, $\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$ a positice compatible complex structure is given by $j_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$ and any other one is obtained by conjugating by an element in the symplectic group $Sp(V, \Omega)$.

Positive compatible almost complex structures are sections of the symplectic twistor bundle.

Geometric interpretation of the Ricci-type condition

Positive compatible almost complex structures are sections of the symplectic twistor bundle

$$J^+(M,\omega) = B(M,\omega) \times_{Sp(V,\Omega)} (Sp(V,\Omega)/U(V,\Omega,j_0)) = B(M,\omega)/U(V,\Omega,j_0)$$

with $B(M,\omega)_p = \{\xi_p : (V,\Omega) \xrightarrow{\simeq} (T_pM,\omega_p)\}$, $V = \mathbb{R}^{2n}$, $\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$, $j_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$.

A symplectic connection ∇ on (M,ω) defines an almost complex structure $\mathcal{J}^\nabla$ on the twistor space $J^+(M,\omega) = B(M,\omega)/U(V,\Omega,j_0)$:

$$\mathcal{J}^\nabla_J|_{H^\nabla_J} = (\pi *_J|_{H^\nabla_J})^{-1} \circ J \circ (\pi *_J|_{H^\nabla_J}) \qquad \mathcal{J}^\nabla|_{V_J} = J\cdot$$

where H^∇_J is the horizontal space at the point J induced by the horizontal space defined by ∇ on the frame bundle $B(M,\omega)$, thus $\pi *_J|_{H^\nabla_J} : H^\nabla_J \simeq T_pM$ with $\pi : J^+(M,\omega) \to M$ the bundle projection and $p = \pi(J)$, and where $V_J = \mathrm{Ker}(\pi_*)_J$ hence

$$V_J \simeq \{C \in \mathrm{sp}(T_pM,\omega_p)\ i.e.\ \omega_p(CX,Y) + \omega_p(X,CY) = 0 \,|\, CJ + JC = 0\}.$$

Theorem (Burstall, Rawnsley; Vaisman) : The symplectic connection ∇ is of Ricci-type iff the almost complex structure $\mathcal{J}^\nabla$ is integrable.

Special symplectic connections

Another view of the construction of a local model

$\left(\mathbb{R}_0^{2n+2}, \Omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}\right)$ can be viewed as the orbit of $e_2 := \begin{pmatrix} 0 & \begin{pmatrix} 1 & 0\ldots0 \\ 0 & 0\ldots0 \\ . & .. \\ 0 & 0\ldots0 \end{pmatrix} \\ 0 & 0 \end{pmatrix}$ in $\mathfrak{g} = \mathfrak{sp}(\mathbb{R}^{2n+2}, \Omega)$ with its natural symplectic structure.

The bracket in $\mathfrak{sp}(\mathbb{R}^{2n+2}, \Omega)$ by the element $h_0 := \begin{pmatrix} \begin{pmatrix} 1 & 0\ldots0 \\ 0 & 0\ldots0 \\ . & .. \\ 0 & 0\ldots0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} -1 & 0\ldots0 \\ 0 & 0\ldots0 \\ . & .. \\ 0 & 0\ldots0 \end{pmatrix} \end{pmatrix}$ has for eigenvalues $0, \pm1, \pm2$ and gives an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \quad \text{with} \quad \mathfrak{g}_2 = \mathbb{R}e_2, \dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2 = 1.$$

The Lie algebra of the stabilizer of e_2 is $\mathfrak{p}' = \mathfrak{h} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}$ with $\mathfrak{h} = (h_0)^\perp \subset \mathfrak{g}_0$.

Let $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}$ and let $\mathrm{P} \subset \mathrm{G}$ be the corresponding connected Lie subgroup.

Then $\mathcal{C} := \mathrm{G}/\mathrm{P}$ can be viewed as the oriented projectivisation of the adjoint orbit of e_2 i.e. We view $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ where $\mathbb{P}^o(V)$ denotes the set of oriented lines through 0 of a vector space V, so that $\mathbb{P}^o(V)$ is the sphere in V.

Cahen-Schwachhöfer local models

Each $a \in \mathfrak{g}$ induces an action field a^* on $\mathcal{C}$ with flow $T_a := \exp(\mathbb{R}a) \subset \mathrm{G}$.

$\mathcal{C} = \mathrm{G}/\mathrm{P}$ carries a G-invariant contact structure determined by $\mathfrak{g}_{-1} \mod \mathfrak{p} \subset \mathfrak{g}/\mathfrak{p} \cong T\mathcal{C}$.

Let $\mathcal{C}_a \subset \mathcal{C}$ be the open subset on which a^* is transversal to the contact distribution.

We cover $\mathcal{C}_a$ by open sets U such that the local quotient $M_U^{red} := T_a^{loc} \backslash U$, i.e. the quotient of U by a sufficiently small neighbourhood of the identity in T_a, is a manifold.

The action of T_a preserves the contact structure on $\mathcal{C}$. There is a unique contact form α so that a^* is its Reeb vector field (i.e. α is a 1-form on $\mathcal{C}_a$ vanishing on the contact distribution and $\alpha(a^*) = 1$).
The quotient M_U^{red} inherits a canonical symplectic structure ω^{red} such that $\pi^*(\omega^{red}) = d\alpha$ for the canonical projection $\pi : U \to M_U^{red}$.

This generalizes whenever one has a real simple Lie algebra $\mathfrak{g}$ (of dim ≥ 14) which contains the root space of a long root. This means that $\mathfrak{g}$ is 2-gradable,

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \quad \text{with} \quad \dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2 = 1.$$

One proceeds as above with $h_0 \in [\mathfrak{g}_2, \mathfrak{g}_{-2}]$ the grading element, with $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}$ and $\mathrm{P} \subset \mathrm{G}$ the corresponding connected Lie subgroup.

Cartan Geometry

"I was led to introduce the notion of new geometries, ... The vast synthesis that I realized in this way depends of course on the ideas of Klein formulated in his celebrated Erlangen programme while at the same time going far beyond it since it includes Riemannian geometry, which had formed a completely isolated branch of geometry, within the compass of a very general scheme in which the notion of group still plays a fundamental role." (Elie Cartan, 1939)

A Cartan geometry is a space equipped with a Cartan connection

Let G be a Lie group and H a Lie subgroup; a Cartan geometry modeled on G/H, also called a (G, H)-Cartan geometry, is a right H-principal bundle $P \to M$ together with a 1-form α' on P with values in the Lie algebra $\mathfrak{g}$ of G, called a Cartan connection, verifying:

- at each $\xi \in P$, α'_ξ is an isomorphism from $T_\xi P$ to $\mathfrak{g}$;
- for each $h \in H$, $R_h^* \alpha' = \mathrm{Ad}(h^{-1})\alpha'$;
- for each $X \in \mathfrak{h}$ (Lie algebra of H), $\alpha'(X^*) = X$ for X^* the fundamental vector field associated to the action $X^*_\xi = \frac{d}{dt}|_0 \xi \cdot \exp tX$.

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Examples of Cartan Geometry

Homogeneous spaces G/H are the simplest examples of a (G,H)-Cartan geometry : one considers the principal H bundle over the homogeneous space G/H given by the natural projection

$$G \to G/H$$

and the Maurer-Cartan form on G, $\alpha'_g(L_{g*}X) := X$ for all $g \in G$ and all $X \in \mathfrak{g}$ defines a Cartan connection in this situation.

Every linear connection ∇ on a manifold M of dimension m induces a $(Gl(m,\mathbb{R})\cdot\mathbb{R}^m, Gl(m,\mathbb{R}))$-Cartan geometry: the $Gl(m,\mathbb{R})$-principal bundle over M is the frame bundle $B(M) \xrightarrow{p} M$, and the Cartan connection α' with values in $\mathfrak{g}l(m,\mathbb{R}) + \mathbb{R}^m$ is given by $\alpha + \theta$ where α is the principal connection corresponding to ∇ and θ is the soldering form $\theta_\xi(X) = \xi^{-1}(p_*X)$.

A parabolic subalgebra $\mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$ is a subalgebra containing a maximal solvable subalgebra (a Borel subalgebra).

A parabolic geometry is a Cartan (G,P) geometry with G semisimple and P a parabolic subgroup, i.e. a subgroup whose Lie algebra is a parabolic subalgebra of $\mathfrak{g}$.

Construction of the special symplectic connection

Let $pr : \mathrm{G} \to \mathcal{C} = \mathrm{G}/\mathrm{P}$. Choose $e_2 \in \mathfrak{g}_2, e_{-2} \in \mathfrak{g}_{-2}$ so that $\beta(e_{-2}, e_2) = 1$ for β the Killing form. Then

$$pr^{-1}(\mathcal{C}_a) = \{g \in G \,|\, \beta\left(e_2, \mathrm{Ad}(g^{-1})a\right) \neq 0\,\} \quad \text{because} \quad a^*_{gP} = pr_{*g}(L_g)_*\left(\mathrm{Ad}(g^{-1})a\right).$$

Recall that $\mathfrak{h} = (h_0)^{\perp} \subset \mathfrak{g}_0$; let H be the corresponding connected Lie subgroup of G; define

$$\Gamma_a := \{g \in G \,|\, \mathrm{Ad}(g^{-1})a - \tfrac{1}{2}e_{-2} \in \mathfrak{p}' = \mathfrak{h} + \mathfrak{g}_1 + \mathfrak{g}_2\,\} \subset pr^{-1}(\mathcal{C}_a) \subset \mathrm{G}.$$

Then $pr_{|\Gamma_a} : \Gamma_a \to \mathcal{C}_a$ becomes a principal H-bundle.

Consider the Maurer-Cartan form $\mu := g^{-1}dg \in \Lambda^1(\mathrm{G}) \otimes \mathfrak{g}$; the restriction of the components $\mu_{\mathfrak{h}} + \mu_{-1} + \mu_{-2}$ to Γ_a yields a pointwise linear isomorphism $T\Gamma_a \to \mathfrak{h} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$.
Define the differential forms $\kappa \in \Lambda^1(\Gamma_a)$, $\theta \in \Lambda^1(\Gamma_a) \otimes \mathfrak{g}_{-1}$ and $\eta \in \Lambda^1(\Gamma_a) \otimes \mathfrak{h}$ by the equation

$$\mu_{-2} + \mu_{-1} + \mu_{\mathfrak{h}} = -2\kappa\,\left(\frac{1}{2}e_{-2} + \rho\right) + \theta + \eta$$

where $\rho : \Gamma_a \to \mathfrak{h} : g \mapsto \left(\mathrm{Ad}(g^{-1})a\right)_{\mathfrak{h}}$.

The Maurer-Cartan form and hence κ, θ and η are invariant under the left action of $T_a \subset \mathrm{G}$, so we consider θ and η as the tautological and the connection 1-form on the would-be principal bundle $T_a\backslash\Gamma_a \to T_a\backslash\mathcal{C}_a$. In general $T_a\backslash\mathcal{C}_a$ is neither Hausdorff nor locally Euclidean, so, as before, one considers *local* quotients : one covers $\mathcal{C}_a$ by sufficiently small open subsets $U \subset \mathcal{C}_a$ for which the local quotient $T_a^{loc}\backslash U$ *is a manifold*

What special connections are obtained this way

- Connections of Ricci-type
- Bochner–Kähler and Bochner-bi-Lagrangian connections
  If the symplectic form is the Kähler form of a (pseudo-)Kähler metric, then its curvature decomposes into the Ricci curvature and the Bochner curvature . If the latter vanishes, then (the Levi-Civita connection of) this metric is called Bochner–Kähler.
  Similarly, if the manifold is equipped with a bi-Lagrangian structure, i.e. two complementary Lagrangian distributions, then the curvature of a symplectic connection for which both distributions are parallel decomposes into the Ricci curvature and the Bochner curvature. Such a connection is called Bochner-bi-Lagrangian if its Bochner curvature vanishes.
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pseudo-Kähler curvature

(M, ω, J) (pseudo)-Kähler, with associated metric $g(X,Y) = \omega(X, JY)$. The Levi Civita connection ∇^g preserves g, ω and J. Its curvature tensor R^g satisfies the algebraic identities:

$$R^g(X,Y)Z = -R^g(Y,X)Z \qquad \bigoplus_{X,Y,Z} R^g(X,Y)Z = 0 \qquad g(R^g(X,Y)Z,T) = -g(R^g(X,Y)T,Z).$$

and $R^g(X,Y) \circ J = J \circ R^g(X,Y)$.

One considers, the Ricci tensor $\mathrm{ric}^g(X,Z) := \mathrm{Trace}[Y \to R^g(X,Y)Z]$ converted into an endomorphism R^{ic} via $g(X, RicY) = ric^g(X,Y)$ and $s^g := \mathrm{Trace}\, R^{ic}$ the scalar curvature.

Decomposition of the (pseudo)Kähler curvature under the (pseudo)unitary group :

$$R^g = S + E + B \qquad dimM = 2n \geq 4 \qquad (Bochner)$$

where $S(X,Y)Z = \frac{s^g}{2n(2n+2)}\left(g(X,Z)Y - g(Y,Z)X + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\right)$ and
$E(X,Y)Z = \frac{1}{2(n+2)}(g(X,Z)R^{ic}_0Y - g(R^{ic}_0Y,Z)X - g(Y,Z)R^{ic}_0X + g(R^{ic}_0X,Z)Y + g(JX,Z)R^{ic}_0JY$
$-g(R^{ic}_0JY,Z)JX - g(JY,Z)R^{ic}_0JX + g(R^{ic}_0JX,Z)JY + 2g(R^{ic}_0JX,Y)JZ + 2g(JX,Y)R^{ic}_0JZ)$
with R^{ic}_0 the traceless part of the ricci endomorphism R^{ic}.

B is called the Bochner tensor. A pseudo-Kähler manifold (M, Ω, J) is said to be Bochner-(pseudo-)Kähler if the Bochner tensor of the Levi Civita curvature of the associated metric g_J vanishes.

Bochner Kähler manifolds

A complex space form is a space of constant holomorphic sectional curvature c; it corresponds to $B = 0$ and $R^{ic}_0 = 0$ i.e.

$$R^g(X,Y)Z = \tfrac{c}{4}\Big(g(X,Z)Y - g(Y,Z)X + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\Big)$$

The models of complex space form are the complex Euclidean space $\mathbb{C}^n$, the complex projective space $\mathbb{CP}^n$ and the complex hyperbolic space $\mathbb{CH}^n$, depending on $c = 0, c > 0$ or $c < 0$.

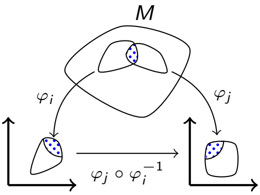
The symmetric Bochner-Kähler manifolds are n-dimentional complex space forms of constant sectional curvature c, M^n_c, and -as constructed by Tachibana and Liu in 1970- products of the form $M^p_c \times M^{n-p}_{-c}$.

Local models for Bochner–pseudo–Kähler manifolds

Thm: (Cahen-Schwachhöfer) The connections constructed with a 2-gradable simple Lie algebra are Bochner-pseudo-Kähler connections for metrics of signature $(2p, 2q)$ when $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ and these models exhaust locally all Bochner-pseudo-Kähler connections.

In particular (Robert Bryant), the germ of the connection at a point up to order 3 determines the connection, so the local moduli space of those connections is finite dimensional.

Almost complex and and complex structures



A smooth complex manifold is a topological space that is locally modeled on open subsets of $\mathbb{C}^m$ with holomorphic coordinate changes.

An almost complex structure J is a smooth section of $\mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$

The almost complex structure J associated to a complex structure is defined in local complex coordinates $\{z^1 = x^1 + iy^1, \ldots, z^n = x^n + iy^n\}$, with holomorphic changes of coordinates, by $J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \quad J\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}$.

An almost complex structure is *integrable* if it comes from a complex structure on M.

The Nijenhuis tensor N^J (or Nijenhuis torsion) of an almost complex structure J is defined by

$$N^J(X,Y) := [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y].$$

Theorem(Newlander, Nirenberg) : An almost complex structure J is integrable iff its Nijenhuis tensor N^J vanishes.

Remark: If ∇ is a torsion free connection, then

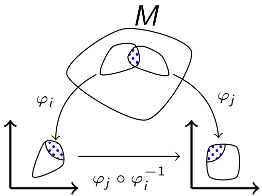
$$N^J(X,Y) := (\nabla_{JX}J)Y - (\nabla_{JY}J)X - J(\nabla_X J)Y + J(\nabla_Y J)X.$$

so that an almost complex structure which is parallel for a torsion free connection is necessarily integrable.

Quaternions

Hamilton discovered quaternions in 1843, on his way to the Royal Irish Academy,
"...I foresaw immediately .. many long years to come of definitely directed thought and work...

the fundamental formula with the symbols $i, j, k; \quad i^2 = j^2 = k^2 = ijk = -1.$ "



A first natural definition of *a smooth quaternionic manifold* would be a topological space that is locally modeled on open subsets of $\mathbb{H}^m$ with smooth quaternionic coordinate changes.

Too restrictive; a smooth map on an open set of $\mathbb{H}^m$ into $\mathbb{H}^{m'}$ whose differential at each point is $\mathbb{H}$-linear, is itself linear.

An almost hypercomplex structure $\mathfrak{h} = \{i, j, k\}$ on a smooth manifold M is a triple of smooth fields of endomorphisms of the tangent bundle TM which satisfy the identities

$$i^2 = j^2 = k^2 = ijk = -\mathrm{Id}. \tag{2}$$

It is said to be integrable and called a hypercomplex structure if furthermore i, j and k are integrable.

Quaternionic manifolds

Given a hypercomplex structure $\mathfrak{h} = \{i, j, k\}$ on a smooth manifold M, there is a torsion free connection (called the Obata connection) which leaves i, j and k parallel and there is a whole sphere of complex structures on M.

The quaternionic projective space $\mathbb{HP}_n$ of (right) quaternionic lines through the origin in $\mathbb{H}^{n+1}$ has no hypercomplex structure.

A quaternionic manifold is a real $4n$-dimensional manifold M, endowed with a rank 3 subbundle in $End(TM)$ locally spanned by $\{i, j, k\}$ with the $i^2 = j^2 = k^2 = ijk = -\mathrm{Id}$, and a torsion free linear connection on M for which the subbundle is parallel (S. Salamon).

Quaternion-Symplectic vector spaces.

A quaternionic n dimensional symplectic vector space (V,Ω) is a left vector space V of dimension n over the field of quaternions $\mathbb{H}$, endowed with a real non degenerate skewsymmetric $\mathbb{R}$–bilinear form $\Omega : V^{(\mathbb{R})}\times V^{(\mathbb{R})}\to\mathbb{R}$ (where $V^{(\mathbb{R})}$ is V viewed as a real $4n$ dimensional vector space) which is compatible in the sense that

$$\Omega(\tilde{i}v,\tilde{i}w)=\Omega(\tilde{j}v,\tilde{j}w)=\Omega(\tilde{k}v,\tilde{k}w)=\Omega(v,w)\qquad \forall v,w\in V,$$

(where $\tilde{i},\tilde{j}$ or $\tilde{k}$ denotes the linear endomorphism of $V^{\mathbb{R}}$ defined by left scalar mutiplication by i,j or k.)

Equivalently, a quaternionic symplectic vector space (V,Ω) is a left vector space V over $\mathbb{H}$, endowed with a skew-hermitian sesquilinear non degenerate product $\omega_\Omega : V\times V\to\mathbb{H}$,

$$\omega_\Omega(v,w):=\Omega(v,w)+i\Omega(v,\tilde{i}w)+j\Omega(v,\tilde{j}w)+k\Omega(v,\tilde{k}w),$$

$\overline{q=q_0+iq_1+jq_2+kq_3}=q_0-iq_1-jq_2-kq_3.$

Or endowed with an α_j-hermitian non degenerate scalar product $h_\Omega : V\times V\to\mathbb{H}$,

$$h_\Omega(v,w):=\Omega(v,\tilde{j}w)+i\Omega(v,\tilde{k}w)-j\Omega(v,w)-k\Omega(v,\tilde{i}w),$$

$\alpha_j(q=q_0+iq_1+jq_2+kq_3)=q_0+iq_1-jq_2+kq_3=j\bar{q}j^{-1}.$

The space (V,h_Ω) admits orthonormal basis.

The group $SO^*(2n) = SO(n, \mathbb{H})$.

The group of automorphisms of a quaternionic symplectic vector space is the group of invertible linear endomorphisms of $V^{\mathbb{R}}$ which are symplectic and commute with $\tilde{i}, \tilde{j}$ and $\tilde{k}$; those are the elements of $Gl(n, \mathbb{H})$ which preserve h_Ω or -equivalently- ω_Ω; in an orthonormal basis basis of (V, h_Ω), it is given by

$$\{A \in Gl(n, \mathbb{H}) \,|\, A \,{}^\tau\alpha_j(A) = \mathrm{Id}\}.$$

Viewing V as a $2n$-dimensional complex vector space $V^{\mathbb{C}}$, a matrix $A \in Gl(n, \mathbb{H})$ corresponds to the matrix in $B \in Gl(2n, \mathbb{C})$ given by $B = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ (for $A^r_s = \alpha^r_s + \beta^r_s j$) and ${}^\tau\alpha_j(A)$ corresponds to $\begin{pmatrix} {}^\tau\alpha & -{}^\tau\overline{\beta} \\ -{}^\tau\beta & {}^\tau\overline{\alpha} \end{pmatrix} = {}^\tau B$ so the group of automorphisms corresponds to

$$\{B = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in Gl(2n, \mathbb{C}) \,|\, B \,{}^\tau B = \mathrm{Id}\}.$$

One defines

$$SO^*(2n) = \{B = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SO(2n, \mathbb{C})\}$$

Quaternion-Symplectic manifolds

A hypercomplex-symplectic manifold is a triple $(M^{4n}, \omega, \mathfrak{h})$, where ω is a symplectic form, $\mathfrak{h} = \{i, j, k\}$ is a a hypercomplex structure so that i, j and k are compatible with the symplectic structure ω.

It yields a reduction of the frame bundle to a $SO^*(2n)$ structure.

Remark: There is a necessary condition for a Lie algebra $\mathfrak{g}$ to appear as the irreducible holonomy algebra of a torsion-free connection (Berger criterion); it is not satisfied by $\mathfrak{so}^*(2n)$,

A (torsionfree) quaternion-symplectic manifold is a quadruple $(M^{4n}, \omega, \nabla, \mathcal{E})$ where ω is a symplectic form, ∇ is a torsionfree connection preserving ω and $\mathcal{E} \subset \mathrm{End}(TM)$ is a rank 3 subbundle, parallel for ∇, locally spanned by almost complex structures $\{i, j, k\}$ which are compatible with the symplectic structure ω and satisfy $ij = k$.
It is equivalent to a $Sp(1)SO^*(2n)$ structure which is parallel for a torsionfree linear connection.
Its holonomy is contained in $Sp(1)SO^*(2n)$ and the construction of Cahen-Schwachhöfer gives local models for these, taking $\mathfrak{g} = \mathfrak{so}^*(2n+2)$.

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Thank you for your attention!