Families of 4-dimensional integrable systems with S^1-symmetries

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joint with Y. Le Floch and S. Hohloch

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Plan

- Based on two papers with Yohann Le Floch:
  - *Semitoric families*, `arXiv:1810.06915` (to appear in the Memoirs of the AMS)
  - *Families of four-dimensional integrable systems with S^1-symmetries*, `arXiv:2307.10670`

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(I will also mention two papers with Sonja)

Symplectic manifolds and integrable systems

- Let (M,ω) be a symplectic manifold and $f\colon M \to \mathbb{R}$.
- Denote by $\mathcal{X}_f$ the Hamiltonian vector field of f, which satisfies

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- Flows of $\mathcal{X}_{f_1}, \ldots \mathcal{X}_{f_n}$ induce (local) $\mathbb{R}^n$-action.

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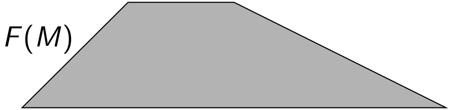
- i.e. a global Hamiltonian T^n-action.
- $F\colon M \to \mathbb{R}^n$, Atiyah, Guillemin-Sternberg (1982) showed that in this case the image $F(M) \subset \mathbb{R}^n$ is a convex polytope.

Toric integrable systems: the classification



Theorem (Delzant, 1988)

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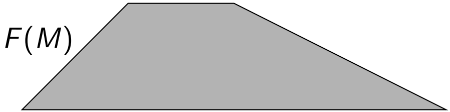

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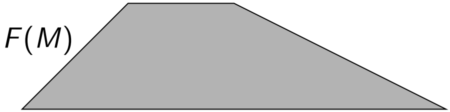

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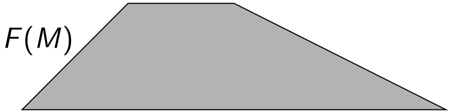

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- The toric system can be reconstructed from Δ by symplectic reduction on $\mathbb{C}^d$.

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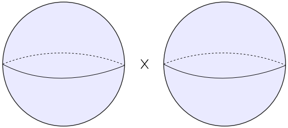

- $M = S^2 \times S^2, \quad \omega = \omega_1 \oplus 2\omega_2$
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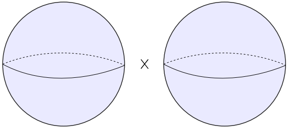

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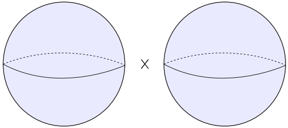

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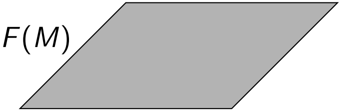

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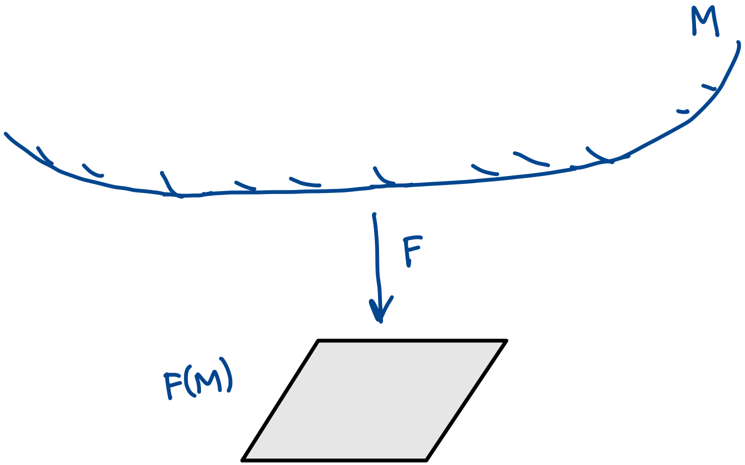

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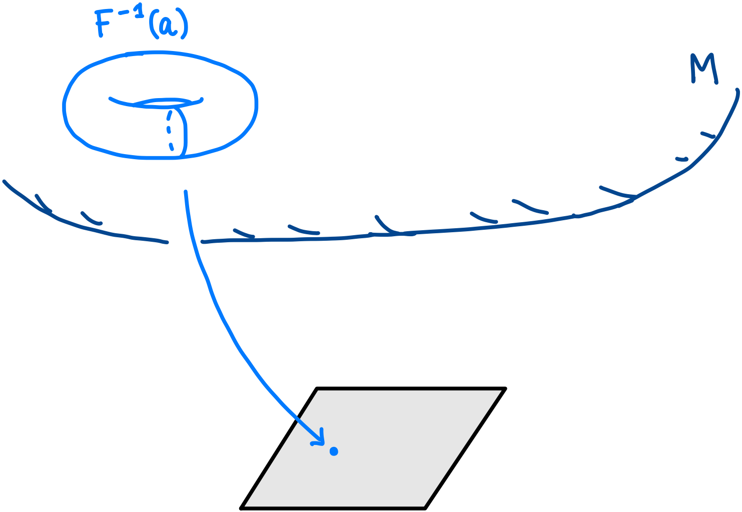

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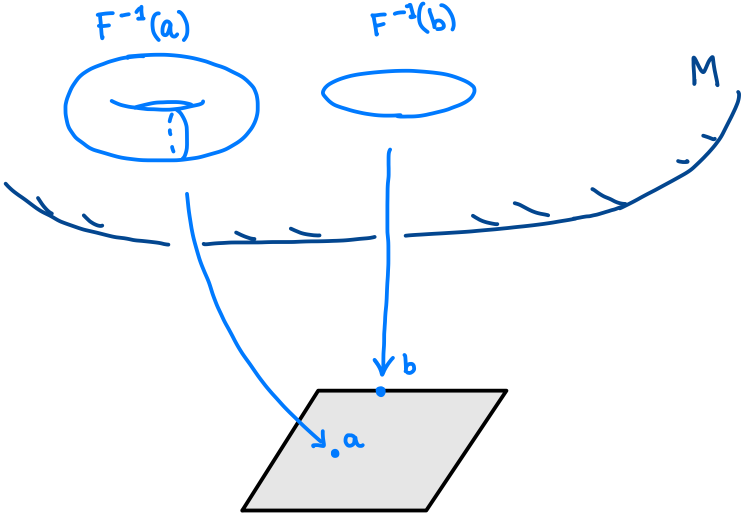

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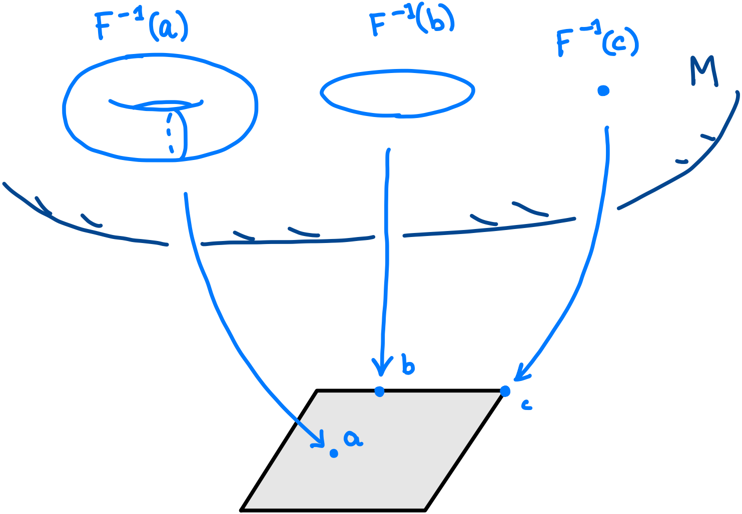

Semitoric integrable systems: definition

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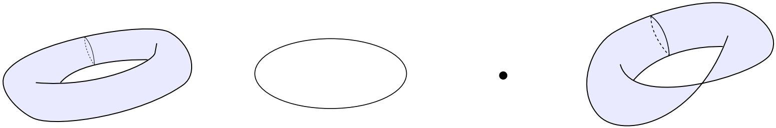
- In particular, (M, ω, J) is a Hamiltonian S^1-space
  - studied by Karshon [1999].

Semitoric integrable systems: fibers

Fibers in simple semitoric systems:

regular elliptic-regular elliptic-elliptic focus-focus



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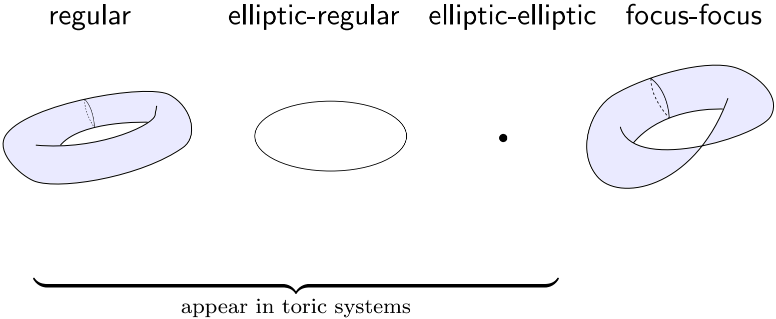

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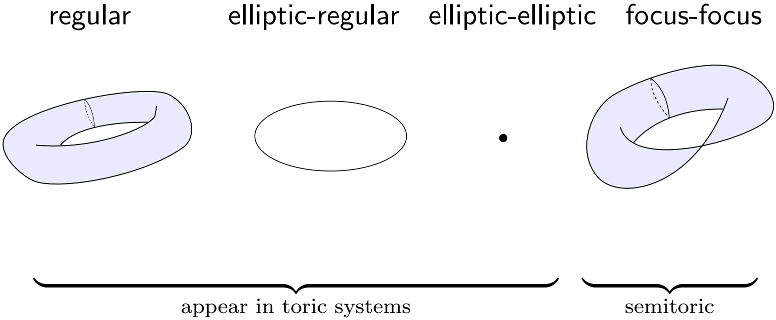

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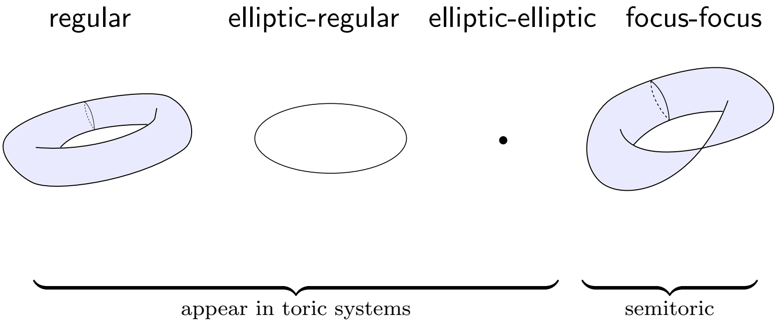

- A system is called *simple* if there is at most one focus-focus point per fiber of J.

Semitoric invariants

- Five invariants of semitoric systems:
  (1) number of focus-focus points;
  (2) semitoric polygon (Vũ Ngọc, Symington);
  (3) height invariant;
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Simple semitoric systems are classified by (1)-(5).

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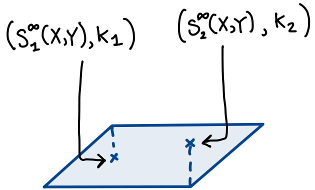

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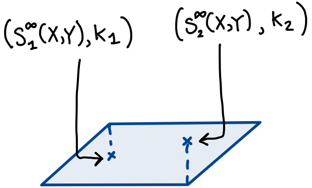

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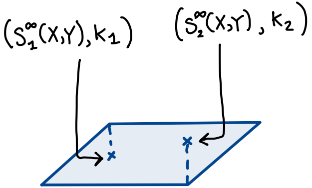

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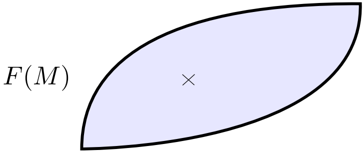
- For this talk, we will be concerned with the marked polygon.

Semitoric invariants: the marked polygon invariant

- $F\colon M \to \mathbb{R}^2$ produces a singular Lagrangian torus fibration.



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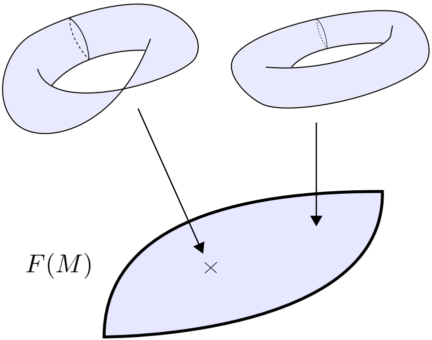

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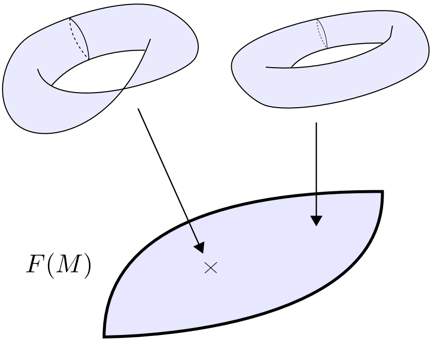

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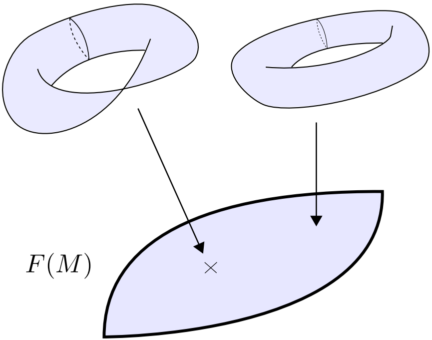

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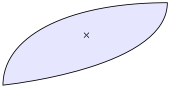

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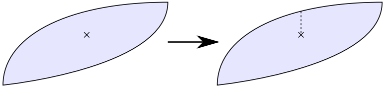

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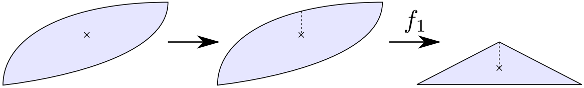

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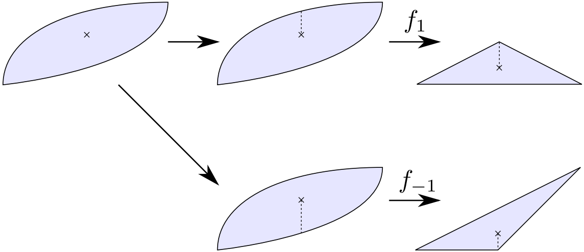

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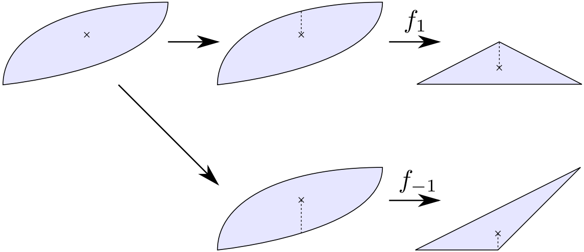

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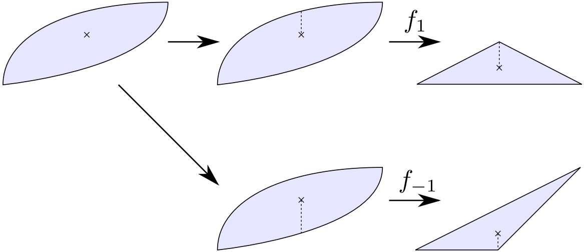

- Marked semitoric polygon: Family of polygons including marked points.
- Any marked polygon satisfying certain conditions (similar to the Delzant conditions) is the polygon for a semitoric system.

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- Toric integrable systems can be explicitly constructed from the polytope by performing symplectic reductions on $\mathbb{C}^d$ by a torus action.
- Semitoric integrable systems can be constructed from invariants by gluing of semi-local normal forms.

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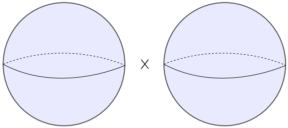
Goal

Given specified (unlabeled) marked semitoric polygon invariant try to find an explicit system with that invariant.

Example: Coupled angular momenta

[Sadovskií and Zĥilinskií, 1999]



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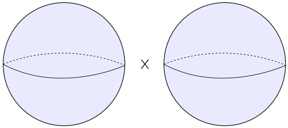

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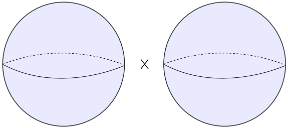

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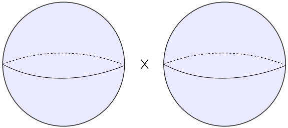

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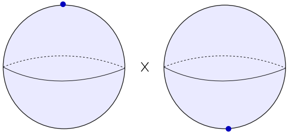

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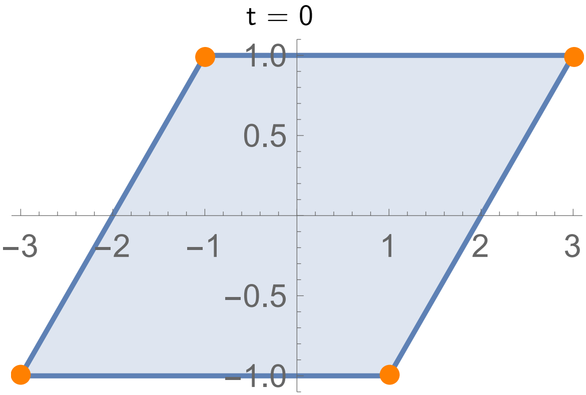
Coupled angular momenta: moment map image



Toric
(figure made in Mathematica)

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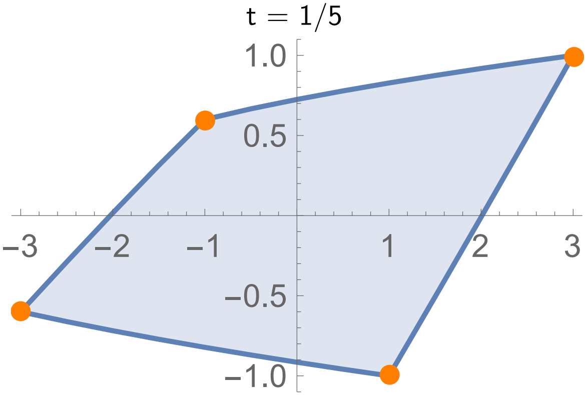

Semitoric with zero focus-focus points
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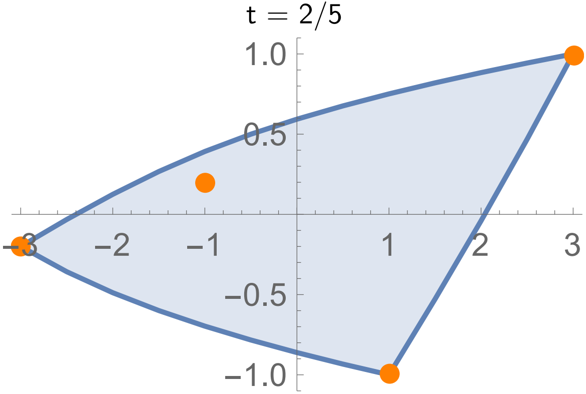

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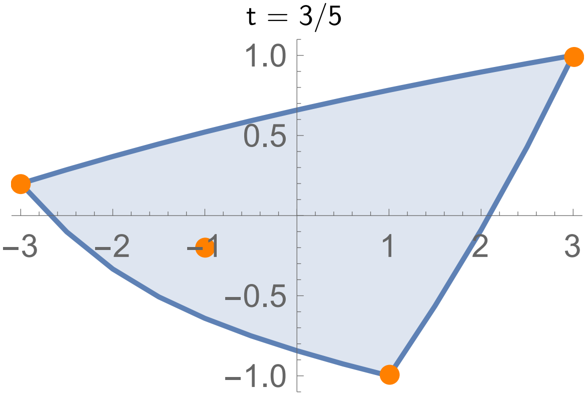

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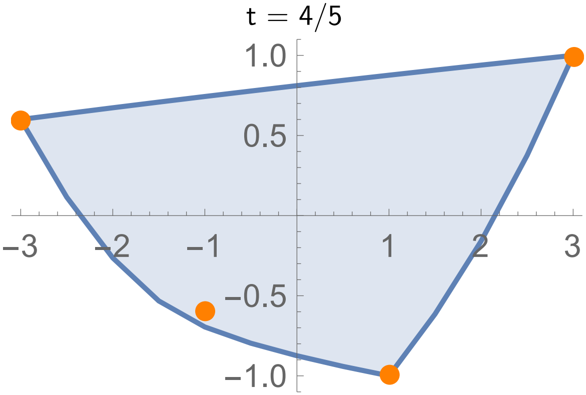

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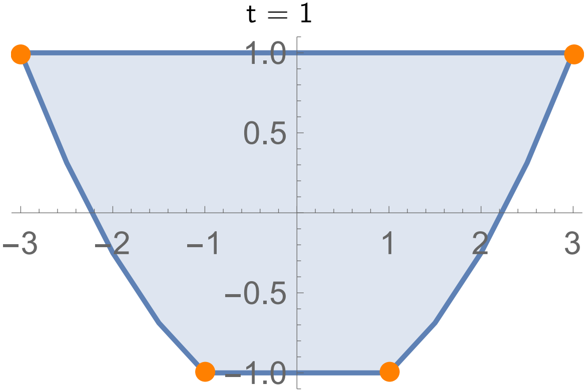

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Coupled angular momenta: semitoric polygon

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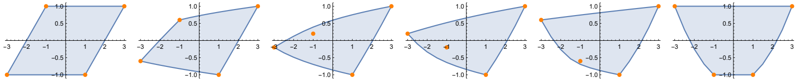

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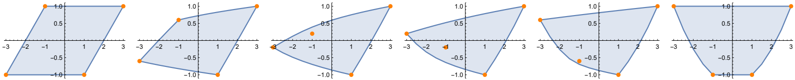

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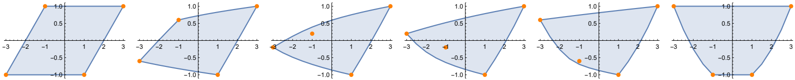

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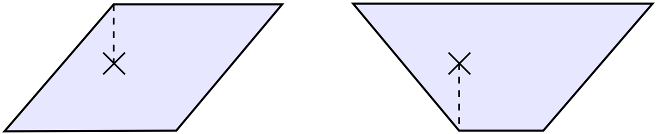

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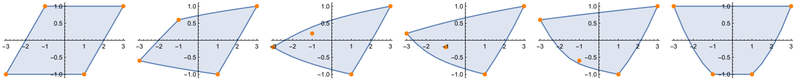

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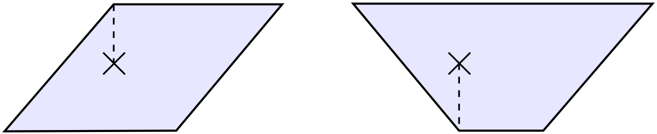

Idea

Given polygons, interpolate between systems "related to the semitoric polygons" to find desired semitoric system.

Semitoric families

Definition (Le Floch-P., 2018)

A semitoric family is a family of integrable systems (M, ω, F_t), $0 \leq t \leq 1$, where

- $\dim(M) = 4$;
- $F_t = (J, H_t)$;
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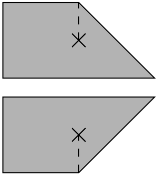
Lemma (Invariance of polygons, Le Floch-P. 2018)

The (unmarked) semitoric polygon in a semitoric family can only change at the values of t for which the system is degenerate.

Polygons in a semitoric family



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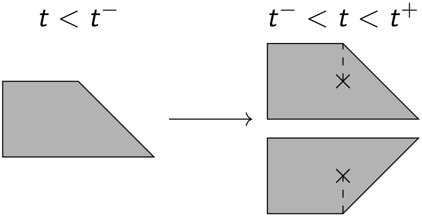

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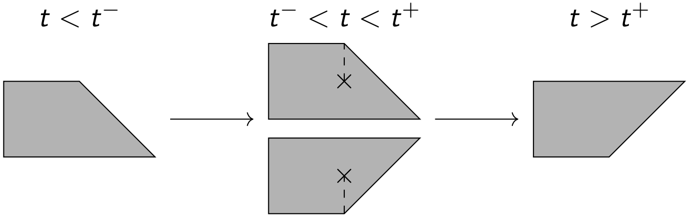

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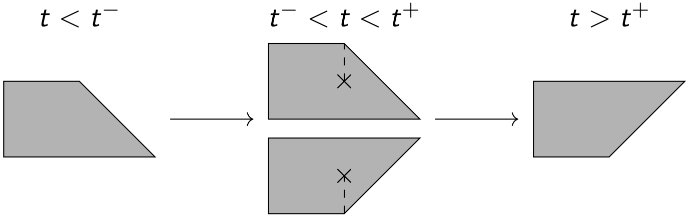
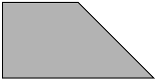


- For example, can we construct a system with the above polygons?

The first Hirzebruch surface



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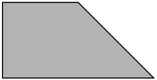

- Applying the Delzant construction we obtain the first Hirzebruch surface: $\mathbb{C}^4$ reduced by the T^2-action generated by

$$N = (1/2)\left(|u_1|^2 + |u_2|^2 + |u_3|^2, |u_3|^2 + |u_4|^2\right) - (2,1).$$

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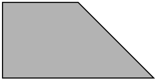

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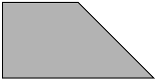

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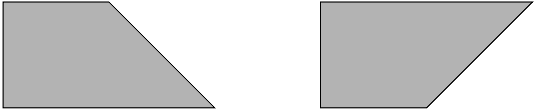

Example on W_1, the first Hirzebruch surface

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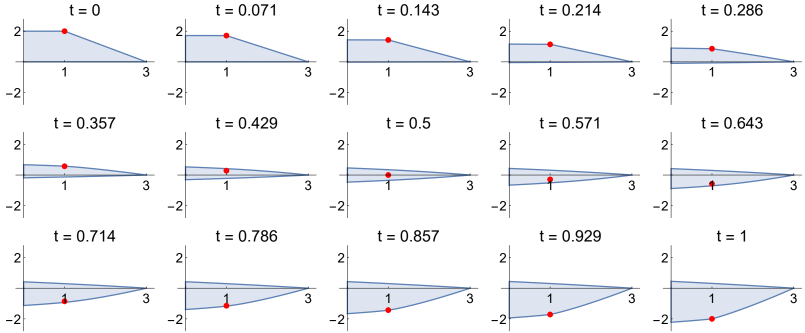

A system with two focus-focus points

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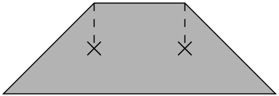

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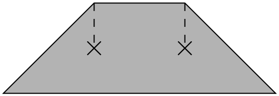

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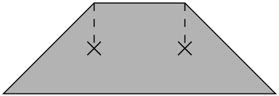

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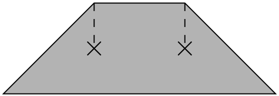

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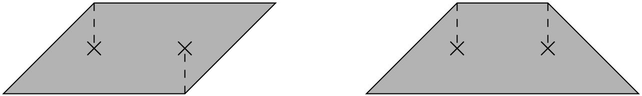

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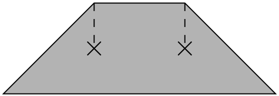

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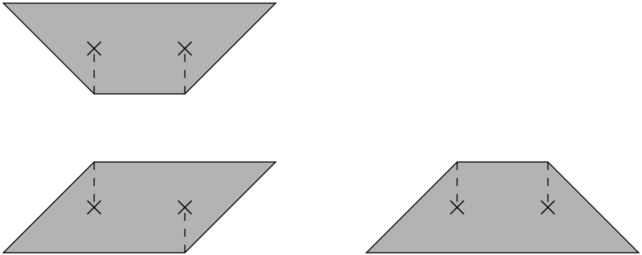

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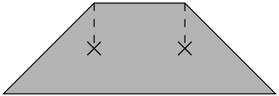

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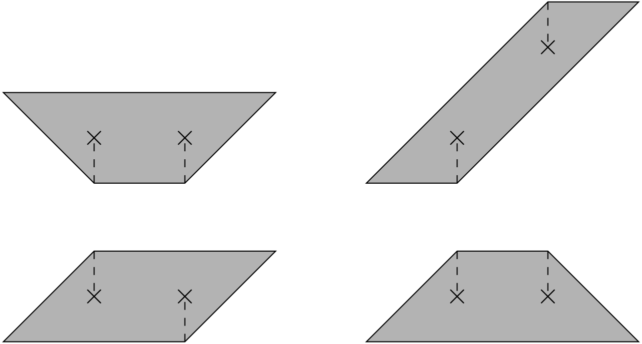

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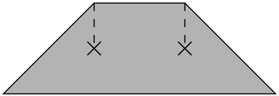

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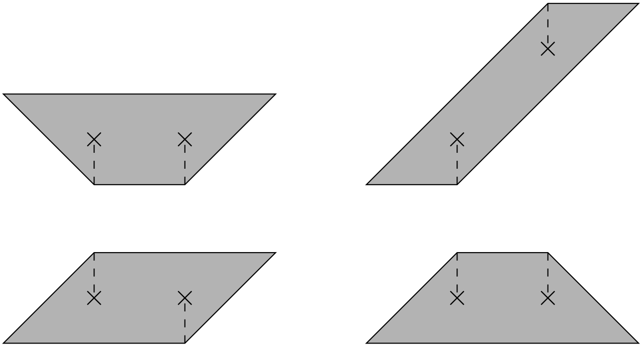

A two parameter family

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for $s_1, s_2 \in [0,1]$.

The momentum map image

Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$

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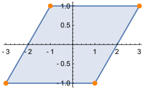

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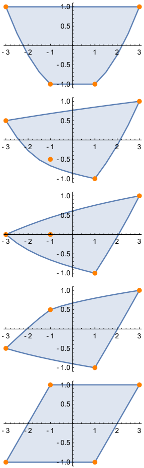

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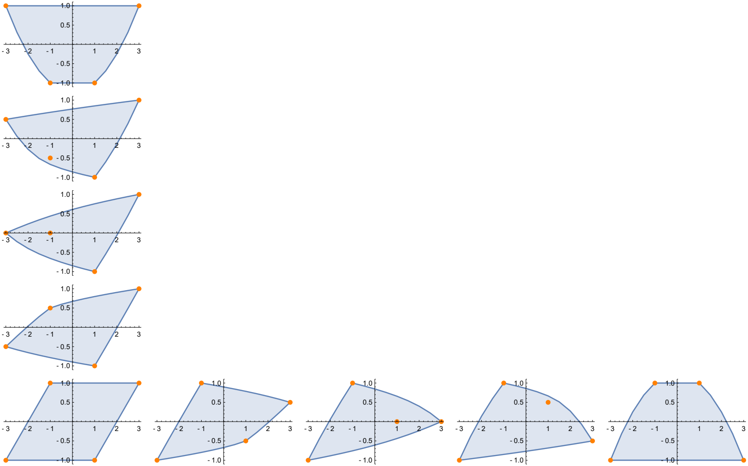

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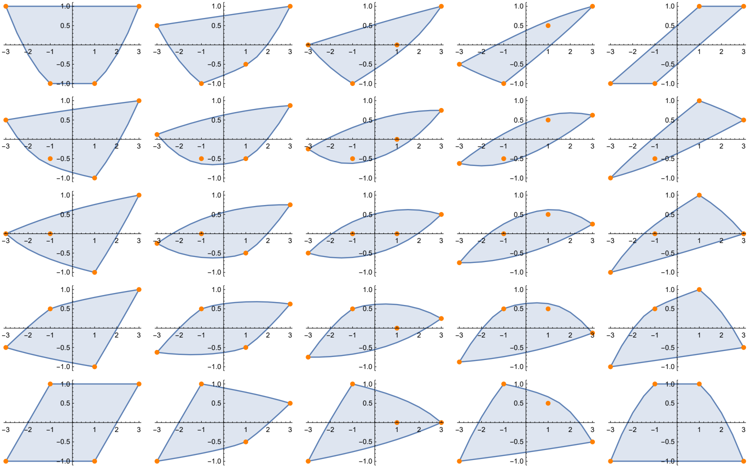
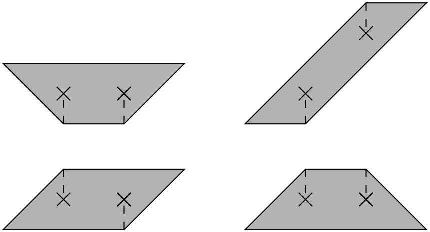


Theorem



Theorem (Hohloch-P., 2018)

The system $(J, H_{\frac{1}{2},\frac{1}{2}})$ is a semitoric integrable system with exactly two focus-focus points which has the above polygons as its semitoric polygon invariant.

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Proposition [Le Floch-P, 2023]

Let $J = \pm\left(\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2\right)$. Then $H \in \mathcal{C}^\infty(\mathbb{C}^2, \mathbb{R})$ is a Hamiltonian such that $\{J, H\} = 0$, $H(0) = 0$ and $\mathrm{d}H(0) = 0$ if and only if there exists $\mu_1, \mu_2, \mu_3, \psi \in \mathbb{R}$ such that

$$H(z_1, z_2) = \mu_1 \Re(e^{i\psi} z_1 z_2) + \mu_2 |z_1|^2 + \mu_3 |z_2|^2 + \text{higher order.}$$

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Moreover, such a (J, H) is integrable if and only if $(\mu_1, \mu_2 + \mu_3) \neq (0, 0)$ and in this case the singular point $(0, 0)$ is

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- This tells us how to chose H_t

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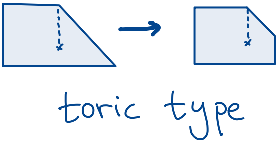

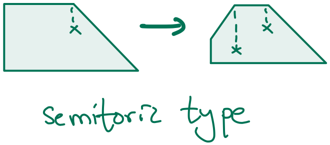

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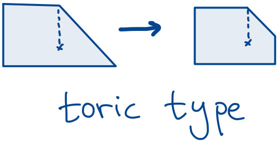

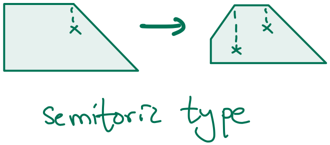

- A semitoric system is called minimal if it does not admit a toric type blowup, and strictly minimal if it does not admit either type.

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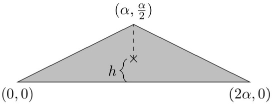
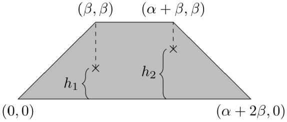
Proposition (P.-Le Floch, 2023):

A semitoric system is strictly minimal if and only if its marked polygon is one of the ones shown on the next slide.

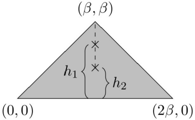
The strictly minimal polygons



(a) Type(1). $0 < h < \frac{\alpha}{2}$.



(b) Type (2a). $0 < h_1, h_2 < \beta$.



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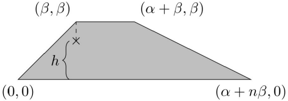
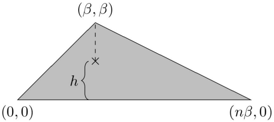
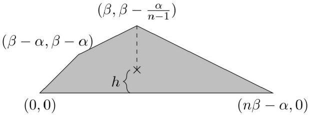


(d) Type (3a). $n \geq 1$, $0 < h < \beta$.

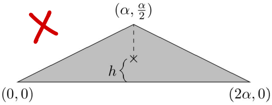


(e) Type (3b). $n \geq 2$, $0 < h < \beta$.



(f) Type (3c). $n \geq 2$, $\alpha < \beta$, $0 < h < \beta - \frac{\alpha}{n-1}$.

The strictly minimal polygons - progress so far



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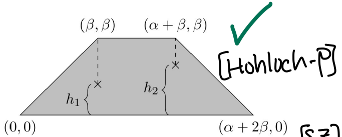

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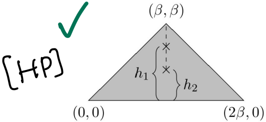

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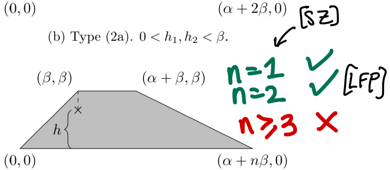

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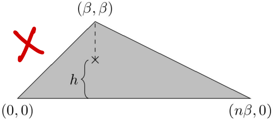

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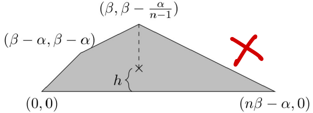

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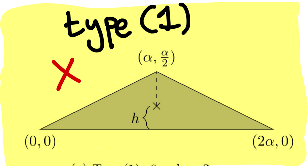

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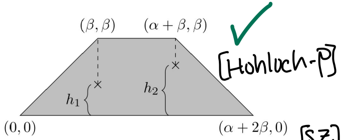

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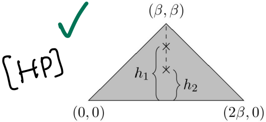

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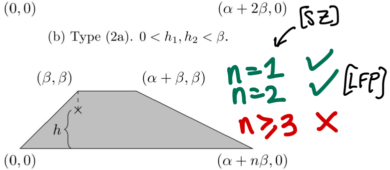

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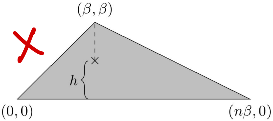

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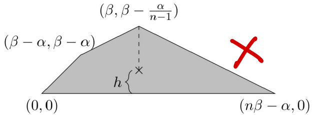

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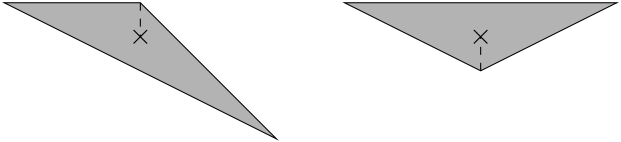

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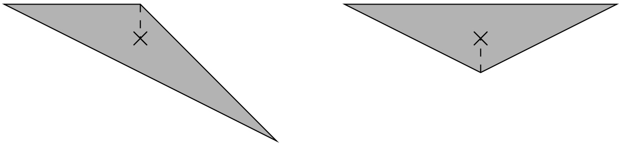

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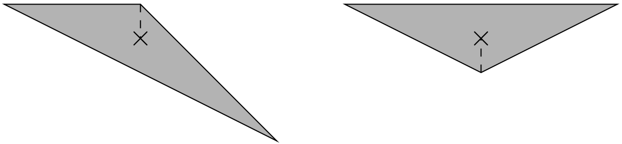

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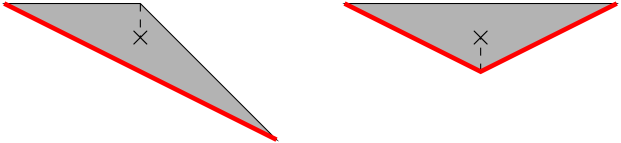

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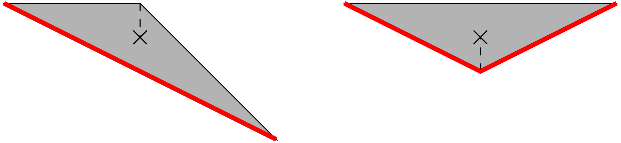

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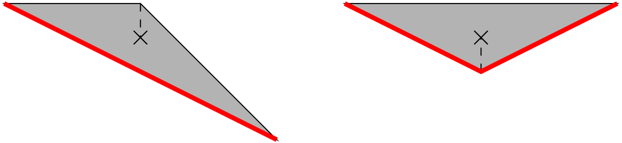

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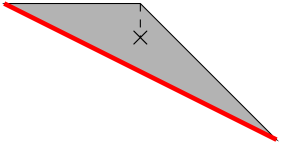
  - Those points **must** be sent to the boundary in a semitoric system.

Towards a system of type (1)

- Type (1) polygons:



- The right polygon does not correspond to a toric system!
  - No problem, just use the other polygon for the $t = 0$ system, and abandon hope that something toric will appear when $t = 1$
- The preimage of this "line" has a special property with respect to the S^1-action generated by J ($\mathbb{Z}_2$ isotropy).
  - Those points **must** be sent to the boundary in a semitoric system.
- Hohloch-Sepe-Sabatini-Symington studied the relationship between semitoric systems and the S^1-space [2015-present]

A type (1) system on $\mathbb{CP}^2$

- λ, δ, γ are parameters satisfying $0 < \gamma < \frac{1}{4\lambda}$ and $\delta > \frac{1}{2\gamma\lambda}$.

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- Last term is to keep the $\mathbb{Z}_2$-sphere on the boundary.

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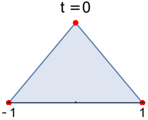

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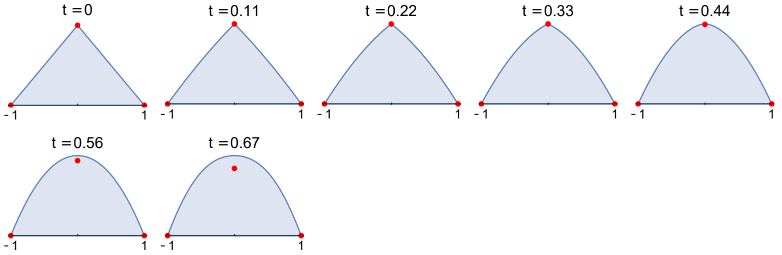

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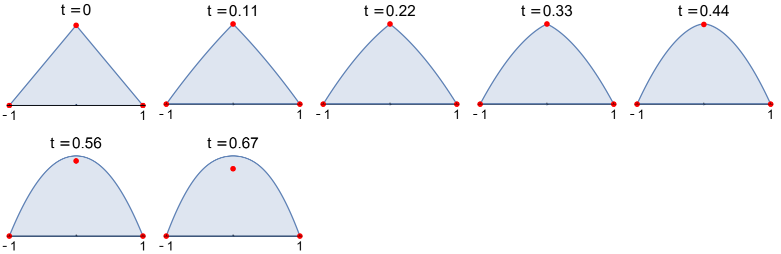

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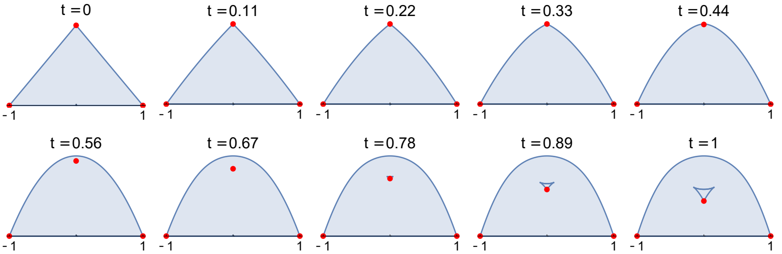

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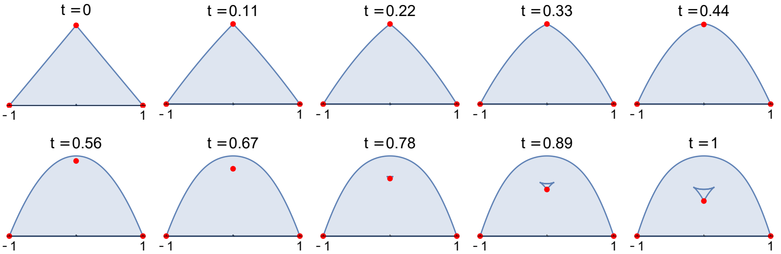

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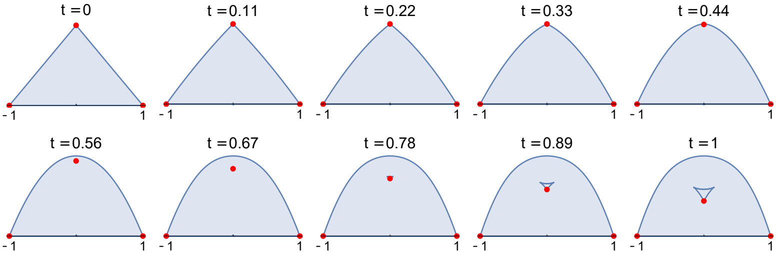
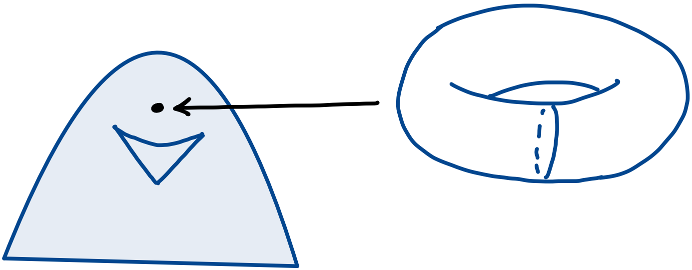


- For $t \approx \frac{1}{2}$ the system is semitoric with a focus-focus point!
- For large t the system becomes hypersemitoric.
- Hypersemitoric systems were introduced in [Hohloch-P., 22] to study the integrable systems with prescribed S^1-actions.

Fibers of hypersemitoric systems



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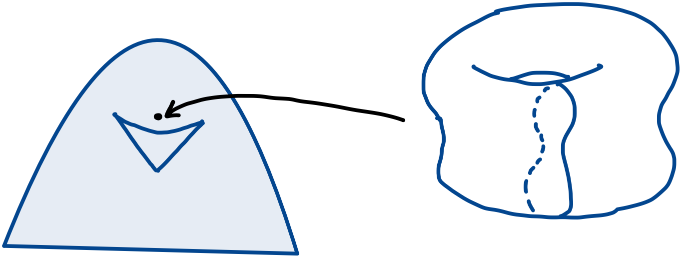

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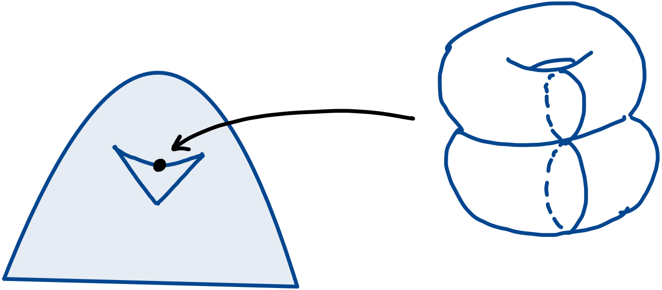

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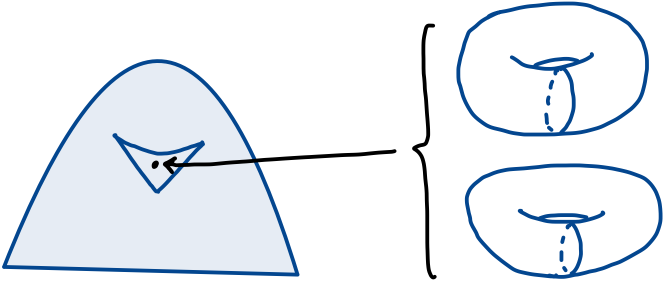

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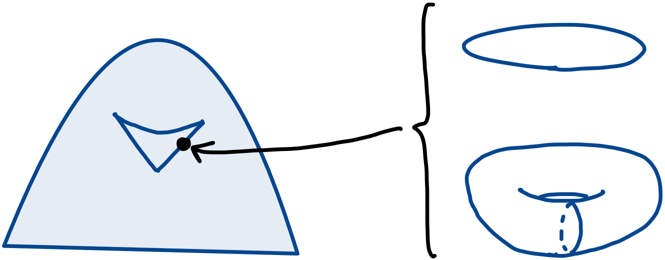

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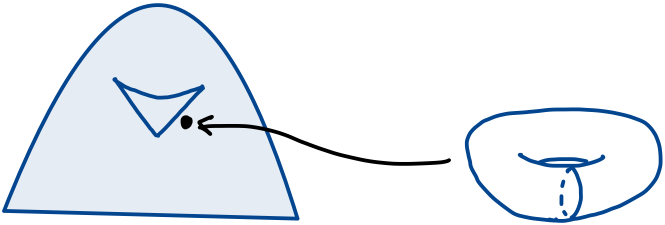

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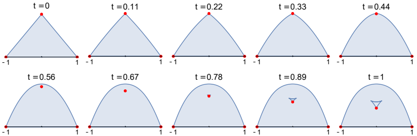

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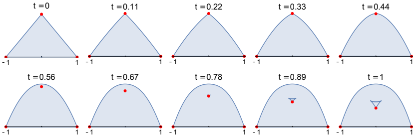

Theorem (Le Floch-P., 2023)

The family $F_t = (J, H_t)$ is

- *semitoric with zero focus-focus points when* $0 \leq t < t^-$,

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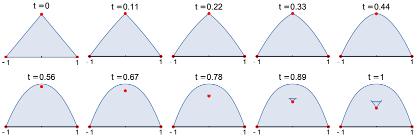

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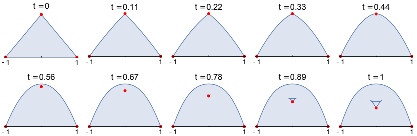

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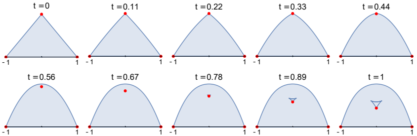

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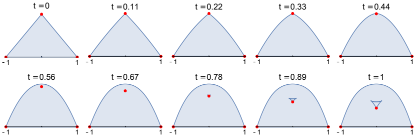

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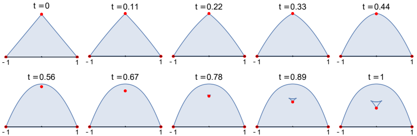

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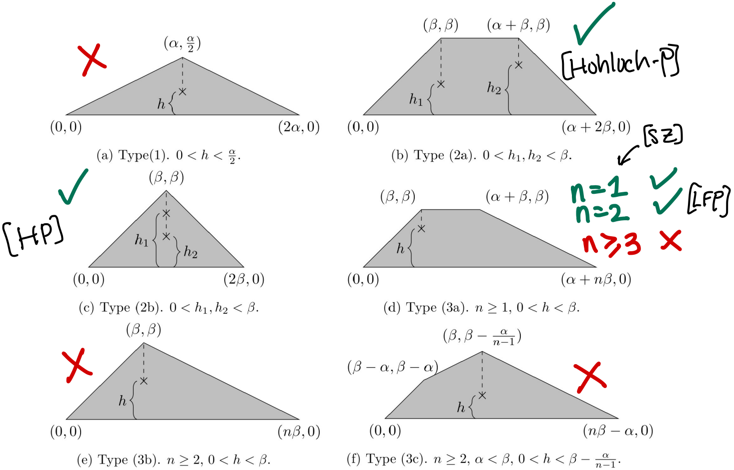
- An example of type (1) was also studied by Chiscop-Dullin-Efstathiou-Waalkens (2019).

The strictly minimal polygons - progress



(a) Type(1). $0 < h < \frac{\alpha}{2}$.

(b) Type (2a). $0 < h_1, h_2 < \beta$.

(c) Type (2b). $0 < h_1, h_2 < \beta$.

(d) Type (3a). $n \geq 1$, $0 < h < \beta$.

(e) Type (3b). $n \geq 2$, $0 < h < \beta$.

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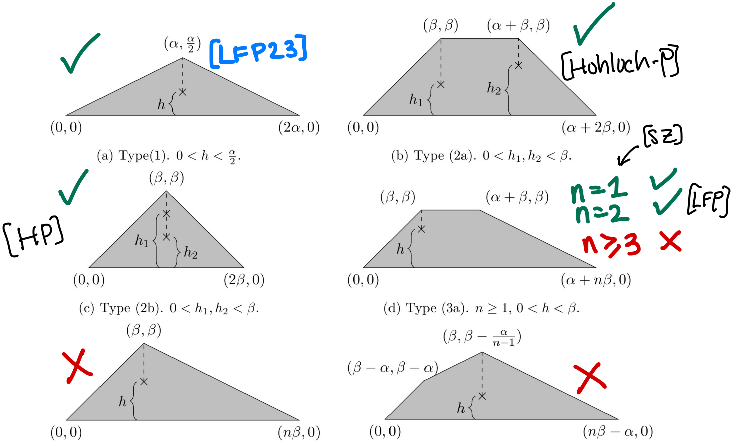

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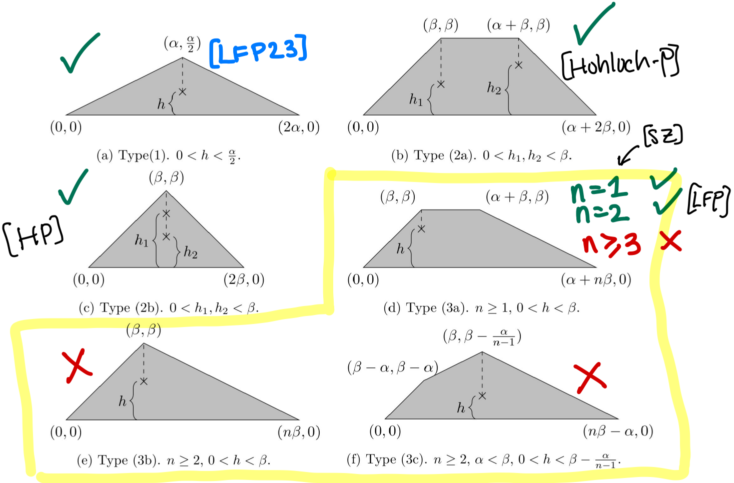

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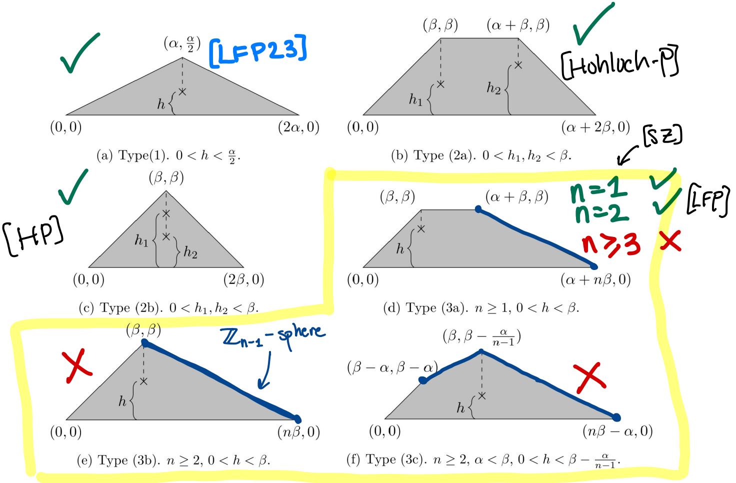

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Strictly minimal systems of type (3)

- For the following slides, we consider the $(n-2)^{th}$ Hirzebruch surface, obtained as the symplectic reduction of $\mathbb{C}^4$ by

$$N = \frac{1}{2}\left(|z_1|^2 + |z_3|^2 + (n-2)|z_4|^2, |z_2|^2 + |z_4|^2\right)$$

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- Let $\mathcal{X} = \Re(z_1 z_2 \bar{z}_3^{n-1} z_4)$ and $R = \frac{1}{2}\left(|z_1|^2 + (n-2)|z_4|^2\right)$

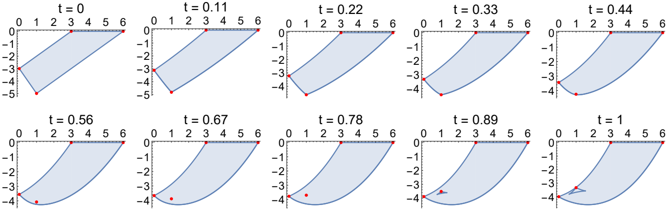
Strictly minimal systems of type (3a)

Theorem (Le Floch-P., 2023)

For certain $\alpha, \beta, \gamma, \delta > 0$, the system

$$J = \tfrac{1}{2}(|z_1|^2 + |z_2|^2), \qquad H_t = \frac{(2t-1)}{2}|z_3|^2 + 2\gamma t(\mathcal{X} + \delta R^2) - 2\gamma\delta t((n-1)\beta + \alpha)^2.$$

is semitoric of type (3a) when $t^- < t < t^+$.



Strictly minimal systems of type (3b)

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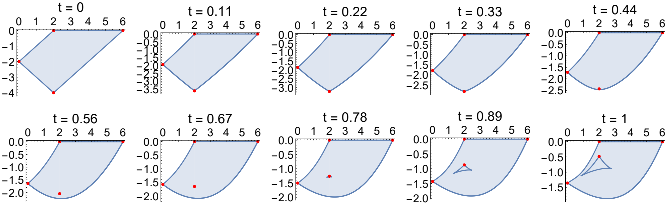

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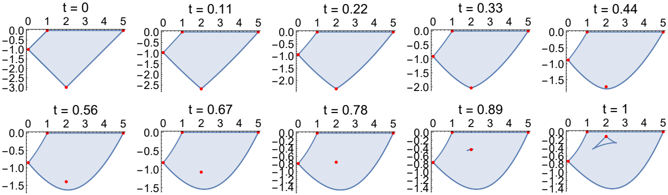
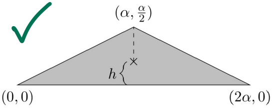


The strictly minimal polygons - progress



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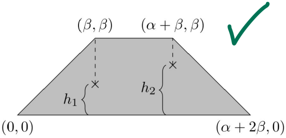
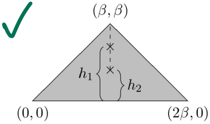


(b) Type (2a). $0 < h_1, h_2 < \beta$.



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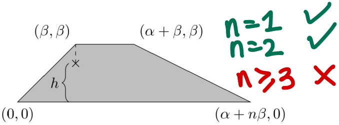
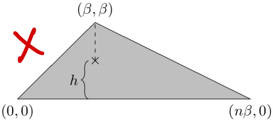
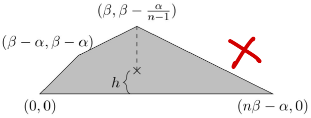


(d) Type (3a). $n \geq 1$, $0 < h < \beta$.

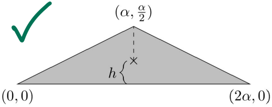


(e) Type (3b). $n \geq 2$, $0 < h < \beta$.



(f) Type (3c). $n \geq 2$, $\alpha < \beta$, $0 < h < \beta - \frac{\alpha}{n-1}$.

The strictly minimal polygons - progress



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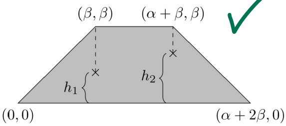

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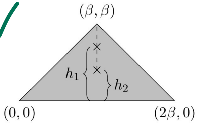
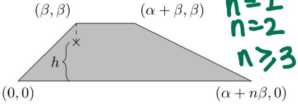
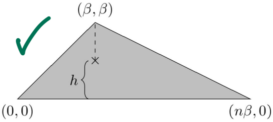


(c) Type (2b). $0 < h_1, h_2 < \beta$.

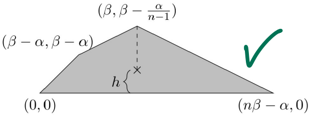


(d) Type (3a). $n \geq 1$, $0 < h < \beta$.

$n=1$
$n=2$
$n \geq 3$



(e) Type (3b). $n \geq 2$, $0 < h < \beta$.



(f) Type (3c). $n \geq 2$, $\alpha < \beta$, $0 < h < \beta - \frac{\alpha}{n-1}$.

The explicit constructions of strictly minimal systems

Theorem

For each strictly minimal (unmarked) semitoric polygon, there exists an explicit system in the literature:

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- **type (2a/b):** *Hohloch-P. (2018), Le Floch-P. (2022)*
- **type (3a), $\mathbf{n = 1}$:** *Coupled angular momenta (SZ,1999)*
- **type (3abc), $\mathbf{n = 2}$:** *Le Floch-P. (2022)*
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- Moreover, these examples each come from a family of systems $(M, \omega, (J, H_t))$, starting with a toric system at $t = 0$ and either becoming toric again or developing a flap by $t = 1$.

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Thanks for listening!