Hamiltonian group actions and fixed points

Mantra: act globally, compute locally

Poisson 2026
Summer School
3-7 August 2026

Tara S. Holm
Cornell University

Outline - Lectures 1 & 2

- Symplectic geometry (with lots of examples)
- Group actions & fixed points (with lots of examples)
- Localization (with lots of examples)

Outline - Lectures 3 & 4

- More on equivariant cohomology calculations
- Classification results: toric symplectic manifolds
- Symplectic reduction (how to take a quotient)
  (with lots of examples)
- Complexity one and how much cohomology conveys
- Symplectic embeddings (if time)

Equivariant cohomology

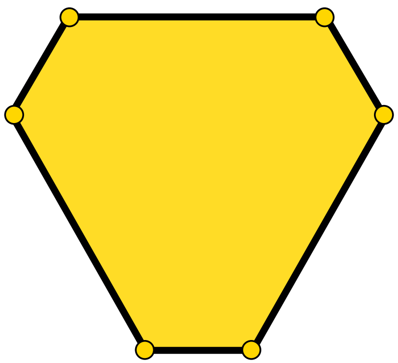
M^T = isolated points

$T^n \curvearrowright M^{2d}$

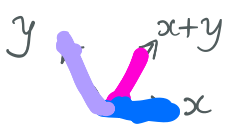
$$\boxed{M^T \hookrightarrow M \rightsquigarrow H_T^*(M;R) \xrightarrow{\star} H_T^*(M^T;R)}$$

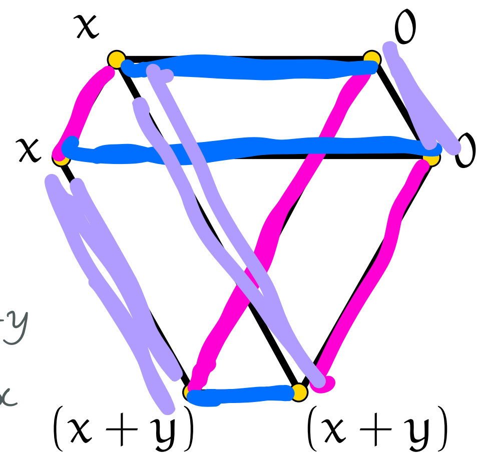
$\deg x_i = 2$

$$= \bigoplus_{p \in M^T} \underbrace{\mathbb{Q}[x_1, \dots, x_n]}_{S(\mathfrak{t}^*)}$$

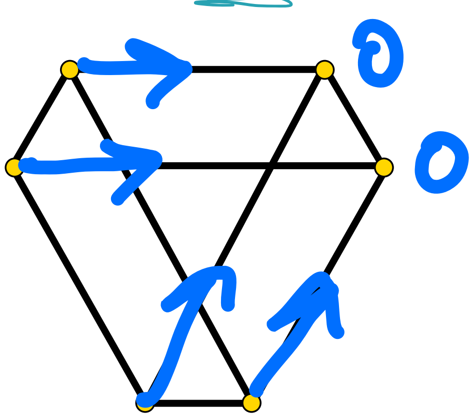


$\rightsquigarrow$

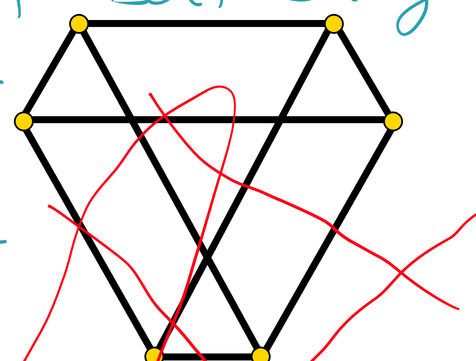




$M \xrightarrow{\Phi} \mathfrak{t}^*$



Im(☆) = tuples of polys that satisfy compat. for ea. edge



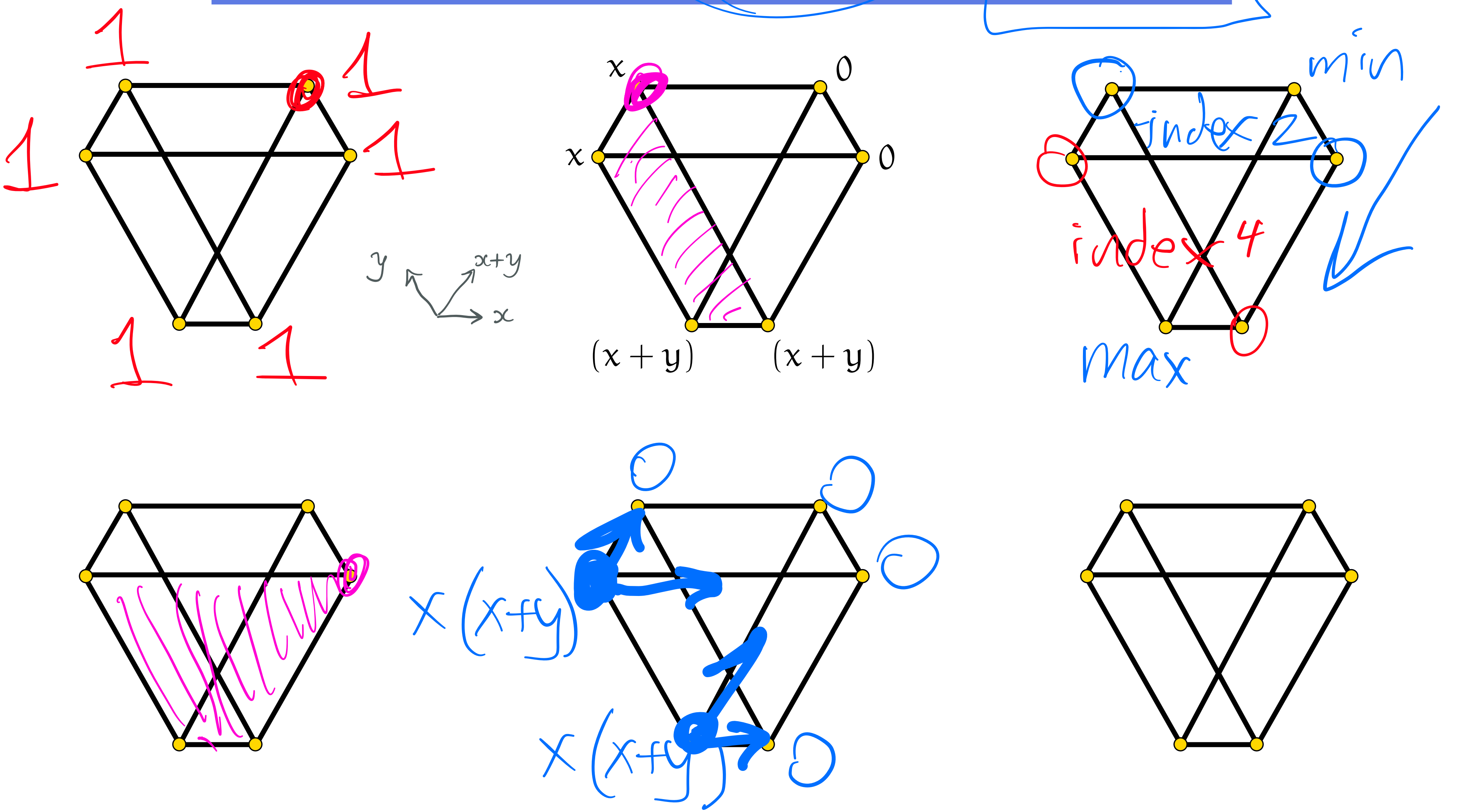
$x+y$ 3

not a $H_T^*(M)$ class!

Equivariant cohomology

$$\boxed{M^T \hookrightarrow M \rightsquigarrow H_T^*(M;R) \longrightarrow H_T^*(M^T;R)}$$

Free $H_T^*(pt)$-module



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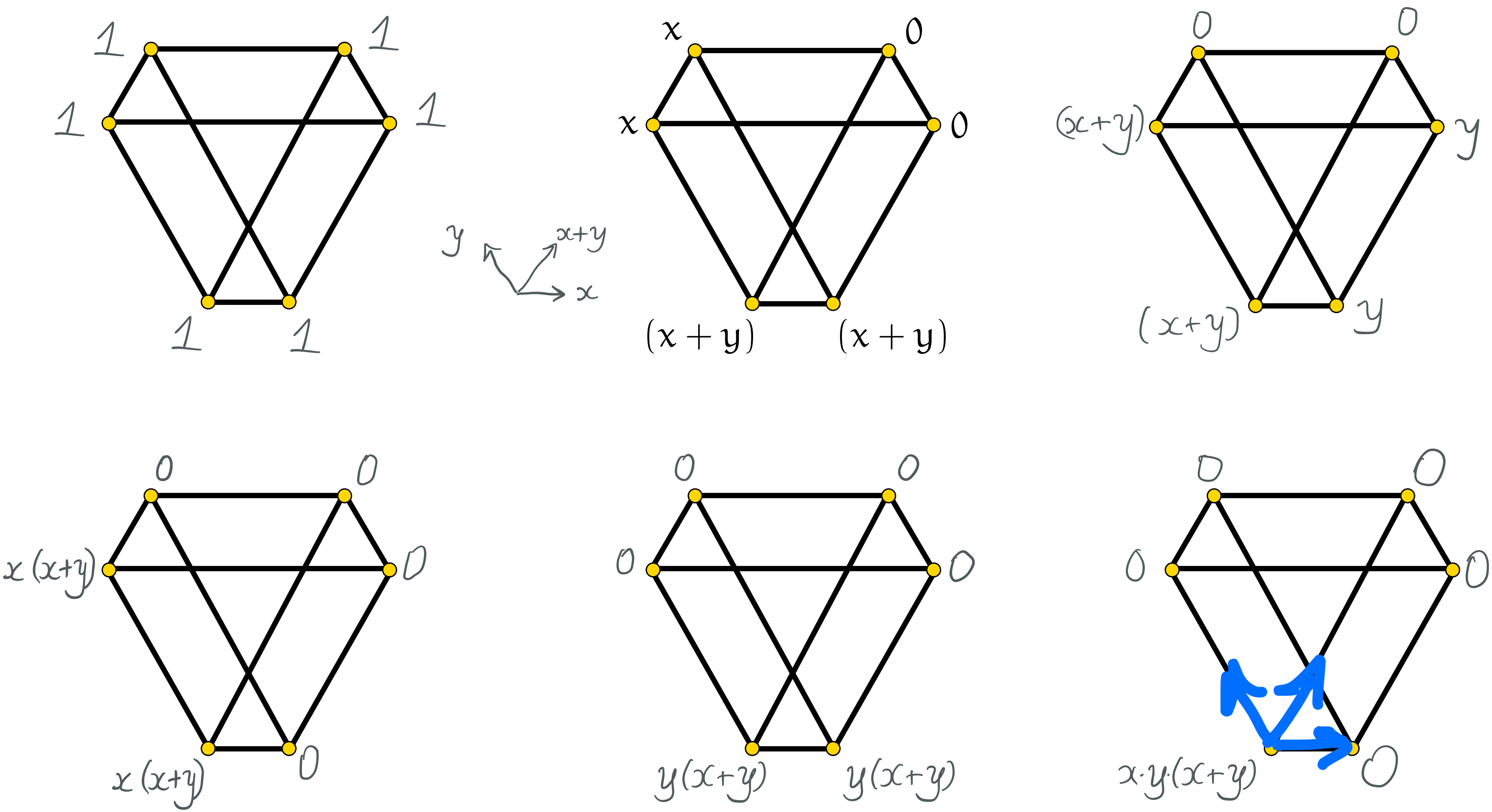

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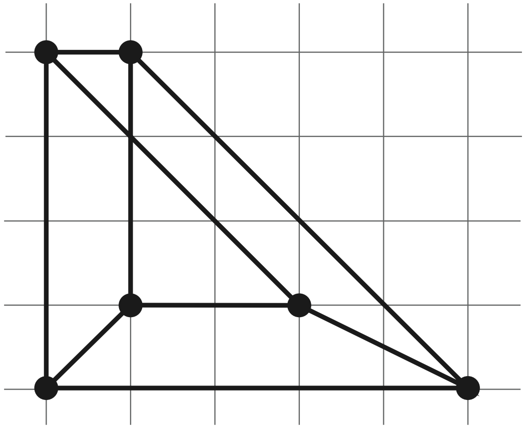

Equivariant cohomology

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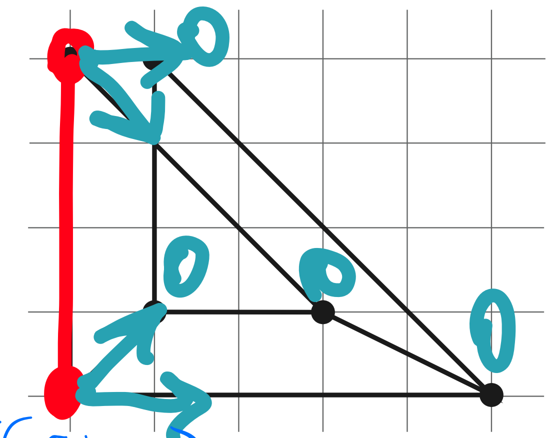
$(x)(x-y)$



$x(x+y)$

$S^2 \subset M$

$\nu(S^2 \subset M)$

$e_T(\nu(S^2 \subset M))$

$T \circlearrowright V$

$$\begin{array}{c} V \\ \downarrow \\ pt \end{array}$$

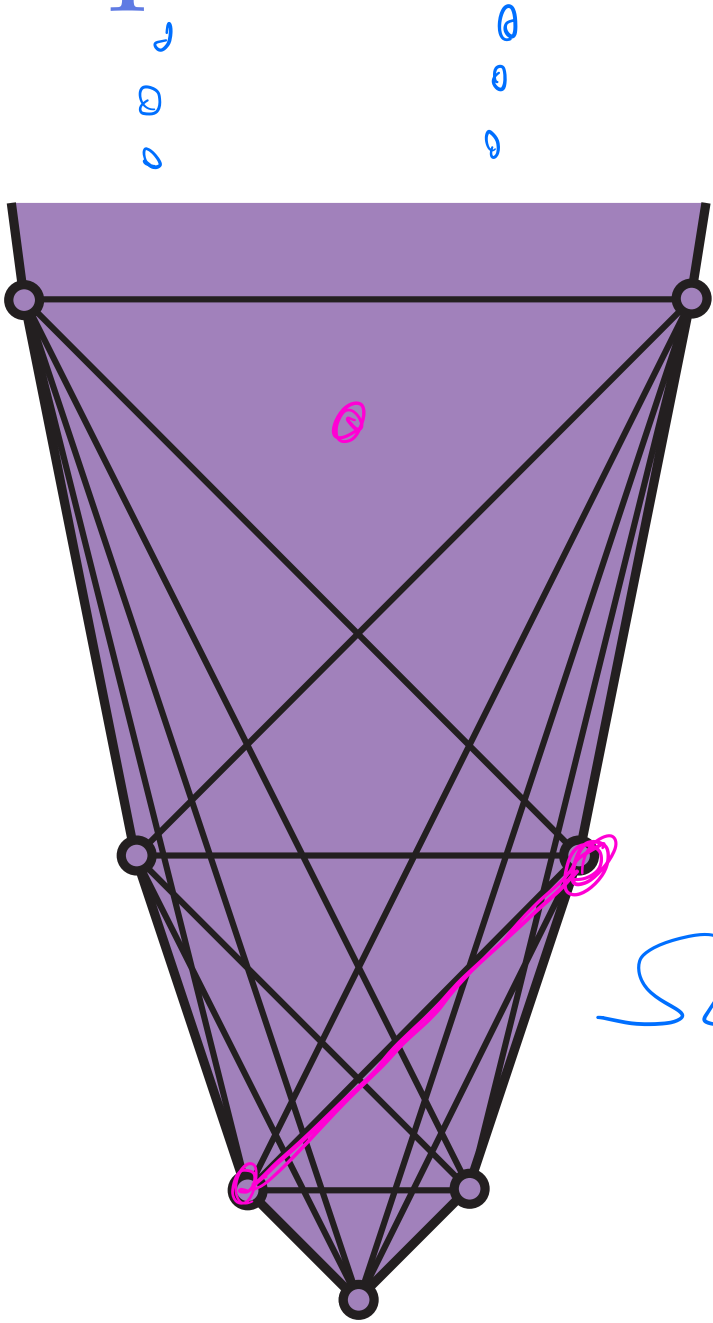
Chern classes

Euler classes...

$e_T(V) = \prod \alpha$

α wt of T-action

Equivariant invariants of ΩG



<u>**Theorem**</u> [Harada-Henriques-H.]:
The GKM description works, even in infinite dimensional cases, for

- Equivariant cohomology $H^*_T(M;\mathbb{Q})$
  (Sometimes integrally!) $H^*_T(M;\mathbb{Z})$
- Equivariant K-theory $K^*_T(M)$
- Equivariant cobordism $MU^*_T(M)$

$\Omega SU(2) \circlearrowleft S^1 \times S^1$

max torus in SU(2)

spins the based loop

Localization and integration

For any compact T-manifold M, we may define an equivariant integral

$$\int_M : H_T^*(M;\mathbb{R}) \to H_T^*(pt;\mathbb{R}).$$

!

Lowers deg by $-\dim M$.

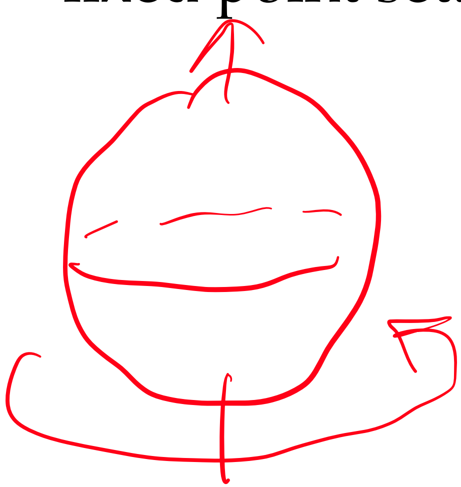
"integration over fiber"

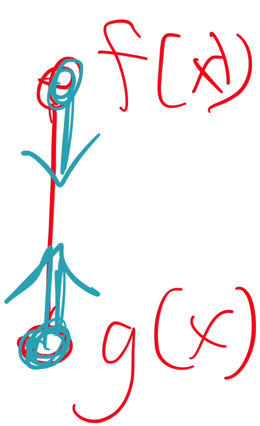
Integration formula: (Atiyah-Bott, Berline-Vergne)

For any class $\alpha \in H_T^*(M;\mathbb{R})$,

$$\int_M \alpha = \sum_{F \subset M^T} \int_F \frac{\alpha|_F}{e_T(\nu(F \subset M))},$$

where the summation runs over all connected components of the fixed point set.





$$\int_M : \alpha \to H_T^*(p)$$

$\| $ poly ring

$$\int \alpha = \frac{f(x)}{-x} + \frac{g(x)}{x} = \frac{g(x) - f(x)}{x} \in \mathbb{Z}[x]$$

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$$H_T^*(M;R) \longrightarrow H_T^*(M^T;R)$$

Localization and integration

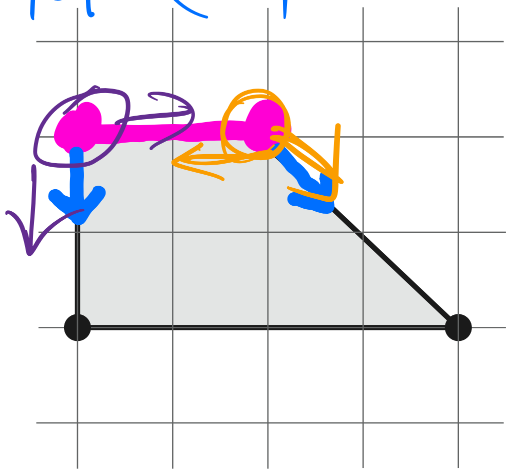
$= -x/x = -1$
$-y - (x-y)$
$= x$

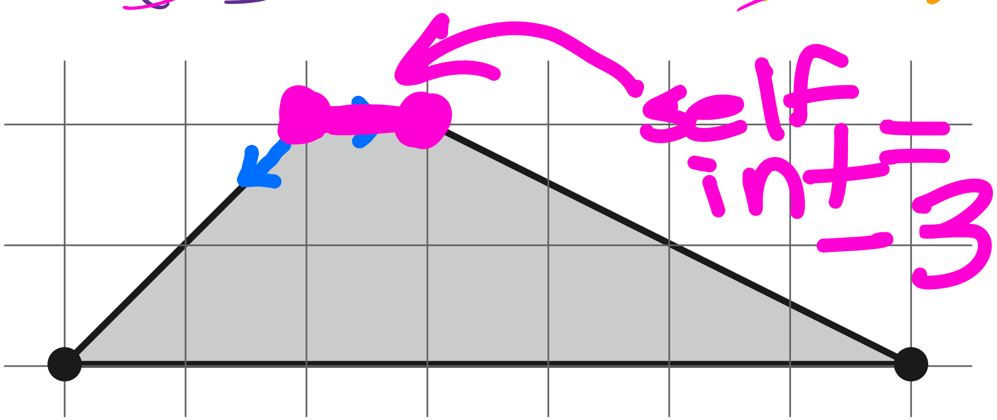
Integration formula: (Atiyah-Bott, Berline-Vergne)

$$\int_M \alpha = \sum_{F \subset M^T} \int_F \frac{\alpha|_F}{e_T(\nu(F \subset M))},$$

$$= \frac{-y}{x} + \frac{(x-y)}{-x}$$

$$\int_M (e_T(S^2))^2 = \frac{(-y)^{\cancel{2}}}{x\cancel{(-y)}} + \frac{(x-y)^{\cancel{2}}}{-x\cancel{(x-y)}}$$





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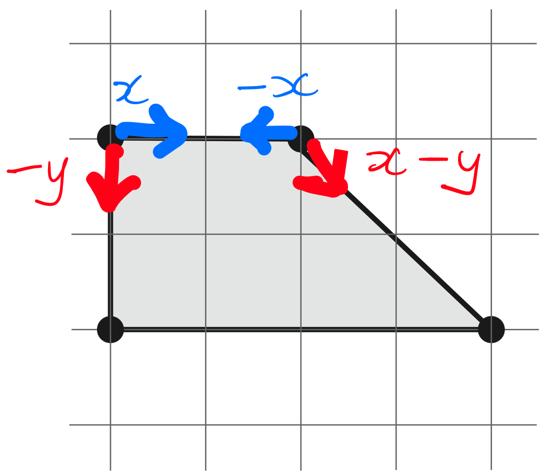

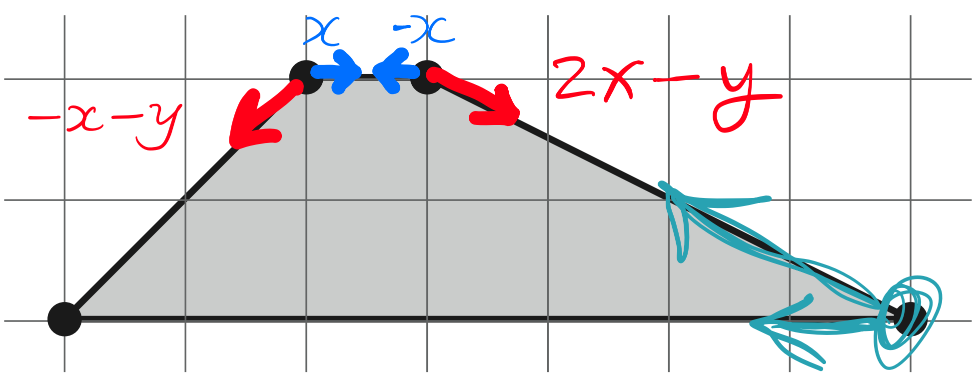

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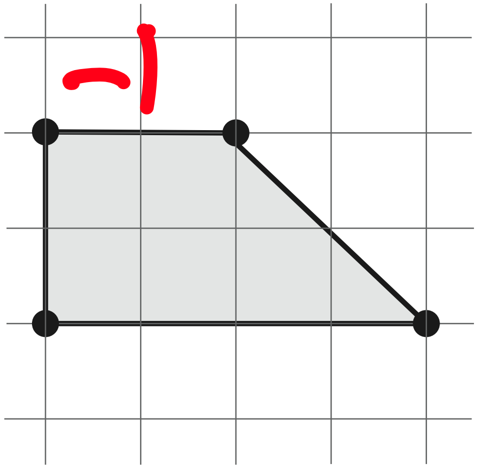
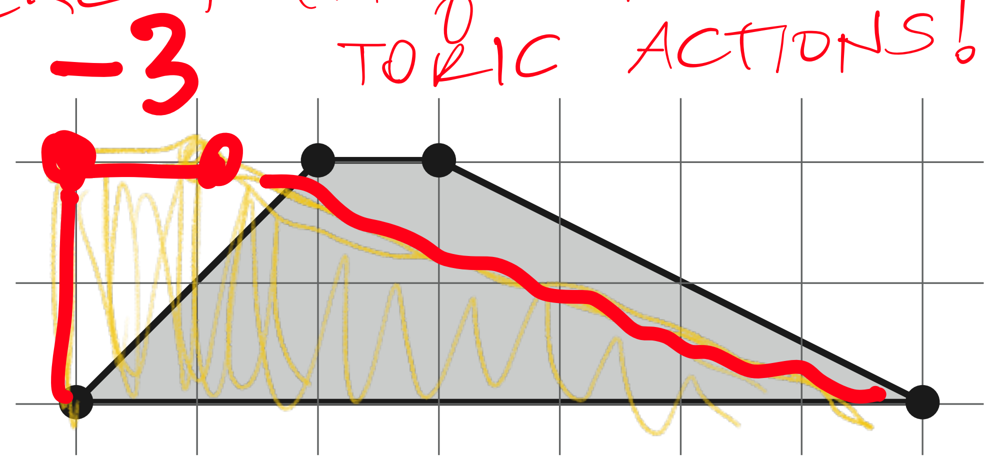
These are the same symplectic manifold

DIFFERENT (inequivalent) TORIC ACTIONS!





$S^2 \tilde{\times} S^2$

Toric symplectic manifolds

Let $T^d = S^1 \times \cdots \times S^1 \circlearrowright M^{2n}$ be an effective Hamiltonian action.

$$\boxed{\text{Then } d \leqslant n.}$$

When $d = n$, the action is called **toric**.

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Let M be a compact toric symplectic manifold. The momentum map

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at each vertex, $\exists$ d edges.

→ d edges point in $\mathbb{Z}^d$ directions

→ form $\mathbb{Z}$-basis of lattice

<u>Delzant's Theorem</u>:

$$\left\{ \begin{array}{c} \text{compact toric} \\ \text{symplectic manifolds} \end{array} \right\} \overset{1:1}{\leftrightsquigarrow} \left\{ \begin{array}{c} \text{simple smooth} \\ \text{rational polytopes} \end{array} \right\}$$

Toric symplectic manifolds

up to translation & $SL_n\mathbb{Z}$

<u>Delzant's Theorem</u>:

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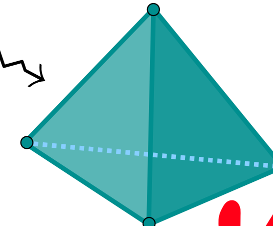
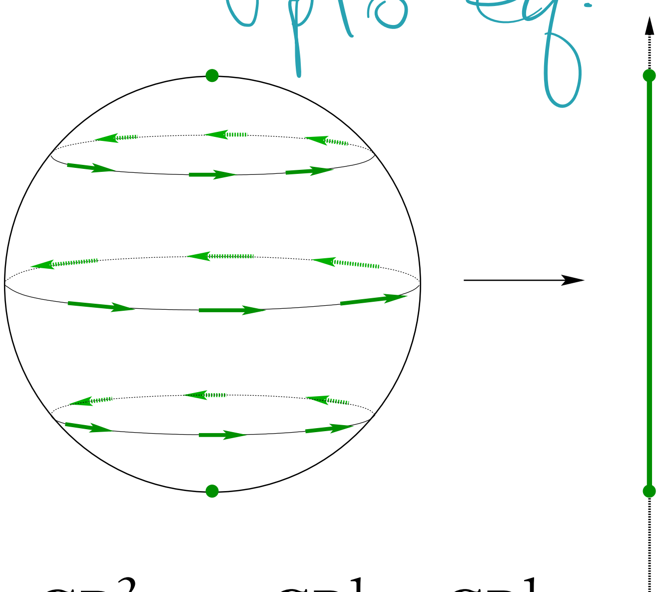
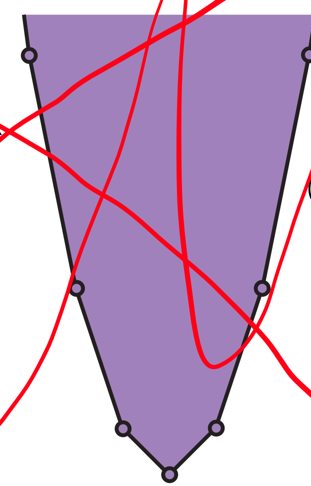
up to eq. symplectom

$\mathbb{C}P^3$

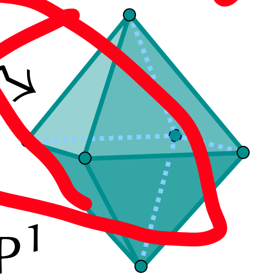




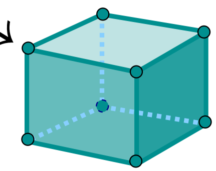
$\Omega SU(2)$



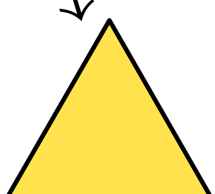
$\mathcal{G}r(2,\mathbb{C}^4)$



$\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$



$\mathbb{C}P^2$



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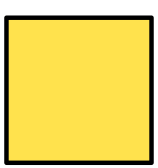
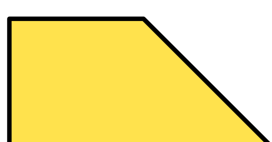
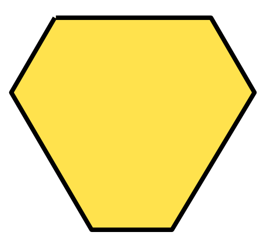
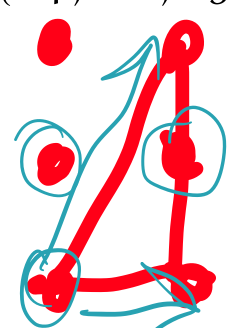
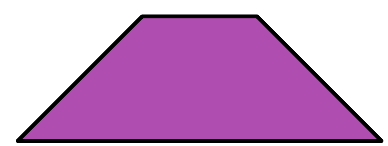






$\mathcal{P}ol_3(a_1,\ldots,a_5)$





Topology of toric manifolds

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- M simply connected

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- $H^*_{\mathbb{T}}(M) =$ Stanley-Reisner ring of $M/\mathbb{T}$
  GKM description
  generated in degree 2

Quotient ring

"subring"

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Danilov
–Jurkiewicz

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- More is known in 4 dimensions $k \geq 0$
  - Each toric symplectic 4-manifold is a k-fold **symplectic blowup** of

$$\mathbb{C}P^2 \;,\; \mathbb{C}P^1 \times \mathbb{C}P^1 \;,\; \text{or } \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$$

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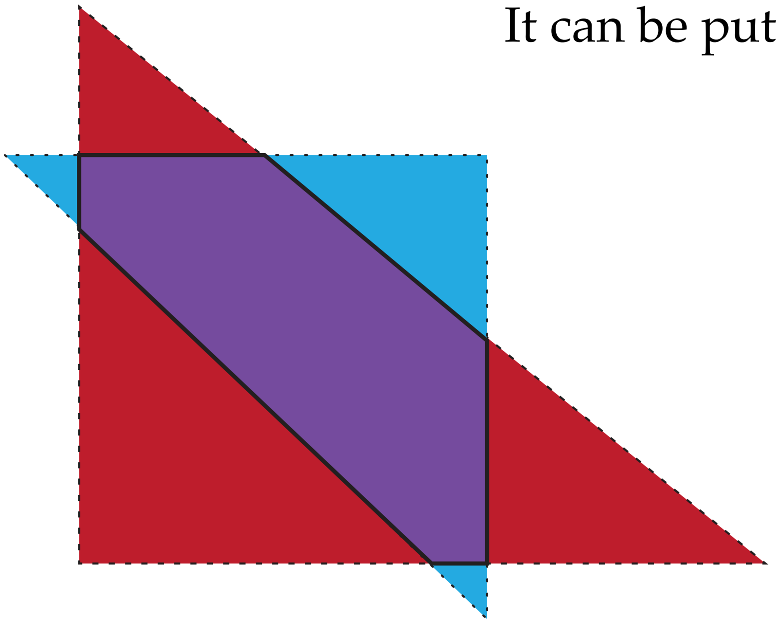

Toric symplectic manifolds

The nontrivial $\mathbb{C}P^1$-bdle over $\mathbb{C}P^1$

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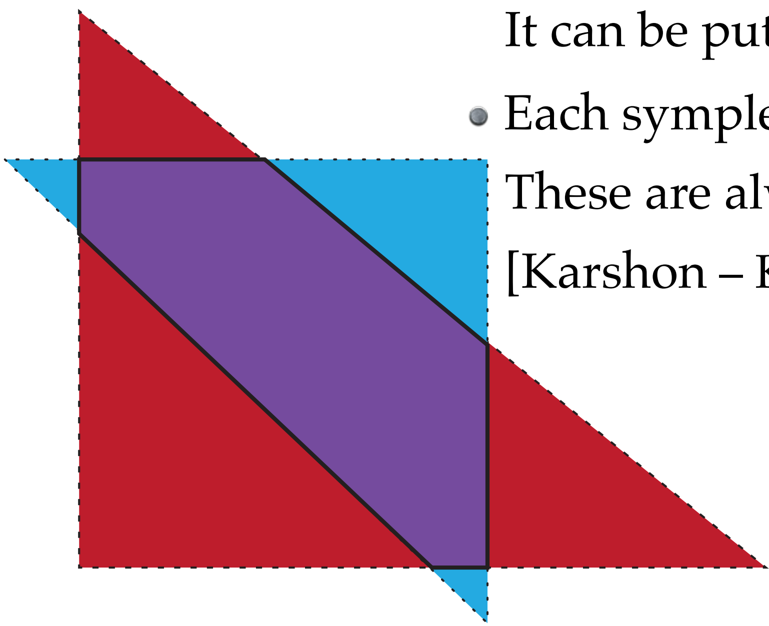
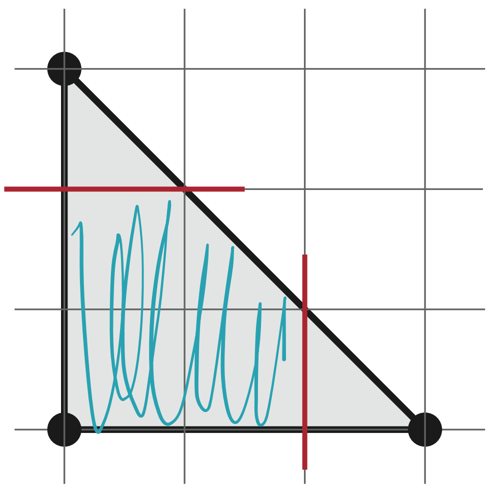
  - Each symplectic M^4 admits only finitely many toric and circle actions
    These are always achieved as **equivariant symplectic blowups**.
    [Karshon – Kessler – Pinsonnault]

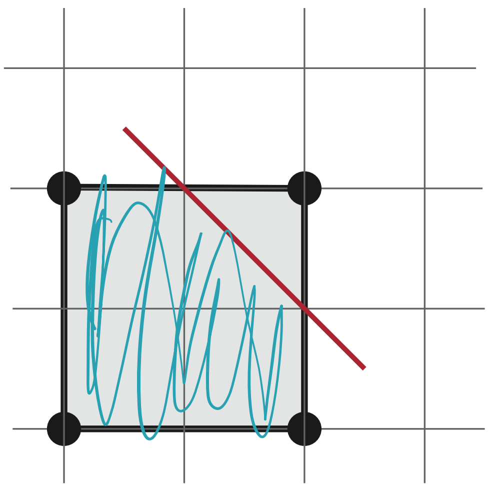


More examples

Delzant's Theorem:

$$\left\{ \begin{array}{c} \text{compact toric} \\ \text{symplectic manifolds} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{simple rational} \\ \text{smooth convex polytopes} \end{array} \right\}$$





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    [Karshon – Kessler – Pinsonnault]
- The cohomology rings are well understood
  - The manifolds are simply connected with no odd cohomology
  - **Danilov** and **GKM** descriptions of $H_T^*(M;\mathbb{Z})$
  - $H_T^*(M;\mathbb{Z})$ determines the toric structure [Masuda]