Courant algebroids

Def (Liu-Weinstein-Xu)

A Courant algebroid (CA) is a vector bundle $E \to M$

+ $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{R}_M$ symmetric and non-deg pairing

+ $\rho : E \to TM$ anchor

+ $[\![\cdot,\cdot]\!] : \Gamma E \times \Gamma E \to \Gamma E$ bracket s.t:

1) $[\![e_1, f e_2]\!] = \rho(e_1)(f)\, e_2 + f [\![e_1, e_2]\!]$

2) $\rho(e_1)(\langle e_2, e_3\rangle) = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$

3) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$

4) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}(\langle e_1, e_2\rangle)$

for $f \in C^\infty(M)$, $e_i \in \Gamma E$ and $\mathcal{D} : C^\infty(M) \to \Gamma E$

$$\langle \mathcal{D} f, e\rangle = \rho(e)(f)$$

Def

For a C.A. $(E \to M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \rho)$

A Dirac structure is a subbundle

$$L \subseteq E \quad \text{s.t.} \quad \begin{cases} \langle L, L\rangle = 0 \\ \operatorname{rk} L = \frac{1}{2} \operatorname{rk} E \\ [\![\Gamma L, \Gamma L]\!] \subseteq \Gamma L \end{cases}$$

Examples

① Quadratic Lie algebras

If $M = pt \rightsquigarrow \rho = 0 \Rightarrow$ 4) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = 0$

$\overset{3)}{\Rightarrow} [\![\cdot,\cdot]\!]$ Lie bracket

$\to (\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ Lie algebra + symmetric non-deg pairing $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ satisfying

$$\langle [e_1,e_2], e_3\rangle + \langle e_2, [e_1,e_3]\rangle = 0 \quad \Leftrightarrow \quad \text{ad-invariant}$$

Concrete examples:

positive definite
- ↳ Semisimple Lie algebras + Killing form
- ↳ Lie algebras integrating a compact Lie group + biinvariant metric. (always exist by averaging)

signature 0 ⇝
- ↳ Drinfeld doubles.

② Exact CA

Let M be a manifold and $H \in \Omega^3(M)$ with $dH = 0$

$\cdot\, E = \mathbb{T}_H M = TM \oplus T^*M$, $\langle X+\alpha, Y+\beta\rangle = i_X\beta + i_Y\alpha$

$\rho = pr_{TM}$, $[\![X+\alpha, Y+\beta]\!]_H = [X,Y] + \mathcal{L}_X\beta - i_Y d\alpha + i_X i_Y H$

Def A CA is exact if $0 \to T^*M \to E \xrightarrow{\rho} TM \to 0$

- ρ surjective
- $\mathrm{Ker}\,\rho = T^*M$.

Prop (Severa)

C.A. automorphism

If E exact C.A then $E \simeq \mathbb{T}_H M$ for some $H \in \Omega^3_{cl}(M)$

and $\mathbb{T}_H M \simeq \mathbb{T}_{H'} M \Leftrightarrow H - H' = dB$

so $\{\text{Exact C.A. over } M\} \leftrightarrow H^3(M)$

Comments

① $H=0$ ⇝ Standard C.A. First example (Dorfman, Courant)

↳ Dirac structures $\supseteq$ graph(ω) with $d\omega = 0$

graph(π) with $[\pi,\pi]=0$

(2) If $H \in H^3(M, \mathbb{Z})$ then $\mathbb{T}_H M$ encode the symmetries of the gerbe classified by H.

(Severa, Bressler, Krepski)

(3) Doubles of Lie bialgebroids

A Lie bialgebroid $(A \to M, \rho_A, [\cdot,\cdot]_A, d_{A^*})$

$d_{A^*} : \Gamma \wedge^\bullet A \to \Gamma \wedge^{\bullet+1} A$ s.t

$v, w \in \Gamma \wedge^\bullet A$

$$\begin{cases} \cdot\ d_{A^*}(v \wedge w) = (d_{A^*} v) \wedge w + (-1)^{|v|} v \wedge d_{A^*} w \\ \cdot\ (d_{A^*})^2 = 0 \\ \cdot\ d_{A^*}[v, w]_A = [d_{A^*} v, w]_A - (-1)^{|v|} [v, d_{A^*} w]_A \end{cases}$$

Thm [Mackenzie-Xu]

$\mathrm{Lie} : \{\text{Poisson groupoids}\} \longrightarrow \{\text{Lie bialgebroids}\}$

the double $E = A \oplus A^*$

$$\langle v+\alpha, w+\beta \rangle = i_v \beta + i_w \alpha$$

$$\rho(v+\alpha) = \rho_A(v) + \rho_{A^*}(\alpha)$$

$$[\![v+\alpha, w+\beta]\!] = [v,w]_A + \mathcal{L}^{A^*}_\alpha w - i_\beta d_{A^*} v$$
$$+ [\alpha,\beta]_{A^*} + \mathcal{L}^A_v \beta - i_w d_A \alpha$$

Thm (Liu-Weinstein-Xu)

$(E, \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!])$ is a C.A.

Conversely E a C.A

$L_+, L_- \subseteq E$ Dirac

s.t $E = L_+ \oplus L_-$

$\rightsquigarrow$ L_+ and L_- Lie bialg.

④ Other examples:

↳ Action C.A (Meinrenken)

↳ Heterotic C.A. (Baraglia-Hekmati)

↳ Cotangent of Lie 2-algebroids (Li-Bland)

⋮

Nevertheless the above definition is not good to decide how to integrate C.A. A better point of view is given by

Theorem (Roytenberg)

There is a 1:1 correspondence

$$\{CA\} \rightleftarrows \left\{ \begin{matrix} \text{Deg } 2 \text{ symplectic} \\ \text{dg-manifolds} \end{matrix} \right\}$$

Def

· A degree ℓ vector field on dg-man $\mathcal{M}$, $\mathfrak{X}^1_\ell(\mathcal{M})$, is

$X: C^\infty_\bullet(\mathcal{M}) \to C^\infty_{\bullet+\ell}(\mathcal{M})$ and $X(fg) = X(f)g + (-1)^{\ell|f|} f X(g)$

A deg ℓ differential k-form on $\mathcal{M}$, $\Omega^k_\ell(\mathcal{M})$, is

$$\omega: \mathfrak{X}^1_{i_1}(\mathcal{M}) \times \dots \times \mathfrak{X}^1_{i_k}(\mathcal{M}) \to C^\infty_{i_1+\dots+i_k+\ell}(\mathcal{M})$$

· $\omega(-, X, Y, -) = -(-1)^{|X||Y|}\omega(-, Y, X, -)$

· $\omega(fX, -) = (-1)^{|\omega||f|} f\, \omega(X, -)$

Def

A deg n symplectic Q-manifold $(\mathcal{M}, Q, \omega)$

· $(\mathcal{M}, Q)$ is a Q-manifold

· $\omega \in \Omega^2_n(\mathcal{M})$ s.t $d\omega = 0$ and $\mathcal{L}_Q \omega = 0$

- $\omega^\flat : \mathfrak{X}^1_\bullet(\mathcal{M}) \longrightarrow \Omega^1_{\bullet+n}(\mathcal{M})$ isomorphism of $C^\infty(\mathcal{M})$-modules.

deg $\mathcal{M} = n$ by

Thm (Roytenberg) idea of proof

$(\Longrightarrow)$

$(E \to M, \langle \cdot,\cdot \rangle, \rho, [\![\cdot,\cdot]\!])$

$C_k(E) = \{ \eta : \Gamma E \times \overset{k)}{\ldots} \times \Gamma E \to C^\infty(M) \mid C^\infty(M)\text{-linear on last entry}$

and

$\eta(-, e_i, e_{i+1}, -) + \eta(-, e_{i+1}, e_i, -) = \nabla\eta(-, \hat{e}_i, \hat{e}_{i+1}, -)(\langle e_i, e_{i+1} \rangle)$

for $\nabla : \Gamma E^{k-2} \to \mathfrak{X}(M)$ $\}$

$$(\eta \cup \tau)(e_1, -, e_{k+\ell}) = \sum_{\sigma \in Sh} \operatorname{sign}(\sigma)\, \eta(e_{\sigma(1)}, -, e_{\sigma(k)})\, \tau(e_{\sigma(k+1)} - e_{\sigma(k+\ell)})$$

$$Q(\eta)(e_0, -, e_k) = \sum_{i=0}^{k} (-1)^i \rho(e_i)\big(\eta(e_0, -, \hat{e}_i, -, e_k)\big)$$
$$- \sum_{i<j} (-1)^i\, \eta(e_0, -, \hat{e}_i, -, [\![e_i, e_j]\!], -, e_k)$$

$\rightsquigarrow \mathcal{M} = (M, C_\bullet(E), Q)$ defines a deg 2 dg-manifold.

Keller-Waldmann dga

$\omega \leftrightarrow \{\cdot,\cdot\}$

$\{e_1, e_2\} = \langle e_1, e_2 \rangle$ $\quad \{\eta, f\} = \nabla_\eta(f)$

$\{\eta, e\} = \eta(e)$ $\quad \{\eta_1, \eta_2\} = \eta_1 \circ \eta_2 - \eta_2 \circ \eta_1$

$(\Leftarrow)$

$(\mathcal{M}, \omega, Q)$ with $\mathcal{M} = (M, C^\infty(\mathcal{M}))$

$C^\infty_0(\mathcal{M}) = C^\infty(M)$ and $C^\infty_1(\mathcal{M})$ locally free f.g $C^\infty(M)$-module

$\Rightarrow C^\infty_1(\mathcal{M}) = \Gamma E$ for some v.b. $E \to M$.

$\langle \cdot, \cdot \rangle = \{\ ,\ \} : \Gamma E \times \Gamma E \to C^\infty(M)$

$[\![e_1, e_2]\!] = \{ e_1, Q(e_2) \} \quad \rho(e)(f) = \{ e, Q(f) \}$

Example

① Quadratic Lie algebra

$(\mathfrak{g}, [\cdot,\cdot], \langle \cdot,\cdot \rangle) \rightsquigarrow \mathfrak{g}[1] = (pt, \wedge^\cdot \mathfrak{g}^*)$

locally $\{ \underset{1}{\xi^i} \}$

$\omega = \frac{1}{2} g_{ij}\, d\xi^i d\xi^j \qquad Q = \frac{1}{2} C^k_{ij}\, \xi^i \xi^j \frac{\partial}{\partial \xi^k}$

② Exact C.A.

$\mathbb{T}_H M \rightsquigarrow T^*[2]T[1]M = (M, \mathrm{Sym}(\mathrm{Der}(\Omega(M))[-2]))$

locally $\{ \underset{0}{x^i}, \underset{1}{\theta^i}, \underset{1}{\xi_i}, \underset{2}{p_i} \}$

$\omega = dp_i\, dx^i + d\xi_i\, d\theta^i \qquad Q^H = \theta^i \frac{\partial}{\partial x^i} + (p_i + H_{ijk}\, \theta^j \theta^k) \frac{\partial}{\partial \xi_i}$

③ Cotangents of (Li-bland)

deg 2 dg-man

$(\mathcal{M}, \hat{Q}) \rightsquigarrow (T^*[2]\mathcal{M},\ \omega_{can},\ Q = \hat{Q}^{cot})$