Integrable Systems: from Poisson to Nijenhuis
Lecture 3

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Outline of Lecture 3

- Poisson structures and Hamiltonian systems
- Compatible Poisson structures, bi-Hamiltonian systems
- Recursion operator and its properties
- Poisson-Nijenhuis structures
- Local normal form for PN structures (algebraically generic case)
- Local normal form for PN structures (singular points)
- Integrability via symmetries of a Nijenhuis operator

Compatible Poisson structures and bi-Hamiltonian systems

Definition

Two Poisson structures P and $\widetilde{P}$ are called *compatible*, if their sum $P + \widetilde{P}$ is also a Poisson structure.

Notation. $\mathcal{X}_f = P \,\mathrm{d}\, f$ denotes the Hamiltonian vector field generated by a function f. Similarly, $\widetilde{X}_f = \widetilde{P} \,\mathrm{d}\, f$.

Definition

A vector field $\mathcal{X}$ is called *bi-Hamiltonian*, if it is Hamiltonian with respect to two compatible Poisson brackets P and $\widetilde{P}$, that is,

$$\mathcal{X} = P \,\mathrm{d}\, H = \widetilde{P} \,\mathrm{d}\, \widetilde{H}.$$

Question. Why are we interested in bi-Hamiltonian systems? What are their properties? And why are such systems "better" than the others?

We will assume here that at least one of the compatible Poisson structures P and $\widetilde{P}$ is non-degenerate so that e.g. P^{-1} makes sense.

Basic properties

Consider the linear operator $L = \widetilde{P}P^{-1}$ or, equivalently,

$$P(L^*\mathrm{d}\, f, \mathrm{d}\, g) = \widetilde{P}(\mathrm{d}\, f, \mathrm{d}\, g) \quad \text{for all } f, g \in C^\infty(M).$$

Theorem (F. Magri)

Let P and $\widetilde{P}$ be Poisson structures. Then P and $\widetilde{P}$ are compatible if and only if L is Nijenhuis.

In the theory of bi-Hamiltonian systems, the operator L plays an important role and is known as *recursion operator*.

In particular, notice the following formula for Hamiltonian vector fields:

$$\widetilde{\mathcal{X}}_f = L\,\mathcal{X}_f\,.$$

Also notice that without loss of generality we may, at least locally, assume that L is non-degenerate. Indeed, replacing $\widetilde{P}$ with $\widetilde{P} + \lambda P$ leads to $L \mapsto L + \lambda\,\mathrm{Id}$.

Proof

We first recall that the Jacobi identity for a Poisson structure P can be rewritten, in terms of Hamiltonian vector fields, as follows:

$$\mathcal{X}_{\{f,g\}} = [\mathcal{X}_f, \mathcal{X}_g].$$

Hence, the Jacobi identity for $P + \widetilde{P}$ can be rewritten as follows (below we use the fact that P and $\widetilde{P}$ are Poisson and remove the terms that vanish automatically due to this fact):

$$\widetilde{\mathcal{X}}_{\{f,g\}} + \mathcal{X}_{\widetilde{\{f,g\}}} = [\widetilde{\mathcal{X}_f}, \mathcal{X}_g] + [\mathcal{X}_f, \widetilde{\mathcal{X}_g}]$$

Now we use the fact that $\widetilde{\mathcal{X}_f} = L\mathcal{X}_f$ for any f:

$$L\mathcal{X}_{\{f,g\}} + L^{-1}\widetilde{\mathcal{X}}_{\widetilde{\{f,g\}}} = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Finally we use again the fact that P and $\widetilde{P}$ are both Poisson to get

$$L[\mathcal{X}_f, \mathcal{X}_g] + L^{-1}[L\mathcal{X}_f, L\mathcal{X}_g] = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Multiplying this identity by L we get the Nijenhuis identity.
It remains to recall that P is non-degenerate so that Hamiltonian vector fields generate the whole tangent space.

Basic facts (continued...)

Let $\mathcal{X}$ be a bi-Hamiltonian vector field w.r.t. P and $\widetilde{P}$.

Proposition

The coefficients σ_k of the characteristic polynomial of the recursion operator $L = \widetilde{P}P^{-1}$ are first integrals of $\mathcal{X}$. Moreover, these integrals commute with respect to the both structures P and $\widetilde{P}$.

Proof. 1. First integrals. It is a well known fact that P is preserved by each Hamiltonian vector field $\mathcal{X}_f$, i.e., $\mathcal{L}_{\mathcal{X}_f} P = 0$. Since in our case $\mathcal{X}$ is bi-Hamiltonian, we conclude that $\mathcal{X}$ preserves both P and $\widetilde{P}$.
Hence, $\mathcal{X}$ also preserves the recursion operator $L = \widetilde{P}P^{-1}$, and therefore all of its algebraic invariants including the coefficients σ_k of $\chi_L(t)$.
2. Commutativity. Instead of σ_k, we check commutativity of $f_k = \frac{1}{k}\,\mathrm{tr}\, L^k$, i.e., the identities $\{f_k, f_m\} = 0$. We use the following general property of Nijenhuis operators $\boxed{\mathrm{d}\, f_k = L^* \mathrm{d}\, f_{k-1}}$ which implies:

$$\{f_k, f_m\} = P(\mathrm{d}\, f_k, \mathrm{d}\, f_m) = P(L^*\mathrm{d}\, f_{k-1}, \mathrm{d}\, f_m) = \{\widetilde{f_{k-1}, f_m}\} = \{f_{k-1}, f_{m+1}\}$$

and by induction $\{f_k, f_m\} = \{f_{k-s}, \widetilde{f_{m+s-1}}\} = \{f_{k-s}, f_{m+s}\}$. It remains to notice that for a suitable s we have either $k-s = m+s$ or $k-s = m+s-1$ and the statement follows from skew symmetry of P and $\widetilde{P}$.

Poisson-Nijenhuis structures

Since $\widetilde{P}$ can be recovered from P and L, we can reformulate the compatibility condition for P and $\widetilde{P}$ in terms of L and P only as follows: We will say that a Poisson structure P and Nijenhuis operator L are *compatible* (or define a *Poisson-Nijenhuis structure*) if

(I) the form $\widetilde{P}(\,\cdot\,,\cdot\,) = P(\,L^*\,\cdot\,,\,\cdot)$ is skew-symmetric, i.e., is a bivector,

(II) $\widetilde{P}$ is Poisson (i.e. satisfies Jacobi identity).

For some reason, it will be more convenient to work with the symplectic forms $\omega = P^{-1}$ and $\widetilde{\omega} = \widetilde{P}^{-1}$ (assuming that P and $\widetilde{P}$ are non-degenerate). As above, we will say that a symplectic structure ω and a Nijenhuis operator L are *compatible* if [1]

(I') the form $\widetilde{\omega}(\,\cdot\,,\cdot\,) = \omega(\,L\,\cdot\,,\,\cdot)$ is skew-symmetric, i.e., is a differential 2-form,

(II') $\mathrm{d}\,\widetilde{\omega} = 0$, i.e., this form is closed.

(I') + (II') is equivalent to the fact that P and $\widetilde{P}$ are Poisson and compatible.

[1] This operator L is not the same as before. They are related by inversion, but we prefer to keep the previous notation.

Local normal forms for Poisson-Nijenhuis structures

It is natural to ask about local simultaneous canonical form for ω and L.

In the case when L is diagonalisable and algebraically generic at a point $p \in \mathrm{M}^{2n}$, i.e., its algebraic type remains the same in a certain neighbourhood $U(p)$, the answer is easy to derive from Splitting Theorem.

Notice, first of all, that condition (I') imposes natural algebraic restrictions on the algebraic type of L:

Algebraic fact. If ω and ωL are both skew-symmetric, then in the Jordan decomposition of L all the blocks can be partitioned into pairs of equal blocks (i.e. of the same size and with the same eigenvalue).

In particular, the characteristic polynomial of L is a full square and each eigenvalue has even multiplicity.

The most generic (and simplest) case

Assume that a symplectic structure ω and a Nijenhuis operator L are compatible.

Theorem

If L has $n = \frac{1}{2}\dim M$ distinct real eigenvalues, each of multiplicity 2, and in addition their differentials are linearly independent at a point $\mathrm{p} \in M$, then there exists a local symplectic coordinate system $x_1, \dots, x_n,\ p_1, \dots, p_n$ in which L is diagonal with x_i being its i-th eigenvalue:

$$\omega = \sum_i dp_i \wedge dx_i, \quad L = \mathrm{diag}(x_1, x_2, \dots, x_n, x_1, x_2, \dots, x_n). \tag{1}$$

Equivalently, one can say that the pair (ω, L) is the direct product of two-dimensional blocks (ω_i, L_i) of the form $\omega_i = dp_i \wedge dx_i$, $L_i = x_i \cdot \mathrm{Id}$.

Proof

Step 1. Apply the Splitting Theorem:

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_s = \bigoplus_i L_i$$

Step 2. Use the following algebraic fact.
Let $L = \bigoplus_i L_i$ be a block-diagonal matrix such that L_i and L_j have no common eigenvalues. Let ω be a non-degenerate skew-symmetric matrix such that ωL is skew symmetric too. Then ω is block diagonal, i.e., $\omega = \bigoplus_i \omega_i$.

Step 3. Let ω be a symplectic form such that in some coordinate coordinate system its matrix is block diagonal as above. Then each ω_i is also symplectic and depends on 'its own coordinates' only. In other words, $\omega = \bigoplus_i \omega_i$ is a direct product of symplectic forms.

Step 4. This shows that the Poisson-Nijenhuis structure (ω, L) splits into direct product of two-dimensional Poisson-Nijenhuis structures (ω_i, L_i).

Step 5. It remains to verify the statement in dimension 2 (Exercise).

More general result

Consider 4 elementary examples of compatible pairs:

Type 1. L has one real non-constant eigenvalue of multiplicity 2:

$$\omega = \mathrm{d}\, p \wedge \mathrm{d}\, x, \quad L = \lambda(x, p) \cdot \mathsf{Id}.$$

Type 2. L has one real constant eigenvalue of multiplicity $2k$;

$$\omega = \sum_{j=1}^{k} \mathrm{d}\, p_j \wedge \mathrm{d}\, x_j, \quad L = \lambda \cdot \mathsf{Id}, \ \lambda \in \mathbb{R}.$$

Type 3. L has one pair of non-constant complex conjugate eigenvalues of multiplicity 2:

$$\omega = \mathrm{Re}\,(\mathrm{d}\, z \wedge \mathrm{d}\, w), \quad L = \alpha(z, w) \cdot \mathsf{Id} + \beta(z, w) \cdot J,$$

where $z = x + \mathrm{i}y, w = u + \mathrm{i}v$, J denotes the corresponding complex structure and $\alpha(z, w) + \mathrm{i}\beta(z, w)$ is a holomorphic function in z, w.

Type 4. L has one pair of constant complex conjugate eigenvalues of multiplicity $2k$:

$$\omega = \mathrm{Re}\left(\sum_{j=1}^{k} \mathrm{d}\, z_j \wedge \mathrm{d}\, w_j\right), \quad L = \alpha \cdot \mathsf{Id} + \beta \cdot J,$$

where $z_j = x_j + \mathrm{i}y_j, w_j = u_j + \mathrm{i}v_j$, J denotes the corresponding complex structure and $\alpha + \mathrm{i}\beta \in \mathbb{C}$, $\beta \neq 0$.

More general result

Theorem

Let ω and L be compatible. Suppose that L is semisimple and algebraically generic in a neighbourhood of $\mathrm{p} \in M$. Then the pair (ω, L) locally splits into a direct product of 'elementary blocks' of 4 types described above. If $\mathrm{d}\,\lambda(\mathrm{p}) \neq 0$ or $\mathrm{d}\,(\alpha(\mathrm{p}) + \mathrm{i}\beta(\mathrm{p})) \neq 0$ for some real or complex eigenvalue, then in the corresponding 'elementary block' we may set $\lambda = x$ (see Type 1) and $\alpha + \mathrm{i}\beta = z$ (see Type 3).

Important observation. To admit a compatible symplectic partner ω, a semisimple algebraically generic Nijenhuis operator L should satisfy the following additional condition: non-constant eigenvalues of L must be all of multiplicity two.

If L is algebraically generic but not necessarily semisimple, the description (rather non-trivial) of compatible pairs (ω, L) was obtained by F. Turiel under additional assumptions on the differentials of $\operatorname{tr} L^k$, $k = 1, \dots, n$. These assumptions basically mean that each eigenvalue is either constant or its differential does not vanish.

What about singular points?

If we want to study singularities of Poisson-Nijenhuis structures, then it is natural to ask which Nijenhuis operators admit compatible symplectic structures and what is a simultaneous canonical form for ω and L near a (possibly singular) point $\mathrm{p} \in M$?

What kind of singularities do we mean?
For a symplectic structure ω, all points $\mathrm{p} \in M$ are obviously equivalent to each other (in other words, there are no singular points). However it is not the case for an operator L since its algebraic properties may vary from point to point. Recall that $\mathrm{p} \in M$ is singular for L if the algebraic type of L changes at this point.

Among singular points, we distinguish a subclass of *differentially non-degenerate* singular points $\mathrm{p} \in M$, i.e., such that the differentials of the coefficients σ_k of the characteristic polynomial $\chi_L(t) = \det(t \,\mathrm{Id} - L)$ are linearly independent at p. Here

$$\chi_L(t) = \det(t \cdot \mathrm{Id} - L) = t^m + \sigma_1 t^{m-1} + \cdots + \sigma_{m-1} t + \sigma_m.$$

Differential non-degeneracy in this context

Relation (I'), however, implies that the eigenvalues of L have always even multiplicity, which in turn implies that L cannot be differentially non-degenerate. Recall that the characteristic polynomial is a full square

$$\chi_L(t) = (t^n + h_1 t^{n-1} + \cdots + h_{n-1} t + h_n)^2, \qquad m = 2n = \dim M,$$

and has therefore at most $\frac{1}{2} \dim M$ functionally independent coefficients. A natural analog of differential non-degeneracy in this setup is as follows:

The differentials $\mathrm{d}\, h_1(\mathrm{p}), \ldots, \mathrm{d}\, h_n(\mathrm{p})$ of the functions h_i are linearly independent at a given (singular) point $\mathrm{p} \in M$.

Question. Do such singularities appear in Poisson-Nijenhuis structures? If yes, can we describe local canonical forms for ω and L at such points?

New application of Nijenhuis Geometry

Theorem

Let ω and L be compatible (i.e., define a Poisson-Nijenhuis structure on M^{2n}) and real analytic. Suppose that at a point $\mathrm{p} \in M^{2n}$, the differentials $\mathrm{d}\, h_1(\mathrm{p}), \dots, \mathrm{d}\, h_n(\mathrm{p})$ are linearly independent. Then there exists a local coordinate system $x_1, \dots, x_n, p_1, \dots, p_n$ such that $\omega = \sum_i \mathrm{d}\, x_i \wedge \mathrm{d}\, p_i$ and L is given by the matrix

$$\begin{pmatrix} A & 0_n \\ S & A^\top \end{pmatrix}, \tag{2}$$

where

$$A = \begin{pmatrix} -x_1 & 1 & 0 & \cdots & 0 \\ -x_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -x_{n-1} & \vdots & & \ddots & 1 \\ -x_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -p_2 & -p_3 & \cdots & -p_n \\ p_2 & 0 & 0 & \cdots & 0 \\ p_3 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ p_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{3}$$

0_n is the zero $n \times n$-matrix, and $A^\top$ denotes the transposed of A.

Commuting integrals and symmetries of Nijenhuis operators

Let N be a Nijenhuis operator on a Poisson manifold (M, P).
Assume that $\mathrm{Id}, N, \dots, N^{n-1}$ are linearly independent at each point, and n is the maximal number with this property[2].
Consider the space $\mathcal{S}(N)$ of strong symmetries of N that can be written as linear combinations $Q = h_1 \mathrm{Id} + h_2 N + \dots + h_n N^{n-1}$ with functional coefficients $h_i \in C^\infty(\mathrm{M})$:

$$\mathcal{S}(N) = \left\{ Q = \sum_k h_k N^{k-1} \mid \langle Q, N \rangle = 0 \right\}.$$

Theorem

Let $Q_1, \dots, Q_n$ be a basis of $\mathcal{S}(N)$ in the sense that every $Q \in \mathcal{S}(N)$ can be written as a linear combination $Q = \sum_i f_i Q_i$, $f_i \in C^\infty(\mathrm{M})$.
If N is self-adjoint w.r.t. P, that is, $P(\alpha, N^*\beta) = P(N^*\alpha, \beta)$ for all $\alpha, \beta \in T^*_x\mathrm{M}$, $x \in \mathrm{M}$, then for any $Q \in \mathcal{S}(N)$, the coefficients f_i Poisson commute.

[2] Equivalently, n is the degree of the minimal polynomial of N so that for every $k \geqslant n$ we can write N^k as a linear combination of $\mathrm{Id}, N, \dots, N^{n-1}$ with functional coefficients.

Proof

The condition that $Q = f_1 Q_1 + \cdots + f_n Q_n$ is a strong symmetry of N is equivalent to

$$0 = \langle N, f_1 Q_1 + \cdots + f_n Q_n \rangle = \sum_i \big(f_i \langle N, Q_i \rangle + N^* \mathrm{d}\, f_i \otimes Q_i - \mathrm{d}\, f_i \otimes N Q_i \big).$$

Since $\langle N, Q_i \rangle = 0$, we obtain $\sum_i N^* \mathrm{d}\, f_i \otimes Q_i = \sum_i \mathrm{d}\, f_i \otimes N Q_i$

Notice that $NQ_i \in \mathcal{S}(N)$ and, therefore, we can set $NQ_i = \sum_j c_i^j Q_j$ to get

$$\sum_j \left(N^* \mathrm{d}\, f_j - \sum_i c_i^j \mathrm{d}\, f_i \right) \otimes Q_j = 0.$$

Since $Q_1, \ldots, Q_n$ are linearly independent, we conclude that

$$N^* \mathrm{d}\, f_j = \sum_i c_i^j \mathrm{d}\, f_i.$$

In particular, $\mathrm{Span}(\mathrm{d}\, f_1, \ldots, \mathrm{d}\, f_n)$ is an N^*-invariant subspace.

Lemma

The subspace $\mathrm{Span}(\mathrm{d}\, f_1, \dots, \mathrm{d}\, f_n)$ *is* N^**-cyclic in the sense that there exists* $\alpha = \sum_j \alpha_j \mathrm{d}\, f_j$ *such that* $\mathrm{Span}(\mathrm{d}\, f_1, \dots, \mathrm{d}\, f_n) = \mathrm{Span}\big(\alpha, N^*\alpha, \dots, (N^*)^{n-1}\alpha\big)$.

Proof.

Take $\alpha = \sum_j \alpha_j \mathrm{d}\, f_j \in \mathrm{Span}(\mathrm{d}\, f_1, \dots, \mathrm{d}\, f_n)$. Then

$$N^*\alpha = N^*\left(\sum_j \alpha_j \mathrm{d}\, f_j\right) = \sum_j \alpha_j N^* \mathrm{d}\, f_j = \sum_i \left(\sum_j c_i^j \alpha_j\right) \mathrm{d}\, f_i = \sum_i \tilde{\alpha}_i \mathrm{d}\, f_i = \tilde{\alpha}.$$

Equivalently, this means that $C = \left(c_i^j\right)$ is the matrix of the map $\alpha \mapsto \alpha^* = N^*\alpha$. Thus, it is sufficient to show that the matrix C is cyclic.

But C are defined from $NQ_j = \sum_j c_i^j Q_j$, which means that C is the matrix of the dual map $Q \mapsto NQ$ in the space $\mathrm{Span}\left(\mathrm{Id}, N, \dots, N^{n-1}\right)$.

It remains to notice this map is obviously cyclic with Id being one of its cyclic vectors. $\square$

Lemma

Let P be a skew-symmetric bilinear form on a vector space V and $A : V \to V$ be a linear operator such that

$$P(A\xi, \eta) = P(\xi, A\eta), \quad \text{for all } \xi, \eta \in V. \tag{4}$$

Then every A-cyclic subspace of V is isotropic w.r.t. P.

Proof.

Exercise in Linear Algebra (cf. the slide Basic facts (continued ...)). □

To summarise:
the subspace $\mathrm{Span}(\mathrm{d}\, f_1, \ldots, \mathrm{d}\, f_n)$ is an N^*-invariant cyclic subspace and, therefore, is isotropic w.r.t. the Poisson structure P, that is, $f_1, \ldots, f_n$ Poisson commute, as required.

Construction (Step 1: Yano–Patterson lift)

Let $\theta = p_q \mathrm{d}\, u^q$ be the Liouville form on the cotangent bundle $T^*\mathrm{M}^n$ and $\Omega = \mathrm{d}\,\theta = \mathrm{d}\, p_q \wedge \mathrm{d}\, u^q$ be the canonical symplectic structure on $T^*\mathrm{M}^n$. For any operator field L on M^n, the *deformed* Liouville form θ_L and deformed symplectic form Ω_L are defined by

$$\theta_L = p_q L^q_s \mathrm{d}\, u^s \quad \text{and} \quad \Omega_L = \mathrm{d}\,\theta_L = p_q \frac{\partial L^q_s}{\partial u^r} \mathrm{d}\, u^r \wedge \mathrm{d}\, u^s + L^q_s \mathrm{d}\, p_q \wedge \mathrm{d}\, u^s.$$

The *complete (Yano–Patterson) lift* $\widehat{L}$ of L onto $T^*\mathrm{M}^n$ is defined by

$$\Omega(\cdot, \widehat{L}\cdot) = \Omega_L(\cdot,\cdot). \tag{5}$$

In canonical coordinates, $\widehat{L}$ takes the form

$$\widehat{L} = \Omega^{-1}\Omega_L = \begin{pmatrix} L^\top & R \\ 0_{n\times n} & L \end{pmatrix}, \quad \text{where} \quad R_{ij} = \left(\frac{\partial L^q_i}{\partial u^j} - \frac{\partial L^q_j}{\partial u^i}\right) p_q. \tag{6}$$

Properties.

- If L is Nijenhuis, then $\widehat{L}$ is Nijenhuis also.
- If L is Nijenhuis, then Ω_L and Ω are compatible, i.e. $(\Omega^{-1}, \widehat{L})$ is Poisson-Nijenhuis.
- If M and L are symmetries (strong symmetries) of each other, then $\widehat{M}$ and $\widehat{L}$ are symmetries (strong symmetries) of each other.

Further steps (without technical details)

Main idea: Take a gl-regular Nijenhuis operator on M and consider its lift $\widehat{L}$ onto the cotangent bundle.
As before, let $\mathcal{S}(\widehat{L})$ be the space of strong symmetries of $\widehat{L}$ that are linear combinations of the powers of $\widehat{L}$. For a given L, this space can be described in reasonable terms.
Then choosing a basis in $Q_1, \dots, Q_n$ in $\mathcal{S}(\widehat{L})$ and a particular symmetry $Q \in \mathcal{S}(\widehat{L})$, we construct an integrable Hamiltonian system on T^*M.

Question. How to choose $Q_1, \dots, Q_n$ and Q to get something interesting?

Possible answer. Let $S_1, \dots, S_n$ be symmetries of L, then we can set $Q_i = \widehat{S}_i$.
What about Q? If $Q = \widehat{M}$, where M is a symmetry of L, then we the output of the construction is trivial:

$$f_1\widehat{S}_1 + \dots + f_n\widehat{S}_n = \widehat{M} \quad \Leftrightarrow \quad f_1 S_1 + \dots + f_n S_n = M$$

and therefore, $f_1, \dots, f_n$ are functions on the configuration space. They Poisson commute, but...
Conclusion. We need to take a symmetry $Q \in \mathcal{S}(\widehat{L})$ that is essentially different from $\widehat{M}$. What are they?

The classical (diagonal) example

Let $L = \mathrm{diag}(u^1, \ldots, u^n)$, then $\widehat{L} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$.

The symmetries of L are of the form $S = \mathrm{diag}\big(s^1, ..., s^n\big)$, $s^j = s^j(u^j)$ and $\widehat{S} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$.

Important: strong symmetries $Q \in \mathcal{S}(\widehat{L})$ have the form $Q = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$, with $R = \mathrm{diag}\big(r^1, ..., r^n\big)$, $r^j = r^j(p_j, u^j)$.

Hence, our relation $f_1 \widehat{S}_1 + \cdots + f_n \widehat{S}_n = Q$ reads $f_1 S_1 + \cdots + f_n S_n = R$ or equivalently

$$\begin{pmatrix} s_1^1 & \ldots & s_n^1 \\ \vdots & \ddots & \vdots \\ s_1^n & \ldots & s_n^n \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} r^1 \\ \vdots \\ r^n \end{pmatrix}, \quad s_i^j = s_i^j(u^j),\ r^j = r^j(p_j, u^j),$$

which is exactly classical Stäckel relations. If we assume that s_j^i depends on u^i only and $r^j = p_j^2 + V_j(u^j)$, then we obtain classical Stäckel integrable systems with Poisson commuting integrals f_α quadratic in momenta p_j.

Good choice for Q?

Let L be $\mathfrak{gl}$-regular, f a regular conservation law of L, $u^1, \dots, u^n$ local coordinates, $p_1, \dots, p_n$ the corresponding momenta. Then there exists a unique operator

$$B_i \in \mathrm{Span}(\mathrm{Id}, L, \dots, L^{n-1}) \text{ such that } B_i^* \mathrm{d} f = \mathrm{d}\, u^i.$$

Definition. $P = p_1 B_1 + \dots + B_n p_n$. Less formally, $P \in \mathrm{Span}(\mathrm{Id}, L, \dots, L^{n-1})$ is such that $P^* \, \mathrm{d}\, f = \sum_i p_i \mathrm{d}\, u^i$.

Example

If $L = \begin{pmatrix} u^1 & & & \\ & u^2 & & \\ & & \ddots & \\ & & & u^n \end{pmatrix}$ and $f = \mathrm{tr}\, L$, then $P = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix}$.

If $L = \begin{pmatrix} u^1 & & & \\ u^2 & u^1 & & \\ \vdots & \ddots & \ddots & \\ u^n & \dots & u^2 & u^1 \end{pmatrix}$ and $f = u^n$, then $P = \begin{pmatrix} p_n & & & \\ \vdots & \ddots & & \\ p_2 & \dots & p_n & \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$

Final result: constructing integrable systems from L

Ingredients of the construction:

- L, gl-regular Nijenhuis operator on the configuration space M^n
- $\operatorname{Sym} L$, the symmetry algebra of L (locally near generic points, we have an explicit description of $\operatorname{Sym} L$; these symmetries are parametrised by n functions of one variables)
- $S_1, \dots, S_n \in \operatorname{Sym} L$, a basis of $\operatorname{Sym} L$
- a regular conservation law f of L (locally near generic points, we have an explicit description of conservation laws; they are parametrised by n functions of one variables)
- from f we construct $P = p_1 B_1 + \dots + p_n B_n$ as above (it is a good invariant object which corresponds to a certain strong symmetry $Q = \begin{pmatrix} P^\top & * \\ 0 & P \end{pmatrix} \in \mathcal{S}(\widehat{L})$).
- (optional) $U \in \operatorname{Sym} L$, an additional symmetry (which will later serve as potential energy of the system).

Final result: constructing integrable systems from L

Theorem

Let L be a gl-regular Nijenhuis operator on M^n and $\mathrm{Sym}\, L$ be the space of its symmetries. Choose a basis $S_1, \dots, S_n$ in $\mathrm{Sym}\, L$ and an additional symmetry $U \in \mathrm{Sym}\, L$. Then the functions $h_\alpha(p, u)$, $\alpha = 1, \dots, n$, defined from the relation

$$h_1 S_1 + \dots + h_n S_n = P^2 + U, \tag{7}$$

have the form

$$h_\alpha(p, u) = \tfrac{1}{2} g_\alpha(p, p) + V_\alpha(u), \quad g_\alpha(p, p) = (g_\alpha)^{qs}(u) p_q p_s,$$

and Poisson commute on $T^*\mathrm{M}^n$ with respect to the canonical Poisson structure. These functions are functionally independent and therefore define and integrable Hamiltonian system on $T^*\mathrm{M}^n$ with integrals quadratic in momenta.