New topological invariants of QFDIS

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Liouville integrability

Definition

Let (M, ω, H) be an integrable Hamiltonian system with n functionally independent integrals $f_1, f_2, \ldots, f_n$ in involution $\{f_i, f_j\} = 0$. Then

$$F = (f_1, f_2, \ldots, f_n) \colon M \to \mathbb{R}^n$$

is called the **integral** or the **energy-momentum map** of the system.

One central problem in this context is to **classify such integral maps up to an appropriate equivalence relation**, e.g., topological or symplectic and/or global or (semi-)local.

Examples of integrable systems

- Particle in a central potential (e.g. the Kepler problem and the spherical pendulum);
- Toric, semitoric and hypersemitoric systems;
- Special geodesic flows;
- Euler, Lagrange and Kovalevskaya tops;
- Calogero-Moser systems;
- etc.

Quadratic spherical pendulum

Consider a particle moving on the unit sphere in $\mathbb{R}^3$ in a quadratic potential $V = aq_3^2 + bq_3$.

The Hamiltonian

The corresponding **Hamiltonian** system is defined by the total energy

$$H(q, p) = \frac{1}{2}\langle p, p\rangle + V(q_3)$$

on the cotangent bundle T^*S^2.

Quantum mechanically, we get the **Hamiltonian operator**

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + V(q_3).$$

Integrability

Since the potential V is invariant under rotations about the q_3-axis, the angular momentum

$$J = q_1 p_2 - q_2 p_1 \quad / \quad \hat{J} = -i\hbar(q_1 \partial_{q_2} - q_2 \partial_{q_1})$$

commutes with the Hamiltonian $H/\hat{H}$.

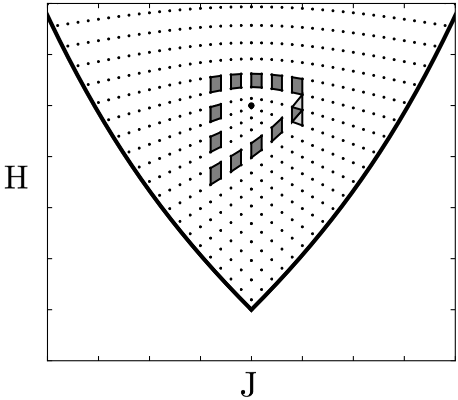
Integral map

The joint map

$$F = (H, J) \colon T^* S^2 \to \mathbb{R}^2$$

is a **torus fibration**! (Arnol'd-Liouville theorem.)

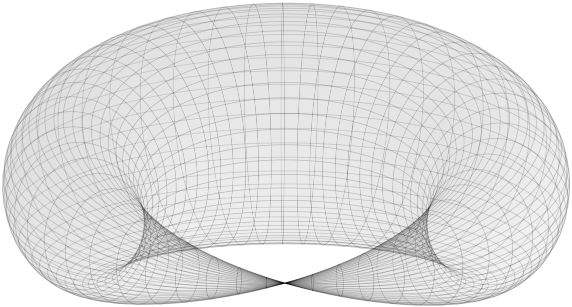
Bifurcation diagram and (quantum) monodromy



J.J. Duistermaat, *On global action-angle coordinates*, Comm. Pure Appl. Math. '80

R.H. Cushman and J.J. Duistermaat, *Quantum mechanical spherical pendulum*, Bulletin of the AMS '88

Pinched torus



Recent developments

- Mathematical foundation of **quantum monodromy** — *S. Vũ Ngọc '99*

- **Fractional monodromy.** Here the matrix is rational rather than integer:

$$M = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \ q \in \mathbb{Q}.$$

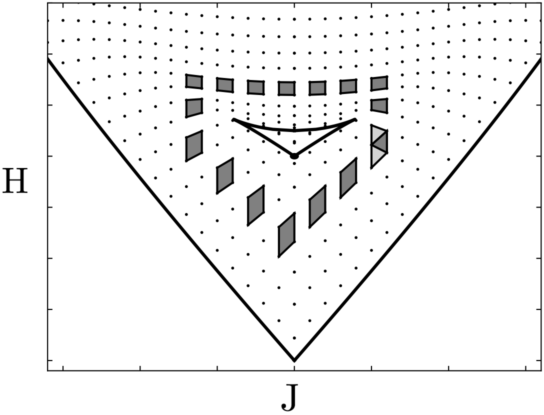
  N.N. Nekhoroshev, D.A. Sadovskii, and B.I. Zhilinskii '06

- **Monodromy in nearly-integrable systems** — *H.W. Broer, R.H. Cushman, F. Fassò, and F. Takens '07*

- **Scattering monodromy** — *R.H. Cushman, L. Bates '07*, etc.

For an overview, see N. Martynchuk, H.W. Broer, K. Efstathiou: *Recent advances in the monodromy theory of integrable Hamiltonian systems*, Indag. Math. '20

Quantum quadratic spherical pendulum



K. Efstathiou, *Metamorphoses of Hamiltonian systems with symmetry*, Springer '05

From KAM to quantum mechanics

Theorem

The topology of a torus fibration survives small perturbations by virtue of KAM theory. In particular, monodromy survives small perturbations.

Question

Work out a symplectic/quantum analogue of the global KAM theorem suitable for monodromy.

H. W. Broer, R. H. Cushman, F. Fassò, and F. Takens, *Geometry of KAM tori for nearly integrable Hamiltonian systems*, Ergodic Theory and Dynamical Systems ’07

Q. S. Phan, Spectral monodromy of non selfadjoint operators, PhD Thesis, Rennes, ’12

Possible applications

- Quantum (fractional) monodromy for nearly-integrable systems;
- Applications to molecular systems (flexible triatomic molecules).

Scattering

Potential scattering

Consider a pair of (integrable) Hamiltonians such that their dynamics is 'the same' at infinity in phase space. For example, one can take

$$H = \frac{1}{2}\|p\|^2 + V(q) \text{ and } H_0 = \frac{1}{2}\|p\|^2 + V_0(q),$$

where $V - V_0$ has finite range (is of compact support).

Scattering map

Scattering map

The idea behind scattering theory is to compare the asymptotic data of the Hamiltonians H and H_0: Let M be g_H^t-invariant. Assume that the composition map

$$S = (A^-)^{-1} \circ A_0^- \circ (A_0^+)^{-1} \circ A^+,$$

where $A^\pm$ and $A_0^\pm$ sends a given point (q, p) to its asymptotics at infinity, is well defined and maps M to itself.

The map S is called the **scattering map** associated to H and H_0. In what follows, by a scattering map we will also mean the corresponding map on the quotient manifold M/g_H^t.

Example

Energy-E scattering map

In the case $M = H^{-1}(E)$, we get the scattering map

$$S_E \colon Q \to Q, \; Q = H^{-1}(E)/g_H^t.$$

The quotient space Q is the cotangent bundle T^*S^{n-1}.

Scattering at different energies

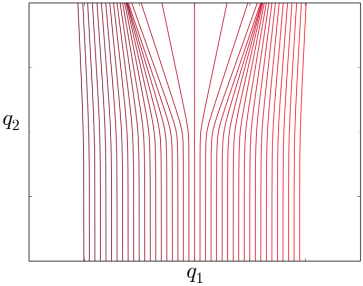


Figure: $\deg(E_{high}) = 0$

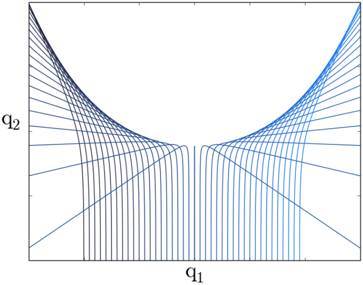
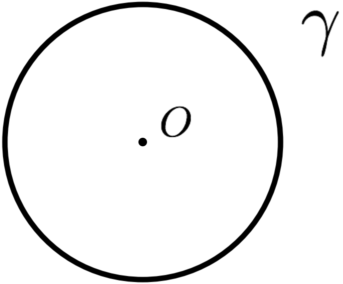


Figure: $\deg(E_{low}) = 1$

A.Knauf *Qualitative Aspects of Classical Potential Scattering,* Regul. Chaotic Dyn. 1999

Bifurcation diagram of $(H = \|p\|^2 - \|q\|^2, J)$



Scattering monodromy

Theorem (Bates and Cushman)

The variation of the deflection angle along γ is given by 2π. This corresponds to a scattering map

$$f\colon T^2 \to T^2 \quad \text{such that} \quad f_\star = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Larry Bates and Richard Cushman, *Scattering monodromy and the A_1 singularity*, Cent. Eur. J. Math., 2007

K.Efstathiou, A.Giacobbe, P. Mardešić, D.Sugny, *Rotation Forms and Local Hamiltonian Monodromy*, 2017

N. Martynchuk, H.R. Dullin, K. Efstathiou, and H. Waalkens, *Scattering invariants in Euler's two-center problem*, Nonlinearity 2019.

Scattering of pairs of integrable systems

Compactification

- Consider a pair of integral maps F and F_0 such that the corresponding scattering map S preserves the fibers of F.
- Then we can use S to (partially) compactify the fibration $F\colon M \to \mathbb{R}^n$, even including certain singularities of F (by extending S by continuity).
- Let M^c and F^c denote the compacitfication of M and F using S.

Definition

The *scattering Fomenko-Zieschang invariants* of F with respect to S is defined as the Fomenko-Zieschang invariants of $F^c\colon M^c \to \mathbb{R}^n$.

Examples, scattering molecule

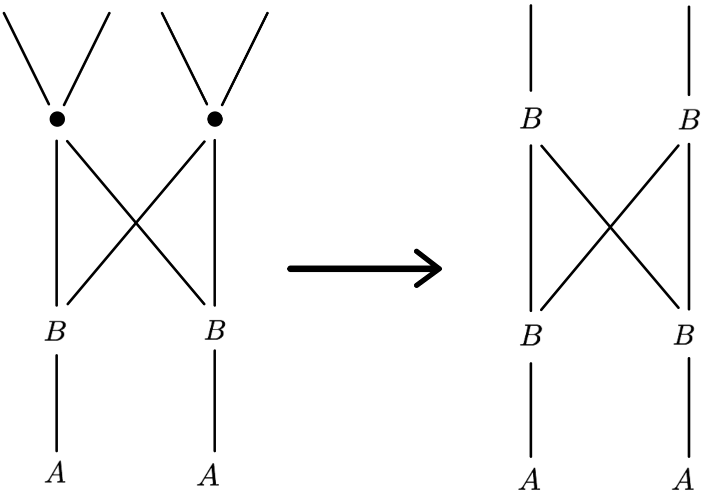
Geodesic flow on a cylinder

Consider a geodesic flow on the cylinder $C^2 = \mathbb{R} \times S^1$ with the Liouville metric

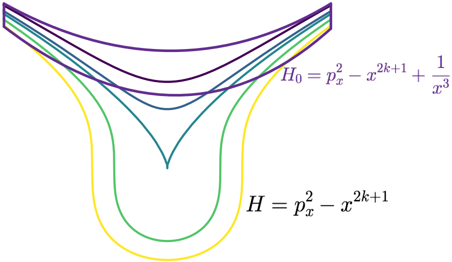
$$ds^2 = (f(\varphi) + g(z))(d\varphi^2 + dz^2),$$

where $f - 2$ is the standard height function on S^1 and $g(z) = z^2 + 1$.

Molecule for a geodesic flow on a cylinder



Example: cuspidal singularity



This is naturally linked to

- A. Bolsinov , L. Guglielmi and E. Kudryavtseva, *Symplectic invariants for parabolic orbits and cusp singularities of integrable systems,* Philos. Trans. Royal Soc. A, 2018
- S. Hohloch and J. Palmer, *Extending compact Hamiltonian S^1-spaces to integrable systems with mild degeneracies in dimension 4,* 2021
- E. Kudryavtseva, N.M., *C^∞ symplectic invariants of parabolic orbits and flaps in integrable Hamiltonian systems,* 2021
- N.M. and S. Vũ Ngọc *On the symplectic geometry of A_k singularities,* arXiv:2305.01814v2, 2023