On the genericity of topology changes in level sets of proper Morse functions when passing critical points of index half the dimension of the manifold

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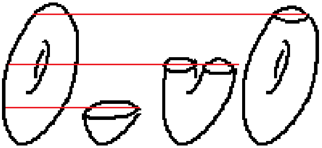
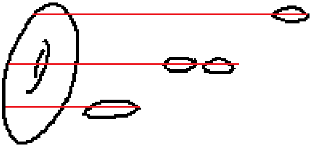
Introduction

Morse theory: topology of below level sets



Integrable manifolds: topology of level sets



Could help to understand global properties of the system.[McC23]

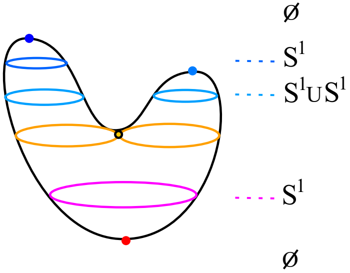
Introduction

M: n-dim manifold
f: Morse function on M
(For now: f proper, to avoid compactification or issues at infinity)

We are interested in the topology of level sets $f^{-1}(\alpha) \subset M$

No critical points between regular levels $\Rightarrow$ diffeomorphic

What happens when passing critical points?



Introduction

What happens when passing critical points?
(Derive global implications from local properties?)

A. Knauf and N. Martynchuk [KM20]:
passing a critical point of index $k \neq n/2$,
passing no other critical points of index $k-1$, $k+1$, or $n-k$,
$\Rightarrow$ change in the homotopy type of the level set.

What happens when passing a single critical point of index $m = n/2$?

Introduction

- Examples

- Idea / statement

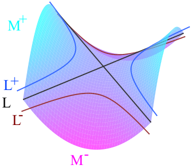
(- Proof)

- Theorem / result

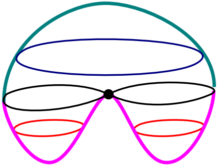
Basic examples

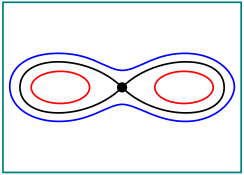
What happens when passing a critical point of index $m = n/2$?

On 2 dimensional surfaces: $m = 1$



Locally: saddle point

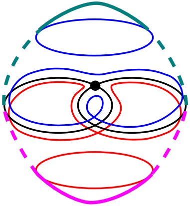


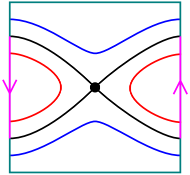


$S^1 \cup S^1 \to S^1$

topology change

(on S^2, orientable)





$S^1 \to S^1$

no topology change

(on $\mathbb{RP}^2$, non-orientable).

Basic examples

What happens when passing a critical point of index $m = n/2$?

From $m = 1$-examples: possibilities determined by orientability?

Example on $\mathbb{CP}^2$ (orientable, $m = 2$):

$$f = \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2 + |\beta|^2 + |\gamma|^2}$$

Symmetry $\Rightarrow$ no topology change.

Basic examples

When passing a critical point of index m

$m = 1$: orientability of M guarantees topology change

$m > 2$: orientability is not enough,

...we need the absence of something in the middle homology group.

Idea

Level set topology at $L^{\pm} = f^{-1}(\pm\varepsilon)$

... is determined by the critical level $L = f^{-1}(0)$,

... and f on a neighbourhood of the critical point (known).

Limited possibilities to extend L on this neighbourhood to all of L,

what are the implications for $L^{\pm}$?

Idea

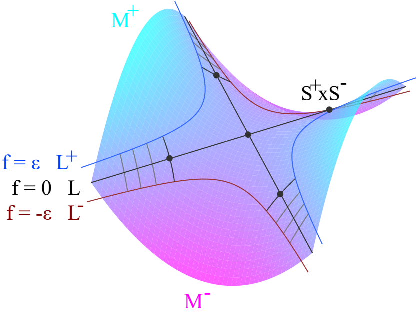
In a neighbourhood D of the critical point (Morse lemma):

$$f = x_1^2 + \cdots + x_m^2 - (x_{m+1}^2 + \cdots + x_{2m}^2)$$

$$\begin{array}{ccccc} D & = & D^m & \times & D^m \\ L^+ \cap D : & & S^{m-1} & \times & D^m \\ L^- \cap D : & & D^m & \times & S^{m-1} \\ \partial(L \cap D) =: S & & S^{m-1} & \times & S^{m-1} \end{array}$$

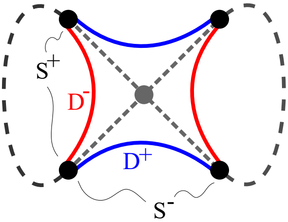
What happens with this product of spheres outside D?

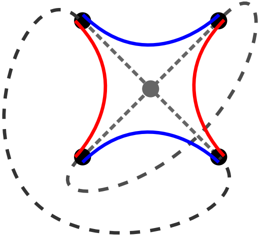
Idea - example for $m = 1$



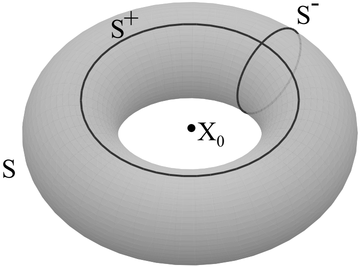
$$\partial(L \cap D) = S = S^0 \times S^0$$

Only two ways to "connect" those





Idea - example for $m = 2$, homology coefficients in $\mathbb{Z}_2$



$\partial(L \cap D) = S^1 \times S^1$

Different things can happen in $\tilde{L} = L \backslash (L \cap D)$:

...

Idea - example for $m = 2$, homology coefficients in $\mathbb{Z}_2$

$\partial(L \cap D) = S^1 \times S^1$

Different things can happen in $\tilde{L} = L \backslash (L \cap D)$,
$S^1 \times S^1$ has two independent cycles:

- both persist independently
  impossible
- both vanish/become trivial/are a boundary
  impossible
- one of them persists, the other becomes trivial
  implies a change in homology
- there is a relation between them (...)
  implies existence of nontrivial $H_m(L, \mathbb{Z}_2)$
  s.t. this implies existence of nontrivial $H_m(M, \mathbb{Z}_2)$

Theorem ($\mathbb{Z}_2$ version)

Let M be a $2m$-dimensional manifold without boundary, with f a proper Morse function on M. Let $L = f^{-1}(0)$ be a critical level set of f with a single critical point x_0 of index m, and let $L^{\pm} = f^{-1}(\pm\varepsilon)$ be the level sets slightly above and below such that there are no other critical points in between.

Theorem

Empty middle homology: $H_m(M, \mathbb{Z}_2) = 0$

$\Rightarrow$

Topology change: $H_m(L^-, \mathbb{Z}_2) \neq H_m(L^+, \mathbb{Z}_2)$

Change in Betti numbers: $|b_m(L^+, \mathbb{Z}_2) - b_m(L^-, \mathbb{Z}_2)| = 1$

Proof

Locally, the only difference between L^+ and L^- is given by which one of the two spheres S^+ and S^- becomes a boundary in $L^\pm \cap D$. So for global differences in topology, we consider how cycles in $\tilde{L}$ are related to S^+ and S^-.

The implications can be described by the following Mayer-Vietoris-like exact sequences

$$\begin{aligned} \dots \quad & \to H_k(S, G) \quad \stackrel{i_k}{\to} H_k(L^\pm \cap D, G) \oplus H_k(\tilde{L}, G) \to H_k(L^\pm, G) \\ & \stackrel{\partial_k}{\to} H_{k-1}(S, G) \quad \to \dots \end{aligned}$$

most of the terms vanish...

Proof

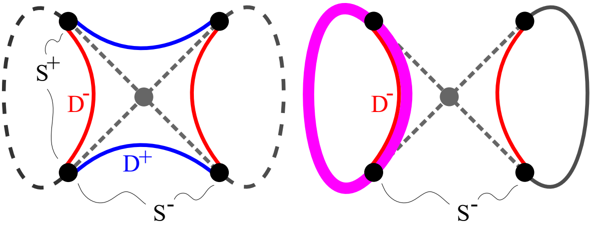
For $m \geq 2$
G free or a field ($G = \mathbb{Z}$ or $\mathbb{Z}_2$)

$$H_m(L^{\pm}, G) \cong H_m(\tilde{L}, G)/\mathrm{im}(i_m) \oplus \mathrm{im}(\partial_m),$$

Here, the only difference between L^+ and L^- can come from the image of $\partial_m : H_m(L^{\pm}, G) \to H_{m-1}(S, G)$.

Which elements of $< [S^+], [S^-] >$ are boundaries in $C_m(\tilde{L}, G)$?

Proof



Suppose $[g^+ S^+]$ is trivial (a boundary) in $H_{m-1}(\tilde{L}, G)$
$\Rightarrow$ cycle in $C_{m-1}(L^-, G)$

If L^- is G-orientable, then by Poincaré duality
$\exists$ cocycle, nonzero on S^-
$\Rightarrow [S^-]$ nontrivial in $H_{m-1}(\tilde{L}, G)$.

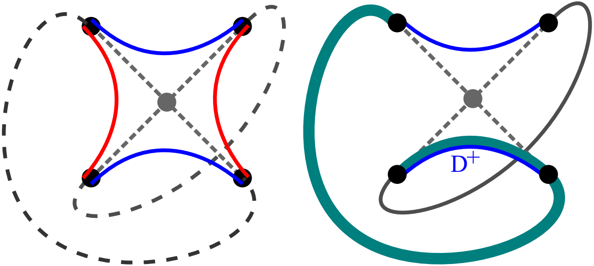
Proof

Suppose $[g^+S^+]$ is never a boundary,
not in $H_{m-1}(\tilde{L}, G)$, but also not in $H_{m-1}(L^+, G)$,

$\Rightarrow$ ...similar type of argument, somewhat more efford... $\Rightarrow$
(Via implications for integral coefficients, construction of a homomorphism $\varphi : H_{m-1}(L^+, \mathbb{Z}) \to G$ which is nonzero on S^+, and the universal coefficient theorem for cohomology, it is possible to construct a $m-1$-cohomology cycle which is nonzero on S^+. If L^+ is G-orientable, then by Poincaré duality this implies existence of an m-cycle which intersects S^+ with nonzero intersection number. Consider its intersection with S and remember that $[g^+S^+]$ is nontrivial in $H_{m-1}(L^+, G)$.)

$\Rightarrow$ some $[g^-S^-]$ is trivial in $H_{m-1}(\tilde{L}, G)$.

Proof



Suppose $[g^+ S^+]$ is nontrivial in $H_{m-1}(\tilde{L}, G)$,
then it is possibile that some $[g'^+ S^+]$ is still trivial in $H_{m-1}(L^+, G)$.

This then implies the existence of some chain in $\tilde{L}$ with boundary $g'^+ S^+ + g'^- S^-$.

Lemma

Suppose both g^+S^+ and g^-S^- are not a boundary in $C_{m-1}(\tilde{L}, G)$ for any nonzero g^+, g^-, but there exists $\tilde{D} \in C_m(\tilde{L}, G)$ with $\partial\tilde{D} = g'^+S^+ + g'^-S^-$ ($g'^+, g'^- \in G$, nonzero, and still $m \geq 2$).

Then there is a nontrivial homology class of $H_m(L, G)$ which can not be represented by a combination of cycles in $C_m(L^+, G)$ and $C_m(L^-, G)$, and this homology class persists in $H_m(M, G)$.

Proof (of Lemma)

Decompose L into $\tilde{L}$ and $L \cap D$
in this sequence:
$\partial_m C_m(L^{\pm}, G) = 0$ while $\partial_m[\tilde{D} - g'^{+}D^{-} - g'^{-}D^{+}] \neq 0$,

and decompose M into $M^{+} \cup M^{-}$ and a small neighbourhood of L,

$[\tilde{D} - g'^{+}D^{-} - g'^{-}D^{+}]$ is not in the image of
$H_m(L^{+} \cup L^{-}, G) = H_m(L^{+}) \oplus H_m(L^{-})$,

$\Rightarrow [\tilde{D} - g'^{+}D^{-} - g'^{-}D^{+}] \in M_m(M, G)$, nonzero.

Theorem

Let M be a $2m$-dimensional manifold without boundary, with f a proper Morse function on M. Let $L = f^{-1}(0)$ be a critical level set of f with a single critical point x_0 of index m, and let $L^{\pm} = f^{-1}(\pm\varepsilon)$ be the level sets slightly above and below such that there are no other critical points in between.

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If we additionally assume *orientability* of the level sets L^+ and L^-, the theorem holds for homology with coefficients in $\mathbb{Z}$, (or in any field if $m \geq 2$).

Corollary

Let M be a $2m$-dimensional manifold without boundary with vanishing m-th homology group $H_m(M, \mathbb{Z}_2)$.

The phenomenon of topology change at critical points
is generic for M

in the sense that if f is a proper Morse function on M such that its critical points are contained on different critical levels, then the topology (at least the homotopy type) of the regular level sets $f^{-1}(h)$ changes at each of these critical points.

Example

Outlook

Multiple critical points

Non-compact level sets

A better description of what happens

Applications

Thank you for listening

References:

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