Integrable Hamiltonian systems

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0. Contents

1. Integrability

There are various notions of integrability. In this chapter, we will define and study *Frobenius* and *Liouville integrability*. The first one is the version of integrability mainly and mostly used in differential geometry. The latter one is tailored for Hamiltonian systems, motivated by energy conservation, and takes additional symmetries of the system into account.

1.1 Hamiltonian systems in $\mathbb{R}^{2n}$

Throughout this section, we will introduce Hamiltonian systems in local coordinates in $\mathbb{R}^{2n}$ in order to provide intuition and motivation without using too many new notions. In the following section, we will remedy this by defining Hamiltonian systems on symplectic manifolds and recovering the local presentation from this section. The definition and basic properties of a flow of an ordinary differential equation are recalled in Definition and Proposition A.17.

Definition 1.1 (**Hamiltonian system in standard coordinates**). *Consider $\mathbb{R}^{2n}$ with the coordinates $z := (q, p) := (q_1, \dots, q_n, p_1, \dots, p_n)$ and let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. The vector field given by*

$$X^H(q,p) := \begin{pmatrix} \partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} := \begin{pmatrix} \partial_{p_1} H(q,p) \\ \vdots \\ \partial_{p_1} H(q,p) \\ -\partial_{q_1} H(q,p) \\ \vdots \\ -\partial_{q_n} H(q,p) \end{pmatrix}$$

is called **Hamiltonian vector field of** *H. The associated ordinary differential equation*

$$z' = X^H(z),$$

reads in (q, p) coordinates

$$\begin{cases} q_i' = \partial_{p_i} H(q,p), \\ p_i' = -\partial_{q_i} H(q,p), \end{cases} \quad \forall\, 1 \le i \le n, \qquad \textit{briefly} \quad \begin{cases} q' = \partial_p H(q,p), \\ p' = -\partial_q H(q,p), \end{cases}$$

and is called **Hamiltonian equation** *or* **Hamiltonian system** *and its solutions* **Hamiltonian solutions**. *The associated flow is called* **Hamiltonian flow** *and usually denoted by Φ^H. In this context, H is usually called* **Hamiltonian function** *or briefly* **Hamiltonian**.

In Section 1.2, we will define Hamiltonian systems in a coordinate free definition via symplectic geometry.

EXAMPLE 1.2. *The Hamiltonian vector field and Hamiltonian system of* $H : \mathbb{R}^2 \to \mathbb{R}$, $H(q,p) := \frac{1}{2}(q^2 + p^2)$ *are given by*

$$X^H(q,p) = \begin{pmatrix} p \\ -q \end{pmatrix} \qquad \textit{and} \qquad \begin{cases} q' = p, \\ p' = -q \end{cases}$$

which is in fact equivalent to $q'' = -q$. *The flow is given by*

$$\Phi_t^H(q,p) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

i.e., the solutions live on concentric circles centered at the origin, parametrised counterclockwise.

Hamiltonian systems have intrinsic geometric properties, which restrict the possible whereabouts of solutions. We will now investigate these properties in detail. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian with usual coordinates $(q,p) \in \mathbb{R}^{2n}$. The derivative $DH = (\partial_q H, \partial_p H)$ is dual to the gradient $\operatorname{grad}_{eu} H$ induced by the euclidean metric via

$$\left\langle \operatorname{grad}_{eu} H, \begin{pmatrix} V_q \\ V_p \end{pmatrix} \right\rangle_{eu} = \begin{pmatrix} \partial_q H, & \partial_p H \end{pmatrix} \begin{pmatrix} V_q \\ V_p \end{pmatrix}$$

which leads to the coordinate expression

$$\operatorname{grad} H|_{(q,p)} := \operatorname{grad}_{eu} H|_{(q,p)} = \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix}.$$

Therefore the Hamiltonian vector field can be considered as **skewgradient** due to the identity

$$X^H(q,p) = \begin{pmatrix} \partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathrm{Id} \\ -\mathrm{Id} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial_q H(q,p) \\ \partial_p H(q,p) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathrm{Id} \\ -\mathrm{Id} & \mathbf{0} \end{pmatrix} \operatorname{grad} H|_{(q,p)}$$

where Id is the $(n \times n)$-identity matrix and $\mathbf{0}$ the $(n \times n)$ zero matrix. In particular, we find

$$\langle \operatorname{grad} H, X^H \rangle = 0$$

i.e., the gradient and the Hamiltonian vector field are perpendicular to each other.

DEFINITION 1.3. *Let* $F : \mathbb{R}^m \to \mathbb{R}^k$ *be smooth with* $m \geq k$. *A point* $z \in \mathbb{R}^m$ *is called* **regular** *if* $DF|_z$ *has rank k (which is maximal). If* $\operatorname{rk}(DF|_z) < k$ *then* $z \in \mathbb{R}^m$ *is called* **singular** *or* **critical** *and* $\operatorname{rk}(DF|_z)$ *is said to be the*

rank of the singular *or* **critical point** z. *We denote the set of critical points by*

$$\mathrm{Crit}(F) := \{z \in \mathbb{R}^m \mid z \text{ critical point of } F\}.$$

The set $F(\mathrm{Crit}(F))$ *is called* **bifurcation diagram** *of* F. *For* $r \in \mathbb{R}$,

$$F^{-1}(r) := \{z \in \mathbb{R}^m \mid F(z) = r\}$$

is called **level set** *or* **fiber of** r. *We call* $r \in \mathbb{R}^k$ *a* **regular value of** F *and its fiber a* **regular fiber** *if* $DF|_z$ *has maximal rank for all* $z \in F^{-1}(r)$. *Otherwise* r *is said to be* **singular** *or* **critical** *and its fiber is referred to as* **singular** *or* **critical**.

Given a smooth function $F : \mathbb{R}^m \to \mathbb{R}^k$ with $m \geq k$, Theorem A.28 (Implicite function) implies that the level set $F^{-1}(r)$ of a regular value r has dimension $m - k$. Moreover, Theorem A.29 (Sard) states that being a regular value is a generic property.

Definition 1.4. *Given a Hamiltonian* H, *a Hamiltonian solution* z *is* **regular** *if its tangent vector* $z' = X^H(z)$ *does not vanish. Otherwise the solution is called* **singular** *or* **stationary**.

Hamiltonian solutions are singular if and only if they are constant. Now we come to a crucial property of Hamiltonian solutions.

Lemma 1.5. *Let* $H : \mathbb{R}^{2n} \to \mathbb{R}$ *be smooth.*

1) $\mathrm{grad}\, H$ *is perpendicular to the level sets of* H. *In particular, if the solutions of the gradient equation* $z' = \mathrm{grad}\, H|_z$ *cross the level sets of* H, *then the intersection is perpendicular.*
2) *A Hamiltonian solution stays within one and the same level set for all times. In particular,* X^H *is tangent to the level sets of* H.

Proof. **1)** Let $r \in \mathbb{R}$ with $\emptyset \neq H^{-1}(r)$ and let $\rho : I \to H^{-1}(r)$ be a smooth curve defined on some interval I. The concatenation $H \circ \rho \equiv r$ is constant so that

$$0 = (H \circ \rho)' = DH|_\rho.\rho' = \langle \mathrm{grad}\, H, \rho' \rangle. \tag{1.6}$$

Since ρ lies in $H^{-1}(r)$, its tangent vector ρ' is tangent to $H^{-1}(r)$. According to equation (1.6), $\mathrm{grad}\, H$ and ρ' are perpendicular. $\mathrm{grad}\, H$ does not vanish if and only if DH does not vanish, i.e., the gradient is nonzero along regular

level sets and stands perpendicular on the level set, i.e., the gradient solutions cross the level set perpendicularly. If $\operatorname{grad} H$ vanishes, the associated solution is constant.
2) Consider a solution $z : I \to \mathbb{R}^{2n}$ of $z' = X^H(z)$ defined on some interval I and differentiate the concatenation $H \circ z$. We obtain

$$(H \circ z)' = DH|_z.z' = DH|_z.X^H(z) = \langle \operatorname{grad} H|_z, X^H(z)\rangle = 0.$$

Therefore the function $H \circ z : I \to \mathbb{R}$ is constant, i.e., $z(t)$ stays in the same level set of H for all $t \in I$. This implies in particular that $z' = X^H(z)$ lies in the tangent space of the level set. □

Given a Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$, Lemma 1.5 means in particular that a Hamiltonian solution does not roam freely through $\mathbb{R}^{2n}$ but is confined to a level set of H, i.e., a subset of dimension $\leq 2n-1$. Hamiltonian solutions are therefore subject to 'geometric' restrictions purely by being a *Hamiltonian* solution. This can be expressed in terms of physics as follows:

Corollary 1.7 (**Energy conservation**). *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth and consider H as its own energy function, i.e., given $r \in \mathbb{R}$, consider $H^{-1}(r)$ as* **set of energy level** *r. Then Hamiltonian solutions are* **energy conserving**, *i.e., they stay within one and the same energy level set.*

Lemma 1.5 implies that the Hamiltonian solutions of $H : \mathbb{R}^2 \to \mathbb{R}$ are up to parametrization and position of singular points completely determined by the level sets: The dimension of the trajectory of a non-constant solution is one, which agrees with the dimension of regular level sets. Therefore, given the graph of a Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$, we can deduce the 'location' of all solutions up to parametrization.

Question 1.8. *Can we get even more control over the whereabouts of Hamiltonian solutions beyond the fact that they are staying within energy level sets? If yes, are there necessary and/or sufficient conditions?*

Since Hamiltonian solutions preserve level sets, the idea is to find other Hamiltonian functions with 'compatible' level sets: given $H : \mathbb{R}^{2n} \to \mathbb{R}$, set $H =: h_1$ and look for a $h_2 : \mathbb{R}^{2n} \to \mathbb{R}$ such that the solutions of h_1 also stay within the level sets of h_2 and vice versa. If h_1 and h_2 satisfy this property then the solutions of both systems live within the intersection of level sets $h_1^{-1}(r_1) \cap h_2^{-1}(r_2)$ for some values $r_1, r_2 \in \mathbb{R}$. If (r_1, r_2) is a regular point of $F := (h_1, h_2) : \mathbb{R}^{2n} \to \mathbb{R}^2$, i.e., DF has maximal rank (which is equivalent with X^{h_1} and X^{h_2} being linearly independent) then the *implicite function theorem* implies

$$\dim(h_1^{-1}(r_1) \cap h_2^{-1}(r_2)) = 2n - 2.$$

Therefore the solutions of h_1 and h_2 may only roam within $(2n - 2)$-dimensional sets, i.e., we just lowered the dimemsion of the set they may roam in by one. Finding more such functions $h_1, \dots, h_k$ leads intuitively to solutions staying in level sets of dimension $\leq 2n - k$. We will see later that we will not gain any more information if we surpass $n = k$.

QUESTION 1.9. *How do we find such functions h_2, h_3, ..., meaning, given h_1, what are natural candidates for h_2, h_3 etc.?*

The idea hereby is to consider energy conservation as a 'symmetry' of the system and to look for 'more symmetries', i.e., quantities that are preserved by the system. Depending on the system, there may be rotational invariance, preserved angles etc.

EXAMPLE 1.10. *The* **coupled angular momenta** *system lives on $\mathbb{S}^2 \times \mathbb{S}^2$ and rotates both spheres at the same speed around their vertical axes. Since the rotation on both spheres is equally fast, the angle between rotating vectors in each of the two spheres compared to another is preserved during the rotation. This makes the angle a candidate for an additional 'symmetry' of the system. We will see in Example 1.50 how this idea is worked out and how the system is defined in detail.*

Finding natural candidates is only the first step, the second one is

QUESTION 1.11. *Once we found a candidate, how do we verify that it really has the desired properties?*

If we require Hamiltonian functions h_1, h_2, ..., h_k to have 'compatible' level sets with regard to each other's flow, an immediate question to ask is how these flows interact with each other in $h_1^{-1}(r_1) \cap \cdots \cap h_k^{-1}(r_k)$ for $r_1, \dots, r_k \in \mathbb{R}$. Intuition suggests that the flows itself should be 'compatible', meaning they should commute. In Proposition 1.28, we will see that two flows commute if and only if the Lie bracket (for a definition see Example 1.26) of their associated vector fields vanishes. But we lost information (for example constant terms) when passing from functions to their Hamiltonian vector fields. Thus it is likely that having commuting Hamiltonian flows is not enough to guarantee 'compatible' level sets. In fact, we have to come up with some 'Lie bracket for functions'. As we will see in Definition 1.45, this role is filled by the so-called Poisson bracket for smooth functions described in Example 1.34.

1.2 Hamiltonian systems on symplectic manifolds

The natural framework for Hamiltonian systems is symplectic geometry. We will see by means of Theorem 1.21 (Darboux) that Hamiltonian systems on symplectic manifolds look locally precisely as described in Section 1.1.

The notions of *(smooth) manifold* and *nondegenerate* and *closed forms* used in the following are recalled in Definition A.1, Definition A.10 and Definition A.14 in particular and Appendix A.1 and Appendix A.3 in general.

Definition 1.12. *A differential form is* **symplectic** *if it is a smooth, nondegenerate, closed* 2*-form. A smooth manifold is said to be* **symplectic** *if it carries a symplectic form.*

Lemma A.12 implies immediately

Corollary 1.13. *Symplectic manifolds are even dimensional.*

The seminal example of a symplectic manifold is

Example 1.14. *Consider* $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ *with coordinates* $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ *and endow it with the standard symplectic form*

$$\omega_{st} := -\sum_{i=1}^{n} dq_i \wedge dp_i = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

This form is represented by the skewsymmetric $(2n \times 2n)$*-matrix*

$$\begin{pmatrix} \mathbf{0} & -\mathrm{Id} \\ \mathrm{Id} & \mathbf{0} \end{pmatrix}$$

where Id *is the* $(n \times n)$*-identity matrix and* $\mathbf{0}$ *the* $(n \times n)$ *zero matrix.*

Other important examples are

Example 1.15. *1)* $\mathbb{C}^n$ *with coordinates* $(z_1, \ldots, z_n)$ *and* $-\frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$ *is a symplectic manifold. The identification* $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ *via* $z_k = q_k + ip_k$ *recovers Example 1.14.*

2) The 2*-sphere* $\mathbb{S}^2$ *with* $\omega_{\mathbb{S}^2}$ *given by*

$$(\omega_{\mathbb{S}^2})_p(u_p, v_p) := \langle p, u_p \times v_p \rangle_{eucl}$$

for all $p \in \mathbb{S}^2$ *and all* $u_p, v_p \in T_p\mathbb{S}^2$ *is a symplectic manifold.* $\omega_{\mathbb{S}^2}$ *is usually considered as standard symplectic form on* $\mathbb{S}^2$*.*

3) Any volume form of a 2*-dimensional manifold is a symplectic form, i.e., all (orientable)* 2*-dimensional manifolds are symplectic.*

4) Any cotangent bundle carries a natural symplectic form.

5) The $2n$*-torus* $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n} \simeq (\mathbb{R}/\mathbb{Z})^{2n}$ *with the symplectic form induced by Example 1.14 is symplectic.*

Proof. Left to the reader. □

Example 1.16. *The complex projective space* $\mathbb{CP}^n$ *is symplectic for all* $n \in \mathbb{N}$*. Its standard symplectic form* ω_{FS} *is called* **Fubini-Study form**. *If we identify* $\mathbb{CP}^1 \simeq \mathbb{S}^2$ *with* $\omega_{\mathbb{S}^2}$ *from Example 1.15, we obtain*

$$\omega_{FS} = -\frac{1}{4}\omega_{\mathbb{S}^2}.$$

Proof. Denote by $\mathbb{C}^*$ the multiplicative group $(\mathbb{C} \setminus \{0\}, \cdot)$ and set

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* \simeq \mathbb{S}^{2n+1}/\mathbb{S}^1$$

with quotient map $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ where $q(z) =: [z]$ is the equivalence class $[z] = [z_0, \ldots, z_n]$ arisen from the identification $(z_0, \ldots, z_n) \sim$

$(\lambda z_0, \ldots, \lambda z_n)$ where $\lambda \in \mathbb{C}^*$. The 2-form

$$\tilde{\omega}_{FS}|_z := \frac{i}{2\,|z|^4} \sum_{j=0}^{n} \sum_{k \neq j} |z_k|^2 \, dz_j \wedge d\bar{z}_j - \bar{z}_k z_j \, dz_k \wedge \bar{z}_j$$

on $\mathbb{C}^{n+1} \setminus \{0\}$ descends to a unique symplectic form ω_{FS} on the quotient $\mathbb{CP}^n$, satisfying $\tilde{\omega}_{FS} = q^* \omega_{FS}$. □

Attention: Whereas even dimensional tori and all complex projective spaces can be endowed with a symplectic form, this is *not* true for $2n$-spheres with $n > 1$:

Proposition 1.17. *Among all spheres $\mathbb{S}^k$ with $k \in \mathbb{N}_0$, only $\mathbb{S}^2$ is symplectic, i.e., higher dimensional spheres of even dimension are not symplectic.*

Proof. Due to Lemma A.12, odd dimensional spheres cannot be symplectic. Now consider the even dimensional spheres. Let $n > 1$ and assume that $\mathbb{S}^{2n}$ admits a symplectic form ω. Since ω is closed we can see it as cohomology class $\omega \in H^2(\mathbb{S}^{2n})$. Because of $H^2(\mathbb{S}^{2n}) = 0$, ω is in fact exact, i.e., there is a 1-form α with $\omega = d\alpha$. We find $0 \neq \omega^n := \omega \wedge \cdots \wedge \omega \in H^{2n}(\mathbb{S}^{2n})$, i.e., ω^n is a volume form on $\mathbb{S}^{2n}$. Moreover, using $d\omega = 0$, we compute $d(\alpha \wedge (\omega^{n-1})) = d\alpha \wedge (\omega^{n-1}) = \omega^n$. Using Stokes' theorem, we find

$$0 \neq \mathrm{vol}\left(\mathbb{S}^{2n}\right) = \int_{S^{2n}} \omega^n = \int_{\mathbb{S}^{2n}} d(\alpha \wedge (\omega^{n-1})) \overset{Stokes}{=} \int_{\partial\mathbb{S}^{2n}} \alpha \wedge (\omega^{n-1}) = 0$$

since the boundary $\partial\mathbb{S}^{2n}$ of $\mathbb{S}^{2n}$ is the empty set. This contradiction shows that $2n$-spheres with $n > 1$ cannot admit a symplectic form. □

This means in particular

Corollary 1.18.

$$\textit{even dimension} \quad \begin{array}{c} \not\Rightarrow \\ \Leftarrow \end{array} \quad \textit{symplectic}$$

Certain types of submanifolds are of particular interest in symplectic geometry:

Definition 1.19. *Let (M, ω) be a symplectic manifold and $N \subseteq M$ a submanifold.*

1) *N is* **isotropic** *if $\omega_p(u_p, v_p) = 0$ for all $p \in N$ and all $u_p, v_p \in T_pN$, i.e., ω vanishes along N.*
2) *N is* **Lagrangian** *if N is isotropic and $\dim N = \frac{1}{2} \dim M$.*

For the definition of a pullback of a 2-form in the following see Definition and Proposition A.16.

Definition 1.20. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and $\psi : M_1 \to M_2$ a smooth map. ψ is said to be* **symplectic** *if $\psi^*\omega_2 = \omega_1$. If ψ is a symplectic diffeomorphism then we call ψ a* **symplectomorphism**. *Symplectic manifolds between which exists a symplectomorphism are called* **symplectomorphic**.

If $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is symplectic then $\psi^*\omega_2 = \omega_1$ forces $D\psi$ to be injective since the symplectic forms are nondegenerate. In particular, we have $\dim M_1 \leq \dim M_2$. If $\dim M_1 = \dim M_2$ then a symplectic ψ is a local diffeomorphism and thus a local symplectomorphism. In general, symplectomorphisms are for symplectic geometry what isomorphisms are for linear algebra and diffeomorphisms for differential geometry.

We will see now that *locally* all symplectic manifolds look the same:

Theorem 1.21 (**Darboux**). *Let (M, ω) be a symplectic manifold of dimension $2n$. Endow $\mathbb{R}^{2n}$ with coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. Then, for all $x \in M$, there exists an open set U with $x \in U$ and a chart $\psi : U \to \mathbb{R}^{2n}$ such that $\psi(x) = 0$ and*

$$(\psi^{-1})^*\omega = -\sum_{i=1}^{n} dq_i \wedge dp_i$$

i.e., $(U, \omega|_U)$ and $(\mathbb{R}^{2n}, -\sum_{i=1}^{n} dq_i \wedge dp_i)$ are symplectomorphic.

Proof. See for example [**Hofer** & **Zehnder**]. □

In Section 1.1, we defined Hamiltonian systems on $\mathbb{R}^{2n}$ using coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. Now we give a coordinate free version:

Definition 1.22. *Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a smooth function. The equation*

$$\omega(X^H, \cdot) = -dH(\cdot)$$

defines a vector field X^H, called the **Hamiltonian vector field of** *H. The associated ODE*

$$z' = X^H(z)$$

is called **Hamiltonian equation**. *Its flow is referred to as* **Hamiltonian flow** *and usually denoted by Φ^H. In this context, the function H is usually referred to as* **Hamiltonian function**.

Note that the equation $\omega(X^H, \cdot) = -dH(\cdot)$ determines X^H uniquely, namely as the vector field dual to the 1-form dH under the isomorphism between 1-forms and vector fields induced by ω.

Lemma 1.23. *The notions in Section 1.1 for Hamiltonian systems on $\mathbb{R}^{2n}$ are generalized by Definition 1.22 to Hamiltonian systems on symplectic manifolds.*

Proof. Theorem 1.21 (Darboux) says that every symplectic manifold looks (locally) like Example 1.14. Thus it is sufficient to calculate the local expression of X^H on $(\mathbb{R}^{2n}, -\sum_{i=1}^n dq_i \wedge dp_i)$ using the formulation from Definition 1.22. To this aim, write X^H with components (X_q^H, X_p^H) w.r.t. the coordinate splitting (q, p). Then the equation $\omega(X^H, \cdot) = -dH(\cdot)$ transforms into

$$\begin{pmatrix} X_q^H, & X_p^H \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\mathrm{Id} \\ \mathrm{Id} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u_q \\ u_p \end{pmatrix} = -\begin{pmatrix} \partial_q H, & \partial_p H \end{pmatrix} \begin{pmatrix} u_q \\ u_p \end{pmatrix}$$

for all tangent vectors $\begin{pmatrix} u_q \\ u_p \end{pmatrix}$ in the point (q, p). This implies the identity

$$X^H = \begin{pmatrix} X_q^H \\ X_p^H \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix}$$

that we used in Definition 1.1 to define the Hamiltonian vector field. $\square$

1.3 Frobenius integrability

We now want to analyse what kind of 'integrability' commuting flows give rise to. Remember that each vector field X gives, via the ordinary differential equation $x' = X(x)$, rise to a flow Φ_t^X and, vice versa, each flow Φ_t gives rise to a vector field via $X^\Phi := \frac{d}{dt}\Phi_t$. Let us therefore study the behaviour of the span of vector fields.

DEFINITION 1.24. *Let M be a smooth m-dimensional manifold and let $n \in \mathbb{N}$ with $1 \leq n \leq m$.*

1) *An n-**dimensional distribution** $\mathscr{D}$ on M is given by $\mathscr{D} = \bigcup_{x \in M} \mathscr{D}_x$ where $\mathscr{D}_x \subseteq T_xM$ is an n-dimensional subspace of the tangent space T_xM for all $x \in M$.*
2) *A distribution $\mathscr{D}$ is **smooth** if, for each $x \in M$, there exists a neighbourhood U of x and smooth, linearly independent vector fields $X_1, \dots, X_n$ on U such that $\mathscr{D}_y = \operatorname{Span}\{X_1|_y, \dots, X_n|_y\}$ for all $y \in U$.*

We now consider a map that produces in a very special way a new vector field from two given ones.

DEFINITION 1.25. *A **Lie bracket** on an $\mathbb{R}$-vector space V is a mapping*

$$[\cdot,\cdot] : V \times V \to V, \quad (u, v) \mapsto [u, v]$$

that satisfies

(i) $[\lambda u, v] = \lambda[u, v]$ *and* $[u + \tilde{u}, v] = [u, v] + [\tilde{u}, v]$
for all $u, \tilde{u}, v \in V$ and for all $\lambda \in \mathbb{R}$.
(ii) $[u, v] = -[v, u]$
for all $u, v \in V$ (**Skewsymmetry** *or* **anti-commutativity**).
(iii) $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$
for all $u, v, w \in V$ (**Jacobi identity**).

*(i) and (ii) together imply that the Lie bracket is **bilinear**. A vector space equipped with a Lie bracket is said to be a **Lie algebra**.*

In our context, we will work mainly with

EXAMPLE 1.26 (**Lie bracket for vector fields**). *Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{R}^m$ and $A, B : \mathbb{R}^m \to \mathbb{R}^m$ differentiable vector fields given in coordinates by $A = \sum_{k=1}^m A^k e_k$ and $B = \sum_{\ell=1}^m B^\ell e_\ell$ where*

$A^k, B^\ell : \mathbb{R}^m \to \mathbb{R}$ are differentiable coefficient functions. Then

$$[A, B] := \sum_{k=1}^{m} \Big(A(B^k) - B(A^k)\Big) e_k = \sum_{k,\ell=1}^{m} \Big(A^\ell D_\ell B^k - B^\ell D_\ell A^k\Big) e_k$$

is a vector field $[A, B] : \mathbb{R}^m \to \mathbb{R}^m$, called **Lie bracket of the vector fields** A **and** B.

Let us briefly recall some properties of this Lie bracket: If $f : \mathbb{R}^m \to \mathbb{R}$ is smooth and $A : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth vector field, then $A(f) := Df.A$ is the derivative of f w.r.t. the vector field A. The Lie bracket of two vector fields A and B satisfies

$$[A, B](f) = A(B(f)) - B(A(f),$$

briefly $[A, B] = AB - BA$, and is therefore also called **commutator** of A and B. On can see the commutator also as differentiating one vector field along another. This motivates the notation $[A, B] = \mathscr{L}_A B$ where $\mathscr{L}_A$ is the so-called **Lie derivative**. Moreover, one can show that

$$[A, B]|_z = \tfrac{d}{dt}\Big|_{t=0} D\Phi^A_{-t}|_{\Phi^A_t(z)} B|_{\Phi^A_t(z)} = \tfrac{d}{dt}\Big|_{t=0} ((\Phi^A_t)^* B)|_z,$$

i.e., there is a way to express the Lie bracket by means of the flows of A and B, for details see for example [**Hohloch2**].

Note that the convention for the definition of the Lie bracket varies throughout the literature by the choice of a sign, i.e., some authors define it as $AB - BA$, others as $BA - AB$.

Definition 1.27. *Let M be a smooth manifold.*

1) *A vector field X on M* **lies in the distribution** $\mathscr{D}$, *briefly $X \in \mathscr{D}$, if $X|_x \in \mathscr{D}_x$ for all $x \in M$.*
2) *A smooth distribution is* **involutive** *or* **completely integrable** *if $[X, Y] \in \mathcal{D}$ for all vector fields $X, Y \in \mathscr{D}$.*

Before we give an example, let us recall

Proposition 1.28. *Let $A, B : \mathbb{R}^m \to \mathbb{R}^m$ be differentiable vector fields with associated flows Φ^A and Φ^B. Then*

$$\Phi^A \circ \Phi^B = \Phi^B \circ \Phi^A \quad \Leftrightarrow \quad AB = BA \quad \Leftrightarrow \quad [A, B] = 0$$

See for intance [**Hohloch2**] for a proof. Therefore $\Phi^A \circ \Phi^B = \Phi^B \circ \Phi^A$ can be seen as 'differentiated version' of $AB = BA$. The seminal example for involutive distributions is

Example 1.29. *Let $A, B : \mathbb{R}^m \to \mathbb{R}^m$ be differentiable, nonvanishing vector fields with commuting flows. Then the distribution given by*

$$\mathscr{D}_x := \mathrm{Span}\{A_x, B_x\} \subseteq T_x\mathbb{R}^m \quad \forall\, x \in \mathbb{R}^m$$

is involutive.

Proof. $[A_x, B_x] = 0_x \in \mathscr{D}_x$ for all $x \in \mathbb{R}^m$ by Proposition 1.28. □

This example motivates

Definition 1.30. *Let $\mathscr{D}$ be a distribution on a manifold M. A submanifold $N \subseteq M$ is an* **integral manifold** *of the distribution $\mathscr{D}$ if $T_xN = \mathscr{D}_x$ for all $x \in N$. A* **maximal integral manifold** *of a distribution $\mathscr{D}$ on M is a connected integral manifold of $\mathscr{D}$ that is not a proper subset of any other connected integral manifold of $\mathscr{D}$.*

The relation between integral manifolds and involutive distributions is characterized by

Theorem 1.31 (**Frobenius**). *Let M be a smooth manifold with a smooth distribution $\mathscr{D}$. Then there are equivalent:*

(i) $\mathscr{D}$ is involutive, i.e., $[X, Y] \in \mathcal{D}$ for all vector fields $X, Y \in \mathscr{D}$.
(ii) Through each point of M, there passes a unique maximal integral manifold of $\mathscr{D}$.

Proof. See [**Warner**]. □

This version of Frobenius' theorem illuminates the relation between vector fields and integrability of distributions. There is also a 'dual' version based on differential forms, see [**Warner**].

Remark 1.32. *Integrability in the sense of Theorem 1.31 (Frobenius) is usually referred to as* **Frobenius integrability**. *It is more general than so-called* **Liouville integrability** *discussed in Remark 1.48.*

Frobenius integrability is a form of integrability tailored for vector fields and their flows. Since Hamiltonian vector fields carry additional information, namely being induced by a function, Frobenius integrability is not 'fine' enough to 'keep track' of this additional information: Frobenius integrability does not 'see' the level sets of the underlying Hamiltonian functions as shown later in Example 1.36.

1.4 Liouville integrability

In this section, we are looking for a notion of integrability particularly suited for Hamiltonian systems. We begin with some kind of 'Lie bracket for functions':

Definition 1.33. *A* **Poisson algebra** *is a triple* $(\mathscr{P}, \diamond, \{\cdot,\cdot\})$ *such that*

1) $(\mathscr{P}, \diamond) := (\mathscr{P}, \diamond, +)$ *is an associative algebra over a field* $\mathbb{K}$ *w.r.t. the (bilinear) multiplication* $\diamond$.

2) There exists a map, referred to as **Poisson bracket***,*

$$\{\cdot,\cdot\} : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$$

satisfying

(i) $\{f+g,h\} = \{f,h\} + \{g,h\}$ *and* $\{cf,h\} = c\,\{f,h\}$ *for all* $f,g,h \in \mathscr{F}$ *and for all* $c \in \mathbb{K}$.

(ii) $\{f,g\} = -\{g,f\}$ *for all* $f,g \in \mathscr{F}$ **(skewsymmetry** *or* **anti-commutativity)**.

(iii) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ *for all* $f,g,h \in \mathscr{F}$ **(Jacobi identity)**.

(iv) $\{fg,h\} = \{f,h\} \diamond g + f \diamond \{g,h\}$ *for all* $f,g,h \in \mathscr{F}$ **(Leibniz rule)**.

(i) and (ii) together imply that the Poisson bracket is **bilinear**.

Thus Poisson brackets are Lie brackets that satisfy in addition the Leibniz rule. The anti-commutativity implies $\{h,h\} = 0$ for all $h \in \mathscr{F}$. We are in particular interested in

EXAMPLE 1.34 (**Poisson bracket for Hamiltonian functions**). *Let (M, ω) be a symplectic manifold of dimension $2n$. Then*

$$\{f, g\} := -\omega(X^f, X^g) : M \to \mathbb{R}$$

with $f, g \in C^\infty(M, \mathbb{R})$ defines a **Poisson bracket on** $C^\infty(M, \mathbb{R})$. *In local coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $(\mathbb{R}^{2n}, -\sum_{i=1}^n dq_i \wedge dp_i)$, this yields for $f, g \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$*

$$\{f, g\} = \partial_q f\, \partial_p g - \partial_p f\, \partial_q g = \left(\sum_{k=1}^n \partial_{q_k} f\, \partial_{p_k} g - \partial_{p_k} f\, \partial_{q_k} g \right) : \mathbb{R}^{2n} \to \mathbb{R}.$$

It relates as follows to the Lie bracket of the Hamiltonian vector fields X^f and X^g:

$$[X^f, X^g] = X^{-\{f,g\}}.$$

In the literatur, the Poisson bracket is sometimes defined with the other sign, i.e., as $\{f, g\} = \omega(X^f, X^g) = \partial_p f \partial_q g - \partial_q f \partial_p g = -(\partial_q f \partial_p g - \partial_p f \partial_q g)$. In this case, we have $[X^f, X^g] = X^{\{f,g\}}$ instead of $[X^f, X^g] = X^{-\{f,g\}}$. The algebraic implication of the choice of sign is explained in the following statement.

LEMMA 1.35. *Let (M, ω) be a symplectic manifold and denote the set of smooth vector fields on M by $Vect(M)$. Then*

1) *$(Vect(M), [\cdot, \cdot])$ is a Lie algebra.*
2) *$(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ is a Poisson algebra. Forgetting the Leibniz rule, $\{\cdot, \cdot\}$ induces the structure of a Lie algebra on $C^\infty(M, \mathbb{R})$.*
3) *The map $h \mapsto X^h$ is in our convention $[X^f, X^g] = X^{-\{f,g\}}$ a Lie algebra anti-homomorphism from $(C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ to $(Vect(M), [\cdot, \cdot])$. In the convention $[X^f, X^g] = X^{\{f,g\}}$, it is a Lie algebra homomorphism. The kernel consists in both cases of the set of constant functions.*

PROOF. Left to the reader. □

The set of pairs of Hamiltonian functions that have Hamiltonian vector fields with vanishing Lie bracket does not coincide with the set of pairs of Hamiltonian functions with vanishing Poisson bracket:

EXAMPLE 1.36. *Consider $\mathbb{R}^{2n}$ with local coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $g, h \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ given by*

$$g(q, p) := q_1 \qquad \text{and} \qquad h(q, p) := p_1$$

Then $[X^g, X^h] = 0$*, but* $\{g, h\} = 1 \neq 0$.

PROOF. We compute $\{g, h\} = 1 + 0 + \cdots + 0 = 1$ which implies immediately $[X^g, X^h] = X^{-\{g,h\}} = 0$ since the Hamiltonian vector field of a constant function always vanishes. □

Combining Example 1.36 and Proposition 1.28, we obtain

COROLLARY 1.37. *Let* f, g *be Hamiltonian functions with Hamiltonian flows* Φ^f*,* Φ^g*. Then*

$$\Phi^f \circ \Phi^g = \Phi^f \circ \Phi^g \quad \Leftrightarrow \quad [X^f, X^g] = 0 \quad \overset{\Leftarrow}{\not\Rightarrow} \quad \{f, g\} = 0.$$

The Poisson bracket is therefore a 'finer comb' than the Lie bracket. Now we will answer the question why the Poisson bracket is the right tool to measure the 'compatibility' of Hamiltonian flows with level sets of other Hamiltonian functions.

DEFINITION 1.38. *Let* (M, ω) *be a symplectic manifold of dimension* $2n$*, let* $h \in C^\infty(M, \mathbb{R})$*, and let* $\gamma :]-\varepsilon, \varepsilon[\to M$ *be a smooth curve. Then the* **evolution** *of* h *along* γ *is given by* $t \mapsto (h \circ \gamma)'(t)$*.*

Consider $\mathbb{R}^{2n}$ with local coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $h \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$. Let $\gamma :]-\varepsilon, \varepsilon[\to \mathbb{R}^{2n}$ be a smooth curve with components $\gamma = (\gamma_q, \gamma_p) = (\gamma_{q_1}, \ldots, \gamma_{q_n}, \gamma_{p_1}, \ldots, \gamma_{p_n})$. In these coordinates, the evolution of h along γ is given by

$$\begin{aligned}(h \circ \gamma)'(t) = Dh|_{\gamma(t)}.\gamma'(t) &= \sum_{i=1}^{n} \partial_{q_i} h|_{\gamma(t)} \gamma'_{q_i}(t) + \partial_{p_i} h|_{\gamma(t)} \gamma'_{p_i}(t) \\ &= \partial_q h|_{\gamma(t)} \gamma'_q(t) + \partial_p h|_{\gamma(t)} \gamma'_p(t).\end{aligned}$$

LEMMA 1.39. *Let* (M, ω) *be symplectic,* $h \in C^\infty(M, \mathbb{R})$*, and* $\gamma :]-\varepsilon, \varepsilon[\to \mathbb{R}^{2n}$ *be smooth. Then the evolution of* h *along* γ *vanishes if and only if* γ *stays within a level set of* h*.*

PROOF.

$$\begin{aligned}\text{The evolution of } h \text{ along } \gamma \text{ vanishes} &\Leftrightarrow (h \circ \gamma)' \equiv 0\\ &\Leftrightarrow (h \circ \gamma) \text{ is constant}\\ &\Leftrightarrow \gamma \text{ stays within a level set of } h\end{aligned}$$

□

For Hamiltonian solutions, Lemma 1.39 implies

COROLLARY 1.40. *Let (M, ω) be symplectic, $f, g \in C^\infty(M, \mathbb{R})$, and let $z^g :]-\varepsilon, \varepsilon[\to M$ a Hamiltonian solution of g. Then*

$$z^g \text{ stays within a level set of } f \quad \Leftrightarrow \quad \{f, g\} = 0.$$

PROOF. According to Lemma 1.39, z^g stays within a level set of f precisely when the evolution of f along z^g vanishes. Thus we compute

$$0 = (f \circ z^g)' = Df.(z^g)' = Df.X^g = -\omega(X^f, X^g) = \{f, g\}.$$

□

Therefore we are interested in pairs of Hamiltonians f, g with $\{f, g\} = 0$ if we want their flows to stay within each others level sets.

DEFINITION 1.41. *Let (M, ω) be symplectic and $f \in C^\infty(M, \mathbb{R})$. A function $g \in C^\infty(M, \mathbb{R})$ satisfying $\{f, g\} = 0$ is said to be an* **integral** *of f. We set*

$$I(f) := \{g \in C^\infty(M, \mathbb{R}) \mid g \text{ integral of } f\}.$$

We note

LEMMA 1.42. *Let f and g be smooth functions. Then*
1) $f \in I(g) \;\Leftrightarrow\; g \in I(f)$.
2) $f \in I(f)$, i.e., the ‘energy’ f is an integral of f.
3) $(I(f), \{\cdot,\cdot\})$ is a Lie algebra. In particular, the Poisson bracket of two integrals is again an integral.

PROOF. **1)** The anti-commutativity of the Poisson bracket implies for all smooth functions f and g that $0 = \{f, g\} = -\{g, f\}$.
2) The anti-commutativity of the Poisson bracket implies $\{f, f\} = 0$ for all smooth functions f.

3) Let $g, h \in I(f)$, i.e., $\{f, g\} = 0 = \{f, h\}$. By adding zero and using the Jacobi identity, we obtain

$$\begin{aligned}\{f, \{g, h\}\} &= \{f, \{g, h\}\} + \{g, 0\} + \{h, 0\}\\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\\ &= 0.\end{aligned}$$

□

Recall that the Hamiltonian vector field X^f of a smooth real valued function f transforms under a symplectomorphism ψ via

$$X^{f\circ\psi}(x) = D\psi^{-1}|_{\psi(x)} X^f(\psi(x)). \tag{1.43}$$

The Poisson bracket behaves under concatenation as follows.

Lemma 1.44. *1) Let ψ be a symplectomorphism and f and g smooth real valued functions. Then $\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$.*
2) Let $h_1, \ldots, h_n$ be smooth, real valued functions with $\{h_i, h_j\} = 0$ for all $1 \le i, j \le n$. Set $h := (h_1, \ldots, h_n)$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then

$$\{f \circ h,\ h_i\} = 0, \quad \forall\ 1 \le i \le n,$$

i.e., $f \circ h \in I(h_i)$ for all $1 \le i \le n$.

Proof. **1)** We calculate

$$\begin{aligned}\{f \circ \psi, g \circ \psi\} &= \omega(X^{f\circ\psi}, X^{g\circ\psi}) = -d(f \circ \psi)(X^{g\circ\psi})\\ &\overset{(1.44)}{=} -Df|_\psi\ D\psi\ (D\psi)^{-1}|_\psi\ X^g|_\psi = -Df|_\psi\ X^g|_\psi = (-Df\ X^g) \circ \psi\\ &= \{f, g\} \circ \psi.\end{aligned}$$

2) We calculate

$$\begin{aligned}\{f \circ h,\ h_i\} &= -d(f \circ h)(X^{h_i}) = Df|_h\ Dh\ X^{h_i}\\ &= -Df|_h \begin{pmatrix} \partial_{q_1} h_1 & \ldots & \partial_{q_n} h_1 & \partial_{p_1} h_1 & \ldots & \partial_{p_n} h_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_{q_1} h_n & \ldots & \partial_{q_n} h_n & \partial_{p_1} h_n & \ldots & \partial_{p_n} h_n \end{pmatrix} \begin{pmatrix} \partial_{p_1} h_i \\ \vdots \\ \partial_{p_n} h_i \\ -\partial_{q_1} h_i \\ \vdots \\ -\partial_{q_n} h_i \end{pmatrix}\\ &= Df|_h \begin{pmatrix} -dh_1(X^{h_i}) \\ \vdots \\ -dh_n(X^{h_i}) \end{pmatrix} = Df|_h \begin{pmatrix} \{h_1, h_i\} \\ \vdots \\ \{h_n, h_i\} \end{pmatrix} = Df|_h \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\\ &= 0.\end{aligned}$$

□

Now we are ready for

DEFINITION 1.45. *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold. A smooth function* $h := (h_1, \dots, h_n) : M \to \mathbb{R}^n$ *is said to be a* **(momentum map of a) completely integrable (Hamiltonian) system** *if*

1) $X^{h_1}, \dots, X^{h_n}$ *are almost everywhere linearly independent.*
2) $\{h_i, h_j\} = 0$ *for all* $1 \le i, j \le n$ **(Poisson commutative)**.

A completely integrable system is often abbreviated by (M, ω, h).

Contrary to standard notions in physics, some mathematicians call the momentum map briefly *moment map*.

The measure theoretic notion 'almost everywhere' does not depend on the choice of $\omega^n := \omega \wedge \cdots \wedge \omega$, the natural n-dimensional volume on (M, ω), or the Lebesgue measure since ω^n coincides with the Lebesgue measure up to scaling by a strictly positive function.

REMARK 1.46. *1) Condition 1) in Definition 1.45 is equivalent to requiring* $\operatorname{rk} Dh = 2n$ *almost everywhere.*

2) Due to condition 2) in Definition 1.45, the Hamiltonian flows $\Phi^{h_1}, \dots, \Phi^{h_n}$ *of a completely integrable system* (M, ω, h) *commute.*

3) If the flows $\Phi^{h_1}, \dots, \Phi^{h_n}$ *are defined on whole* $\mathbb{R}$*, then we get a group action*

$$\begin{aligned} &\mathbb{R}^n \times M \to M, \\ &(t, x) = (t_1, \dots, t_n, x) \mapsto \Phi^{h_1}_{t_1} \circ \cdots \circ \Phi^{h_n}_{t_n}(x) =: \Phi^h_t(x). \end{aligned}$$

Since the flows commute, the definition of Φ^h_t *does not depend on the order of concatenation, i.e., for all permutations* σ*, we have*

$$\Phi^{h_{\sigma(1)}}_{t_{\sigma(1)}} \circ \cdots \circ \Phi^{h_{\sigma(n)}}_{t_{\sigma(n)}}(x) = \Phi^{h_1}_{t_1} \circ \cdots \circ \Phi^{h_n}_{t_n}(x).$$

If one wants to emphasize the 'additional symmetries aspect' of a single Hamiltonian function, Definition 1.45 can be reformulated as follows:

DEFINITION 1.47. *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold. A Hamiltonian* $H : M \to \mathbb{R}$ *is* **completely integrable** *if* H *has* n *integrals* $h_1, \dots, h_n \in I(H)$ *that form an integrable system* $(M, \omega, h = (h_1, \dots, h_n))$.

In this situation, one often uses the 'energy' of H as integral $h_1 := H$.

REMARK 1.48. *Integrability in the sense of Definition 1.45 or Definition 1.47 is usually referred to as* **Liouville integrability**. *Liouville integrability implies Frobenius integrability but not the other way around, see Corollary 1.37.*

Let us have a look at some examples from physics.

EXAMPLE 1.49 (**Uncoupled harmonic oscillator**). *On $\mathbb{R}^4$ with coordinates $(q, p) = (q_1, q_2, p_1, p_2)$ and symplectic form $-\sum_{k=1}^2 dq_i \wedge dp_i$ consider $H : \mathbb{R}^4 \to \mathbb{R}$ given by*

$$H(q, p) := h_1(q, p) + h_2(q, p) := \frac{\nu_1}{2}(q_1^2 + p_1^2) + \frac{\nu_2}{2}(q_2^2 + p_2^2)$$

where $\nu_1, \nu_2 \in \mathbb{R}^{>0}$. Then
1) $h := (h_1, h_2) : \mathbb{R}^4 \to \mathbb{R}^2$ is a completely integrable system.
2) H is completely integrable.

PROOF. Left as an exercise to the reader. □

EXAMPLE 1.50 (**Coupled angular momenta**). *Let $\vec{R} := (R_1, R_2) \in \mathbb{R}^2$ with $0 < R_1 < R_2 < \infty$ and $t \in [0, 1]$. Consider the product of 2-spheres $\mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with Cartesian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ and symplectic form $-R_1\omega_{\mathbb{S}^2} \oplus R_2\omega_{\mathbb{S}^2}$ where $\omega_{\mathbb{S}^2}$ is the standard symplectic form on the 2-sphere, see Example 1.15. Then*

$$h_{\vec{R},t} := (J_{\vec{R}}, H_t) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}^2$$

given by

$$\begin{aligned} J_{\vec{R}}(x_1, y_1, z_1, x_2, y_2, z_2) &:= R_1 z_1 + R_2 z_2, \\ H_t(x_1, y_1, z_1, x_2, y_2, z_2) &:= (1 - t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2) \end{aligned}$$

describes the so-called **coupled angular momenta system**. *It is a completely integrable system which describes the coupled rotation of vectors on the two spheres with angle between the rotating vectors as preserved 'symmetry'. The image $h_{\vec{R},t}(\mathbb{S}^2 \times \mathbb{S}^2) \subset \mathbb{R}^2$ of the momentum map is displayed in Figure 1.1.*

Proof. This is a very special case of the system studied in [**Hohloch** & **Palmer**]. See also the earlier references therein. □

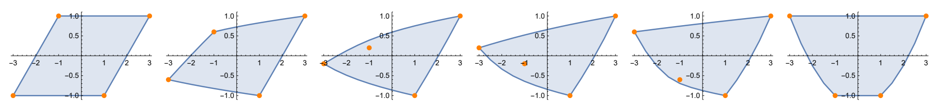


Figure 1.1. Image of the momentum map of the coupled angular momenta system for $\vec{R} = (1,2)$ and t passing from $t = 0$ (at the very left) to $t = 1$ (at the very right). The singular points of rank 0 are marked as red points. When the 'coupling parameter' t changes, one of the rank zero points transitions from being elliptic-elliptic to being focus-focus and then back to elliptic-elliptic.

2. Behaviour of completely integrable systems due to regular and singular points

When we want to describe the flow of a given smooth vector field $X : \mathbb{R}^m \to \mathbb{R}^m$, the theory of ordinary differential equations tell us to distinguish between the behaviour near regular points (= points where the vector field is nonzero) and singular points (= points where the vector field vanishes): given a regular point, then Theorem A.19 (Flow box) states that the flow nearby can be 'straightened' by a smooth change of coordinates to a flow parallel to one of the coordinate axes. For singular points, the situation is more complicated: If the singular point is hyperbolic then, by Theorem A.21 (Hartman-Grobman), the flow is C^0-conjugated to the flow of the linearization at this point. If the singular point is not hyperbolic one either has to require stronger properties (like linear systems conjugated by linear changes of coordinates) or involve higher derivatives to analyse the local dynamics.

Since a completely integrable system $(M, \omega, h = (h_1, \dots, h_n))$ consists of n vector fields $X^{h_1}, \dots, X^{h_n}$ we certainly may apply the above techniques to study each of the n flows $\Phi^{h_1}, \dots, \Phi^{h_n}$ separately. The natural question is if the Poisson commutativity of the flows allows us to combine the results for each of the flows to a result of the integrable system h and its flow Φ^h. The rough answer is *yes* as we will see in this chapter. In fact, we are even able to obtain some 'semilocal' statements, i.e., statements that hold for (connected compontens of) the whole preimage (= fiber) of a regular or singular value.

2.1 The Arnold-Liouville theorem

In this section, we will see how Theorem A.19 (Flow box) extends to completely integrable systems.

DEFINITION 2.1. *Let $(M, \omega, h = (h_1, \dots, h_n))$ be a completely integrable system. The decomposition $M = \bigcup_{r \in h(M)} h^{-1}(r)$ is called* **Liouville foliation** *of (M, ω, h). A connected component of a fiber $h^{-1}(r)$ is said to be a* **leaf**. *A fiber or leaf are called* **regular** *if all of its points are regular. Otherwise the fiber or leaf are called* **singular**.

The fibers and leaves of the Liouville foliation are invariant under the flow of the momentum map. The set of points lying in singular leaves is a zero set in the underlying symplectic manifold.

EXAMPLE 2.2. *The fibers of the Liouville foliation induced by the completely integrable system $h := (h_1, h_2) : \mathbb{R}^4 \to \mathbb{R}^2$ with*

$$h_1(q,p) = \frac{\nu_1}{2}(q_1^2 + p_1^2) \quad \text{and} \quad h_2(q,p) = \frac{\nu_2}{2}(q_2^2 + p_2^2)$$

associated to the **uncoupled harmonic oscillator** *(see Example 1.49) are given by*

$$h^{-1}(r_1, r_2) = \left\{(q_1, p_1) \,\middle|\, q_1^2 + p_1^2 = \frac{r_1 \nu_1}{2}\right\} \times \left\{(q_2, p_2) \,\middle|\, q_2^2 + p_2^2 = \frac{r_2 \nu_2}{2}\right\}.$$

Depending on the value of $r = (r_1, r_2)$, we find

- *If $r_1, r_2 > 0$, then the fibers are 2-tori.*
- *If $r_1 = 0$ and $r_2 > 0$ or $r_1 > 0$ and $r_2 = 0$, then the fibers are 1-tori.*
- *If $r_1 = 0 = r_2$, then the fiber is a single point.*
- *If $r_1 < 0$ or $r_2 < 0$, then the fiber is the empty set.*

PROOF. Left as an exercise to the reader. □

If the fibers of an integrable system (M, ω, h) are not connected, then the *leaf space* (given by collapsing each connected component to a point), differs from the image of the momentum map $h(M)$ which causes many difficulties. Therefore one often imposes conditions which insure the fibers to be connected. For more details on the leaf space, see for example [**Hohloch** & **Sabatini** & **Sepe** & **Symington**] and the references therein.

The Theorem A.28 (Implicite function) implies that a regular fiber or regular leaf is a submanifold whose dimension equals half of the dimension of the underlying symplectic manifold.

LEMMA 2.3. *Let (M, ω, h) be a completely integrable system. Then the fibers and leafs of the Liouville foliation are isotropic and, if regular, Lagrangian submanifolds.*

PROOF. Let $h = (h_1, \dots, h_n)$ and $x \in h^{-1}(r)$. The tangent space $T_x(h^{-1}(r))$ coincides with $\mathrm{Span}_{\mathbb{R}}\{X^{h_1}(x), \dots, X^{h_n}(x)\}$. We compute

$$\omega_x(X^{h_i}(x), X^{h_j}(x)) = -\{h_i, h_j\} = 0 \quad \forall\ 1 \leq i, j \leq n$$

so that the result follows by linearity. □

For this reason, Liouville foliations can be seen as 'singular Lagrangian fibrations'. The following statement refines this to 'Lagrangian 2-torus fibrations' for connected compact regular fibers. Although it is usually called *Arnold-Liouville theorem*, it contains in addition to the works by [**Liouville**] and [**Arnold** & **Avez**] important contributions by [**Mineur 1936**], [**Mineur 1937**], [**Jost**], and [**Markus** & **Meyer**].

THEOREM 2.4 (**Arnold-Liouville**). *Let $(M, \omega, h = (h_1, \dots, h_n))$ be a completely integrable system and $r \in \mathbb{R}^n$ a regular value. If $h^{-1}(r)$ is compact and connected, then*

1) $h^{-1}(r)$ is an embedded n-torus.

2) There exists
- *open sets $D, E \subseteq \mathbb{R}^n$ with $0 \in D$ and $r \in E$,*
- *an open neighbourhood $U := \bigcup_{\rho \in E} h^{-1}(\rho) \subseteq M$ of $h^{-1}(r)$,*
- *a diffeomorphism $\psi : \mathbb{T}^n \times D \to U$,*
- *a diffeomorphism $\mu : E \to D$ with $\mu(r) = 0$*

such that
- *$\psi^*\omega = -\sum_{i=1}^n dq_i \wedge dp_i$, i.e., ψ is a symplectomorphism to the standard model,*

- *$\mu \circ h \circ \psi : \mathbb{T}^n \times D \to D$ satisfies $(\mu \circ h \circ \psi)(q, p) = p$.*

The requirement $h^{-1}(r)$ compact (and connected) is necessary since there are completely integrable systems with regular *noncompact* fibers, like the mathematical pendulum when defined over $\mathbb{R}^2$ instead of the cylinder. These regular fibers are certainly no tori, so we have to exclude them. Theorem 2.4 (Arnold-Liouville) has important consequences:

Corollary 2.5 (**Action-angle coordinates**). *1) The symplectomorphism ψ in Theorem 2.4 (Arnold-Liouville) transforms (M, ω, h) locally into $\mathcal{H} := (\mathcal{H}_1, \ldots, \mathcal{H}_n) := h \circ \psi : \mathbb{T}^n \times D \to \mathbb{R}^n$ with $\mathcal{H}(p, q) = \mathcal{H}(p)$ depending solely on the* **action variables** *$p = (p_1, \ldots, p_n)$, i.e., being independent of the* **angle variables** *$q = (q_1, \ldots, q_n)$.*

2) The Hamiltonian vector field $X^{\mathcal{H}_i}(q, p)$ has $\partial_{p_i}\mathcal{H}_i(p)$ as ith entry with all other entries equal to 0. Thus $\mathcal{H}_i$ induces the Hamiltonian system

$$q_i' = \partial_{p_i}\mathcal{H}(p), \quad p_i' = 0, \quad q_j' = 0, \quad p_j' = 0, \quad \forall\ j \neq i.$$

The components of a Hamiltonian solution $(q(t_i), p(t_i))$ of $\mathcal{H}_i$ are

$$\begin{aligned}
t_i &\mapsto p_i(t_i) = const,\\
t_i &\mapsto q_i(t_i) = q_i(0) + t_i\partial_{p_i}\mathcal{H}(p_i(t_i)),\\
t_i &\mapsto q_j(t_i) = const, \quad \forall\ j \neq i\\
t_i &\mapsto p_j(t_i) = const, \quad \forall\ j \neq i.
\end{aligned}$$

Hence the Hamiltonian flow of $\mathcal{H}_i$ is given by

$$\Phi_{t_i}^{\mathcal{H}_i}(q, p) = (q + t_i\partial_{p_i}\mathcal{H}_i e_i, p)$$

where e_i is the ith standard basis vector of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Thus the flow is linear and leaves each torus $\mathbb{T}^n \times \{p\} \subset \mathbb{T}^n \times D$ invariant.

3) By abuse of notation, we can write as $X^{\mathcal{H}}(q, p) = \binom{\partial_p\mathcal{H}(p)}{0}$ with

$$q' = \partial_p\mathcal{H}(p) \quad \textit{and} \quad p' = 0$$

and

$$\Phi_{t_1}^{\mathcal{H}_1} \circ \cdots \circ \Phi_{t_n}^{\mathcal{H}_n}(q, p) =: \Phi_t^{\mathcal{H}}(q, p) =: (q + t\partial_p\mathcal{H}(p), p).$$

We note

REMARK 2.6. *The 'slope' $\nu_i(p) := \partial_{p_i}\mathcal{H}_i(p)$ of the flow $\Phi^{\mathcal{H}_i}$ is called* **frequency** *of the flow and depends only on p. We write briefly $\nu(p) = \partial_p\mathcal{H}(p)$ i.e., the frequency varies with the 'foot point' p but not within a given torus $\mathbb{T}^n \times \{p\}$.*

We will first prove Corollary 2.5 under the assumption that Theorem 2.4 (Arnold-Liouville) holds true.

PROOF OF COROLLARY 2.5. Since all h_i are integrals of each other, Lemma 1.44 implies that all $h_i \circ \psi$ are integrals of each other. Therefore their flows preserve the fibers $\mathbb{T}^n \times \{p\}$.
According to Theorem 2.4 (Arnold-Liouville), we have $\mu \circ h \circ \psi(q, p) = p$ so that the concatenation with the component function μ_i of $\mu = (\mu_1, \dots, \mu_n)$ yields the coordinate function $\mu_i \circ h \circ \psi(p, q) = p_i$. From Lemma 1.44, it follows that, for all $1 \le i \le n$, the coordinate function p_i is an integral of $h_j \circ \psi$ for all $1 \le j \le n$. This means that their mutual Poisson brackets vanish. Hence we obtain for all $1 \le i, j \le n$

$$0 = \{h_j \circ \psi,\ p_i\} = \sum_{k=1}^{n} \partial_{q_k}(h_j \circ \psi)\, \partial_{p_k} p_i - \partial_{p_k}(h_j \circ \psi)\, \partial_{q_k} p_i = \partial_{q_i}(h_j \circ \psi),$$

i.e., for all $1 \le j \le n$, the function $\mathcal{H}_j := h_j \circ \psi$ does not depend on $q = (q_1, \dots, q_n)$. The formulas of $X^{\mathcal{H}_j}$ and of $\Phi^{\mathcal{H}_j}$ follow immediately. □

Before we approach the proof of Theorem 2.4 (Arnold-Liouville), let us comment on the (lack of) uniqueness of action-angle coordinates.

REMARK 2.7. *The action-angle coordinates in Corollary 2.5 are not unique. We may change the basis of the torus or apply certain translations and still have action-angle coordinates. More precisely, let $A \in GL(n, \mathbb{Z})$ with $\det A = \pm 1$, $c \in \mathbb{R}$ and $w : D \to \mathbb{R}$ a smooth function. Then*

$$\begin{cases} \tilde{q} := A(q + \partial_p w(p)), \\ \tilde{p} := (A^T)^{-1} p + c \end{cases}$$

are also action-angle coordinates.

Let us get an intuition for the origine of the name *action-angle* coordinates.

REMARK 2.8. *ω restricted to $U = \psi(\mathbb{T}^n \times D)$ is the pullback of the standard symplectic form $\omega_{st} := -\sum_{k=1}^n q_k \wedge dp_k$ on $\mathbb{T}^n \times D$ under ψ^{-1}. Since $\omega_{st} = d\sigma_{st}$ is exact with primitive $\sigma_{st} = \sum_{k=1}^n p_k dq_k$ its pullback*

$$\omega|_U = (\psi^{-1})^*\omega_{st} = (\psi^{-1})^* d\sigma_{st} = d((\psi^{-1})^*\sigma_{st}) =: d\sigma$$

is exact, too. Choose a basis consisting of n loops $(\gamma_1, \dots, \gamma_n)$ of the first homology class $H_1(h^{-1}(r) \cap U, \mathbb{Z})$ where $\mu(r) = p$. Then the variables $(p_1, \dots, p_n) = p = \mu \circ h \circ \psi$ can be seen as the 'action integral'

$$p_i = \int_{\gamma_i} \sigma,$$

hence the name 'action variables' for p. Since $q = (q_1, \dots, q_n)$ describes 'angles' in the torus $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$, they are referred to as 'angle variables'.

We will encounter obstructions to the global existence of action-angle variables in Section 3.5. For a detailed analysis 'how global' action-angle coordinates can be defined, we refer the interested reader to [**Duistermaat**].

2.2 Generating functions and the maximal number of independent integrals

Now we will provide some techniques used for the proof of Theorem 2.4 (Arnold-Liouville). We follow hereby closely [**Hofer** & **Zehnder**, Appendix A.1] who use so-called 'generating functions' to characterize symplectic maps locally.

LEMMA 2.9 (**Generating functions I**). *Let $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ be a symplectomorphism and consider ψ as change of coordinates from $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ to*

$$\psi(\xi, \eta) =: \big(\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)\big) =: (x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

1) *If* $\det(\partial_\xi \mathfrak{a}) \neq 0$ *in* $(\hat{\xi}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *then there exists an open neighbourhood* U *of* $(\mathfrak{a}(\hat{\xi}, \hat{\eta}), \hat{\eta}) =: (\hat{x}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and a smooth function* $W : U \to \mathbb{R}$ *such that*
$$\xi = \partial_\eta W(x, \eta) \quad \text{and} \quad y = \partial_x W(x, \eta) \quad \forall\, (x, \eta) \in U$$
with $\det(\partial_x \partial_\eta W) \neq 0$ *in* U.
2) *For all* $(\hat{x}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and all open neighbourhoods* U *of* $(\hat{x}, \hat{\eta})$ *and all smooth functions* $W : U \to \mathbb{R}$ *with* $\det(\partial_x \partial_\eta W) \neq 0$ *in* U, *the map defined on* U *given by*
$$(x, \eta) \mapsto (\xi, y) := (\partial_\eta W(x, \eta), \partial_x W(x, \eta))$$
is a symplectomorphism.

Such a function W *is usually referred to as* **generating function**.

Before we prove this statement, let us discuss it a bit. Note that a symplectic map consists therefore locally of the $(n + n)$ dimensional 'coupling' $(\partial_\eta W, \partial_x W)$, i.e., the $2n$ components of a symplectic map satisfy a functional equation and therefore are not 'independent' from each other.

EXAMPLE 2.10. *The Euclidean scalar product is a generating function for the identity* $\mathrm{Id} : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$.

PROOF. Consider
$$W(x, \eta) = W(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n) := \langle x, \eta \rangle_{eu} := \sum_{k=1}^{n} x_i \eta_i$$
and calculate $\partial_{x_k} W(x, \eta) = \eta_k$ and $\partial_{\eta_k} W(x, \eta) = x_k$. Given the coordinate definition $(\xi, \eta) = \mathrm{Id}(\xi, \eta) =: (x, y)$, we get
$$\xi = x = \partial_\eta W(x, \eta) \quad \text{and} \quad y = \eta = \partial_x W(x, \eta).$$
□

This implies also

EXAMPLE 2.11. *Function of the form* $W(x, \eta) = \langle x, \eta \rangle_{eu} + w(x, \eta)$, *where* w *and its first and second derivatives* Dw *and* $D^2 w$ *have sufficiently small norm, are generating functions for symplectic mappings that are sufficiently close to the identity. Conversely, all symplectic mappings* *close to the identity can locally be described by means of generating functions of the form* $W(x,\eta) = \langle x, \eta\rangle_{eu} + w(x,\eta)$.

PROOF. Left as an exercise to the reader. □

Note that the rotation $(\xi,\eta) \mapsto (\eta, -\xi) =: (\mathfrak{a}(\xi,\eta), \mathfrak{b}(\xi,\eta))$ is symplectic but satisfies $\det(\partial_\xi \mathfrak{a}) = 0$ and $\det(\partial_\eta \mathfrak{a}) \neq 0$. In such situations, we may use

LEMMA 2.12 (**Generating functions II**). *Let* $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ *be a symplectomorphism and consider* ψ *as change of coordinates from* $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ *to*

$$\psi(\xi,\eta) =: (\mathfrak{a}(\xi,\eta), \mathfrak{b}(\xi,\eta)) =: (x,y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

1) *If* $\det(\partial_\eta \mathfrak{a}) \neq 0$ *in* $(\hat{\xi},\hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ *then there exists an open neighbourhood* U *of* $(\hat{\xi}, \mathfrak{a}(\hat{\xi},\hat{\eta})) =: (\hat{\xi},\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and a smooth function* $V : U \to \mathbb{R}$ *such that*
$$\eta = -\partial_\xi V(\xi,x) \quad \textit{and} \quad y = \partial_x V(\xi,x) \quad \forall\ (\xi,x) \in U$$
with $\det(\partial_x \partial_\xi V) \neq 0$ *in* U.
2) *For all* $(\hat{\xi},\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ *and all open neighbourhoods* U *of* $(\hat{\xi},\hat{x})$ *and all smooth functions* $V : U \to \mathbb{R}$ *with* $\det(\partial_x\partial_\xi V) \neq 0$ *in* U*, the map defined on* U *given by*
$$(\xi,x) \mapsto (\eta,y) := (-\partial_\xi V(\xi,x), \partial_x V(\xi,x))$$
is a symplectomorphism.

Such a function V *is usually referred to as* **generating function**.

Note that, for the identity, we only can apply Lemma 2.9 (Generating functions I) but not Lemma 2.12 (Generating functions II). For the symplectic rotation $(\xi,\eta) \mapsto (\eta,-\xi)$, it is percisely the other way around.

The analytic reason behind Lemma 2.9 (Generating functions I) is that $\det(\partial_\xi \mathfrak{a}) \neq 0$ allows us to use the Theorem A.28 (Implicite function) and solve the equation $\mathfrak{a}(\xi,\eta) = x$ for ξ which can be written as $\xi = \alpha(x,\eta)$ for a suitable function α. Now let us make this precise.

PROOF OF LEMMA 2.9 (GENERATING FUNCTIONS I). Consider a given diffeomorphism $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ as change of coordinates

$$(\xi,\eta) \mapsto \psi(\xi,\eta) =: (\mathfrak{a}(\xi,\eta), \mathfrak{b}(\xi,\eta)) =: (x,y)$$

where $\det(\partial_\xi \mathfrak{a}) \neq 0$. Thus Theorem A.28 (Implicite function) guarantees the existence of a local diffeomorphism φ of the form

$$\varphi(x,\eta) = (\alpha(x,\eta),\eta)$$

solving $x = \mathfrak{a}(\xi,\eta)$ locally for ξ via $\xi = \alpha(x,\eta)$ for a suitable mapping α. Now consider $\mathbb{R}^{4n}$ with coordinates (ξ,η,x,y) and embeddings

$$\begin{aligned} &i,j : \mathbb{R}^{2n} \to \mathbb{R}^{4n}, \\ &i(x,\eta) := \big(\alpha(x,\eta),\eta,x,\beta(x,\eta)\big), \\ &j(\xi,\eta) := \big(\xi,\eta,\mathfrak{a}(\xi,\eta),\mathfrak{b}(\xi,\eta)\big) = \big(\xi,\eta,\psi(\xi,\eta)\big). \end{aligned}$$

satisfying $i = j \circ \varphi$, i.e., $\beta(x,\eta) := \mathfrak{b}(\alpha(x,\eta),\eta)$. The 1-form

$$\sigma := ydx + \xi d\eta := \sum_{k=1}^{n} y_k dx_k + \xi_k d\eta_k$$

has differential

$$d\sigma = dy \wedge dx + d\xi \wedge d\eta = dy \wedge dx - d\eta \wedge d\xi.$$

We calculate

$$d(i^*\sigma) = i^* d\sigma = \varphi^*(j^*(d\sigma)) = \varphi^*(\psi^*\omega_{st} - \omega_{st}).$$

Therefore $d(i^*\sigma) = 0$ if and only if ψ is a symplectomorphism. Thus, for symplectomorphisms ψ, Lemma A.15 (Poincaré) implies the local existence of a 0-form (function) W such that $i^*\sigma = dW$. In coordinates, this reads

$$\begin{aligned} &\sum_{k=1}^{n} \beta_k(x,\eta)dx_k + \alpha_k(x,\eta)d\eta_k \\ &\qquad = i^*\sigma = dW = \sum_{k=1}^{n} \partial_{x_k} W(x,\eta)dx_k + \partial_{\eta_k} W(x,\eta)d\eta_k \end{aligned}$$

and yields

$$\begin{aligned} \xi &= \alpha(x,\eta) = \partial_\eta W(x,\eta), \\ y &= \beta(x,\eta) = \partial_x W(x,\eta). \end{aligned}$$

□

Moreover

PROOF OF LEMMA 2.12 (GENERATING FUNCTIONS II). Consider the 1-form

$$\sigma = \sum_{k=1}^{n} y_k dx_k - \eta_k d\xi_k = ydx - \eta d\xi$$

and proceed as in the proof of Lemma 2.9 (Generating functions I). □

Let us have a look at the following special situation.

Example 2.13. *Let $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ be symplectic and write $\psi(\xi, \eta) = (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) =: (x, y)$. Let* $\det(\partial_\xi \mathfrak{a}) \neq 0$ *and assume that $\mathfrak{a}(\xi, \eta) = \mathfrak{a}(\xi)$ does not depend on η. Then ψ is in fact of the form*

$$\begin{aligned} x &= \mathfrak{a}(\xi), \\ y &= \left((\partial_\xi \mathfrak{a}(\xi))^T\right)^{-1} (\eta + \partial_\xi u(\xi)) \end{aligned}$$

for a smooth function $u : \mathbb{R}^n \to \mathbb{R}$. If $\mathfrak{a} = \mathrm{Id}$ *then $y = \eta + \partial_\xi u(\xi)$.*

Proof. Write ψ by means of a generating function U as

$$\begin{aligned} \mathfrak{a}(\xi) &= x = \partial_y U(\xi, y), \\ \mathfrak{b}(\xi, \eta) &= \eta = \partial_\xi U(\xi, y). \end{aligned}$$

Integrating $\mathfrak{a}(\xi) = \partial_y U(\xi, y)$, we find $U(\xi, \eta) = \langle \mathfrak{a}(\xi), y \rangle_{eu} - u(\xi)$ for a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ and $\eta = \partial_\xi U(\xi, y) = (\partial_\xi \mathfrak{a}(\xi))^T y - \partial_\xi u(\xi)$. Solving for y yields the desired expression. □

Now recall from Lemma 1.44 that symplectomorphisms leave the Poisson bracket invariant.

Lemma 2.14. *Consider $(\mathbb{R}^{2n}, \omega_{st})$ with standard coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and Poisson bracket $\{\cdot, \cdot\}$ induced by ω_{st}.*

1) For the coordinate functions $q_1, \ldots, q_n, p_1, \ldots, p_n$ given by $(q, p) \mapsto q_i$ and $(q, p) \mapsto p_i$, we have

$$\begin{aligned} &\{q_i, q_j\} = 0 = \{p_i, p_j\} \quad \forall\ 1 \leq i, j \leq n, \\ &\{q_i, p_i\} = 1 \quad \forall\ 1 \leq i \leq n, \\ &\{q_i, p_j\} = 0 \quad \forall\ i \neq j,\ 1 \leq i, j \leq n. \end{aligned}$$

2) Let $\psi : (\mathbb{R}^{2n}, \omega_{st}) \to (\mathbb{R}^{2n}, \omega_{st})$ be a symplectomorphism and write $\psi = (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}_1, \ldots, \mathfrak{b}_n)$. Then

$$\begin{aligned} &\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0 = \{\mathfrak{b}_i, \mathfrak{b}_j\} \quad \forall\ 1 \leq i, j \leq n, \\ &\{\mathfrak{a}_i, \mathfrak{b}_i\} = 1 \quad \forall\ 1 \leq i \leq n, \\ &\{\mathfrak{a}_i, \mathfrak{b}_j\} = 0 \quad \forall\ i \neq j,\ 1 \leq i, j \leq n. \end{aligned}$$

Proof. **1)** Simple calculation.
2) Write $\mathfrak{a}_i = q_i \circ \psi$ and $\mathfrak{b}_i = p_i \circ \psi$ and use Lemma 1.44 and **1)**. □

This means that, in standard coordinates, the first n components of a symplectomorphism do Poisson commute with each other. The same holds for the last n components. Now we want to investigate the converse, i.e., if one can always extend n Poisson commuting functions to a symplectomorphism.

Theorem 2.15 (**Liouville**). *Consider $(\mathbb{R}^{2n}, \omega_{st})$ with standard coordinates $(\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$ and let $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}^n$ be smooth with $\mathrm{rk}(D\mathfrak{a}) = n$ and $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ for all $1 \leq i, j \leq n$. Then we can extend $\mathfrak{a}$ locally to a symplectomorphism ψ whose first n components coincide with $\mathfrak{a}$.*

Proof. W.l.o.g. assume that $\det(\partial_\xi \mathfrak{a}) \neq 0$ and solve $x = \mathfrak{a}(\xi, \eta)$ locally for $\xi = \alpha(x, \eta)$ with a suitable function α. We want to show that α is locally of the form $\alpha(x, \eta) = \partial_\eta W(x, \eta)$ for a smooth function W and that this W has the properties of a generating function. If this is the case, then part 2) in Lemma 2.9 (Generating functions I) describes how to extend α locally to a symplectomorphism. Taking the inverse delivers a symplectomorphism whose first n coordinate functions are given by $x = \mathfrak{a}(\xi, \eta)$.
Since the second partial derivatives of a smooth function commute, the matrix $\partial_\eta \mathfrak{a}$ has to be symmetric to allow for the existence of any smooth W with $\alpha(x, \eta) = \partial_\eta W(x, \eta)$. Writing Lemma A.15 (Poincaré) in local coordinates shows that symmetry of $\partial_\eta \mathfrak{a}$ is not only necessary but also sufficient for the existence of such a W.
To investigate symmetry, we calculate

$$\begin{aligned} & 0 = \{\mathfrak{a}_i, \mathfrak{a}_j\} = \partial_\xi \mathfrak{a}_i \, (\partial_\eta \mathfrak{a}_j)^T - \partial_\eta \mathfrak{a}_i \, (\partial_\xi \mathfrak{a}_j)^T \\ \Leftrightarrow \quad & \partial_\xi \mathfrak{a}_i \, (\partial_\eta \mathfrak{a}_j)^T = \partial_\eta \mathfrak{a}_i \, (\partial_\xi \mathfrak{a}_j)^T \\ \Leftrightarrow \quad & (\partial_\eta \mathfrak{a}_j)^T \, ((\partial_\xi \mathfrak{a}_j)^T)^{-1} = (\partial_\xi \mathfrak{a}_i)^{-1} \, \partial_\eta \mathfrak{a}_i \\ \Leftrightarrow \quad & ((\partial_\xi \mathfrak{a}_j)^{-1} \, \partial_\eta \mathfrak{a}_j)^T = (\partial_\xi \mathfrak{a}_i)^{-1} \, \partial_\eta \mathfrak{a}_i \end{aligned}$$

and conclude that the matrix $(\partial_\xi \mathfrak{a})^{-1} \, \partial_\eta \mathfrak{a}$ is symmetric. Moreover, differentiating the equation

$$\xi = \alpha(x, \eta) = \alpha(\mathfrak{a}(\xi, \eta), \eta)$$

yields, on the one hand, $\mathrm{Id} = \partial_\xi \xi = \partial_x \alpha \, \partial_\xi \mathfrak{a}$, implying $\partial_x \alpha = (\partial_\xi \mathfrak{a})^{-1}$. On the other hand, we get $0 = \partial_\eta \xi = \partial_\eta \alpha(x, \eta) = \partial_x \alpha \, \partial_\eta \mathfrak{a} + \partial_\eta \alpha$. Altogether, we get $\partial_\eta \alpha = -\partial_x \alpha \, \partial_\eta \mathfrak{a} = -(\partial_\xi \mathfrak{a})^{-1} \, \partial_\eta \mathfrak{a}$ which we already showed to be symmetric. □

Now we give an answer to the question how many ‘independent’ Poisson commuting functions we may find at most on $\mathbb{R}^{2n}$.

Corollary 2.16. *Let* $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}^n$ *and* $\mathfrak{a}_0 : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}$ *be smooth functions with* $\mathrm{rk}(D\mathfrak{a}) = n$ *and* $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ *for* $0 \leq i, j \leq n$. *Then* $\mathfrak{a}_0$ *can be expressed as a function of* $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$. *In particular, there are maximally n independent, Poisson commuting functions on* $\mathbb{R}^{2n}$.

Proof. Consider $(\mathbb{R}^{2n}, \omega_{st})$ with standard coordinates (ξ, η) and $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : (\mathbb{R}^{2n}, \omega_{st}) \to \mathbb{R}^n$ with $\mathrm{rk}(D\mathfrak{a}) = n$ and Poisson commuting components $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ for $0 \leq i, j \leq n$. Theorem 2.15 (Liouville) implies the existence of a local symplectomorphism ψ whose first n components are given by $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$, i.e., $\psi(\xi, \eta) = (\mathfrak{a}(\xi, \eta), \mathfrak{b}(\xi, \eta)) = (x, y)$. We rewrite this as $(\mathfrak{a} \circ \psi^{-1})(x, y) = \mathfrak{a}(\xi, \eta) = x$ and set $f := \mathfrak{a}_0 \circ \psi^{-1}$. Then we get

$$0 = \{\mathfrak{a}_0, \mathfrak{a}_i\} \circ \psi^{-1} \overset{1.44}{=} \{\mathfrak{a}_0 \circ \psi^{-1}, \mathfrak{a}_i \circ \psi^{-1}\} = \{f, x_i\} = -\partial_{y_i} f$$

for all $1 \leq i \leq n$. Therefore f does not depend on $(y_1, \ldots, y_n)$ but is a function solely depending on $(x_1, \ldots, x_n) = x = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$. □

Now we strengthen Theorem 1.21 (Darboux).

Lemma 2.17 (**Liouville coordinates**). *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold and* $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : M \to \mathbb{R}^n$ *be smooth with* $\mathrm{rk}(D\mathfrak{a}) = n$ *and* $\{\mathfrak{a}_i, \mathfrak{a}_j\} = 0$ *for all* $1 \leq i, j \leq n$. *Then, for all* $z \in M$, *there exists an open neighbourhood* $U \subseteq M$ *of* z *and an open neighbourhood* $V \subseteq \mathbb{R}^{2n}$ *of* $0 \in \mathbb{R}^{2n}$ *and a diffeomorphism* $\psi : V \to U$ *such that*

(i) $\psi(0) = z$,
(ii) $\psi^*\omega = \omega_{st}$,
(iii) $(\mathfrak{a} \circ \psi)(q, p) = p$ *for all* $(q, p) \in V$.

Proof. Start with local coordinates as given by Theorem 1.21 (Darboux) and use Theorem 2.15 (Liouville) and Corollary 2.16 to tweak the change of coordinates into the wished form. □

In Liouville coordinates, the flow turns out to be linear:

Corollary 2.18. *In the setting of Lemma 2.17 (Liouville coordinates), the flow* $\Phi_t^{\mathfrak{a}} := \Phi_{t_1}^{\mathfrak{a}_1} \circ \cdots \circ \Phi_{t_n}^{\mathfrak{a}_n}$ *is given by* $\Phi_t^{\mathfrak{a}}(\psi(q, p)) = \psi(q + t, p)$ *for all* $t = (t_1, \ldots, t_n)$ *as long as* (q, p) *and* $(q + t, p)$ *lie in* V.

Proof. Let $t = (t_1, \ldots, t_n)$. The flow of $X^{\mathfrak{a}_i \circ \psi}$ is given by $\psi^{-1} \circ \Phi^{\mathfrak{a}_i}_{t_i} \circ \psi$ for all $1 \leq i \leq n$. Since $(\mathfrak{a}_i \circ \psi)(q, p) = p_i$, we compute $X^{\mathfrak{a}_i \circ \psi}(q, p) = e_i \in \mathbb{R}^{2n}$ where e_i is the ith unit vector. Hence $\Phi^{\mathfrak{a}_i \circ \psi}_{t_i}(q, p) = (q, p) + t_i e_i$ and

$$\Phi^{\mathfrak{a}}_t(q, p) := \Phi^{\mathfrak{a}_1}_{t_1} \circ \cdots \circ \Phi^{\mathfrak{a}_n}_{t_n}(q, p) = (q + t, p)$$

with $t = (t_1, \ldots, t_n)$. □

2.3 The proof of Theorem 2.4 (Arnold-Liouville)

Since Theorem 2.4 (Arnold-Liouville) is a central theorem within the theory of integrable systems, it appears in almost every text book on integrable systems. But since the details of the proof are quite tedious, many authors shorten or even skip them. To our knowledge, the most precise, detailed and nevertheless accessible proof is [**Hofer** & **Zehnder**, Appendix A.1, A.2] which we will follow closely in this section. Other versions can be found for example in [**Arnold 1974**], [**Bolsinov** & **Fomenko**], [**Cushman** & **Bates**], [**Duistermaat**], [**Fassò**], [**Sepe** & **Vũ Ngọc**] etc.

We spread the proof of Theorem 2.4 (Arnold-Liouville) over several successive statements.

Proposition 2.19. *Let $(M, \omega, h = (h_1, \ldots, h_n))$ be a completely integrable system and $r \in \mathbb{R}^n$ a regular value. If $h^{-1}(r)$ is compact and connected, then $h^{-1}(r)$ is an embedded n-torus.*

Proof. Let $r \in \mathbb{R}^n$ be a regular value of h and $h^{-1}(r)$ compact and connected. Since all component functions of h Poisson commute, we have a welldefined local action of $t \in \mathbb{R}^n$ on M given by

$$(t = (t_1, \ldots, t_n), z) \mapsto \Phi^h_t(z) := \Phi^{h_1}_{t_1} \circ \cdots \circ \Phi^{h_n}_{t_n}(z)$$

that preserves the fibers of h. Since $h^{-1}(r)$ is compact, we note that for all $z \in h^{-1}(r)$, the flow lines $t_i \mapsto \Phi^{h_i}_{t_i}(z)$ are defined for all $1 \leq i \leq n$ and all $t_i \in \mathbb{R}$, i.e., the flow is complete on the fiber and we get a welldefined action

$$\mathbb{R}^n \times h^{-1}(r) \to h^{-1}(r), \quad (t = (t_1, \ldots, t_n), z) \mapsto \Phi^h_t(z) := \Phi^{h_1}_{t_1} \circ \cdots \circ \Phi^{h_n}_{t_n}(z).$$

Since for each $1 \leq i \leq n$ and $t_i \in \mathbb{R}$, the flow $\Phi^{h_i}_{t_i} : h^{-1}(r) \to h^{-1}(r)$ is a local diffeomorphism satisfying $\Phi^{h_i}_{t_i} \circ \Phi^{h_i}_{s_i} = \Phi^{h_i}_{t_i+s_i}$ for all $s_i \in \mathbb{R}$, we have $\Phi^h_t \circ \Phi^h_s = \Phi^h_{t+s}$ and $\Phi^h_t : h^{-1}(r) \to h^{-1}(r)$ is a local diffeomorphism. This implies that for all $z \in h^{-1}(r)$, the map

$$F_z : \mathbb{R}^n \to h^{-1}(r), \qquad F_z(t) := \Phi^h_t(z)$$

is a local diffeomorphism. Thus its image is open and closed at the same time. Since the fiber $h^{-1}(r)$ is connected, we obtain $F_z(\mathbb{R}^n) = h^{-1}(r)$. Since $\mathbb{R}^n$ is open and $h^{-1}(r)$ is compact, F_z is surjective but not injective. Since

$$DF_z|_t = (\partial_{t_1}\Phi^{h_1}_{t_1}(z), \ldots, \partial_{t_n}\Phi^{h_n}_{t_n}(z)) = (X^{h_1}(\Phi^{h_1}_{t_1}(z)), \ldots, X^{h_n}(\Phi^{h_n}_{t_n}(z)))$$

has always full rank, F_z is an immersion. Lack of injectivity implies that the isotropy group

$$\Gamma := \{t \in \mathbb{R}^n \mid \Phi^h_t(z) = z\}$$

of the $\mathbb{R}^n$-action on the fiber is nontrivial. Γ does not depend on the chosen point $z \in h^{-1}(r)$ since $\mathbb{R}^n$ acts transitively on $h^{-1}(r)$. Since DF_z has full rank for all $z \in h^{-1}(r)$ and all $t \in \mathbb{R}^n$, the parameter values $t \in \mathbb{R}^n$ with $\Phi^h_t(z) = z$ are isolated points in $\mathbb{R}^n$. Therefore $\Gamma \subset \mathbb{R}^n$ is a discrete subgroup. Hence there are linearly independent $\gamma_1, \ldots, \gamma_k \in \mathbb{R}^n$ with $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \ldots, \gamma_k\}$. For all $\gamma \in \Gamma$, we have $F_z(t+\gamma) = \Phi^h_{t+\gamma}(z) = \Phi^h_t(z) = F_z(t)$ for all $t \in \mathbb{R}^n$ and thus in particular $\mathbb{R}^n/\Gamma \simeq h^{-1}(r)$. Since $h^{-1}(r)$ is compact, we must have $k = n$. Therefore $h^{-1}(r) \simeq \mathbb{R}^n/\Gamma \simeq \mathbb{T}^n$ is an n-torus. □

We remark

Notation 2.20. *Discrete subgroups of $\mathbb{R}^n$ are often called* **lattices**. *A set of linearly independent vectors spanning a lattice is called a* **basis of the lattice**. *The number of basis vectors is called the* **rank of the lattice**. *If $v_1, \ldots, v_n$ is a basis of a lattice, then*

$$\left\{ \sum_{i=1}^{n} \lambda_1 v_i \,\middle|\, \lambda_i \in [0,1] \text{ for } 1 \leq i \leq n \right\}$$

is called a **fundamental domain** *of the lattice.*

Let (M, ω) be a $2n$-dimensional symplectic manifold and $h = (h_1, \ldots, h_n) : M \to \mathbb{R}^n$ an integrable system with regular value $r \in \mathbb{R}^n$ whose fiber $h^{-1}(r)$ is compact and connected.

W.l.o.g. assume within this section that $r = 0 \in \mathbb{R}^n$**.**

In the rest of this section, we will be busy extending Liouville coordinates $\psi : V \subseteq \mathbb{R}^n \times \mathbb{R}^n \to U \subseteq M$ associated to h to larger and larger neighbourhoods until they fulfill the claim of Theorem 2.4 (Arnold-Liouville). The idea is similar to the proof of Theorem A.19 (Flow box) of ordinary differential equations: Using the coordinates ψ from Lemma 2.17 (Liouville coordinates), we consider

$$\theta(x, y) := \Phi^h_x(\psi(0, y))$$

which, by Corollary 2.18, satisfies $\theta(x, y) = \Phi^h_x(\psi(0, y)) = \psi(x, y)$ on V. The next few lemmas are dedicated to showing that θ is in fact welldefined on domains of the form $\mathbb{R}^n \times D$ where $D \subset \mathbb{R}^n$ is a sufficiently small neighbourhood of the origin and that it is invariant under shifts by certain lattices in the first variable.

Lemma 2.21. *Let $\psi : V \to U$ be Liouville coordinates associated to the completely integrable system $h : (M, \omega) \to \mathbb{R}^n$ and let $z = \psi(0) \in h^{-1}(0)$ where $r = 0$ is a regular value of h. Then there exists a small open neighbourhood $E \subseteq \mathbb{R}^n$ of the origin and a compact $K \subset \mathbb{R}^n$ containing a fundamental domain of the rank n lattice $\Gamma = \{t \in \mathbb{R}^n \mid \Phi^h_t(z) = z\}$ such that*

$$\theta : K \times E \to M, \quad \theta(x, y) := \Phi^h_x(\psi(0, y))$$

is welldefined. Moreover, $\theta(K \times \{y\}) \subseteq h^{-1}(y)$ for all $y \in E$. In particular, $\theta(\cdot, 0)$ is defined on all of $\mathbb{R}^n$ with $\theta(\mathbb{R}^n \times \{0\}) = h^{-1}(0)$.

Proof. For $y = 0$, we find $\theta(x, 0) = \Phi^h_x(\psi(0, 0)) = \Phi^h_x(z)$ which, according to (the proof of) Proposition 2.19, is defined for all $x \in \mathbb{R}^n$ and satisfies $h^{-1}(0) = \Phi^h_{\mathbb{R}^n}(z) = \theta(\mathbb{R}^n \times \{0\})$. Since h is invariant under its flow, we compute

$$h(\theta(x, y)) = h(\Phi^h_x(\psi(0, y))) = h(\psi(0, y)) \overset{2.17}{=} y$$

implying $\theta(K \times \{y\}) \subseteq h^{-1}(y)$ for all y where this expression is defined. According to Proposition 2.19, $x \mapsto \Phi^h_x(z) = \theta(x, 0)$ covers $h^{-1}(0)$ surjectively and descends to the quotient $\mathbb{R}^n/\Gamma \simeq h^{-1}(0)$ as a diffeomorphism. Smooth dependence of the flow on initial conditions and 0 being a regular value for h imply that θ must at least be defined on a domain of the form $K \times E$ where $E \subseteq \mathbb{R}^n$ is a small open neighbourhood of the origin and $K \subset \mathbb{R}^n$ is a compact set containing a fundamental domain of Γ. □

We want to extend the lattice $\Gamma = \Gamma(0)$ associated with the $\mathbb{R}^n$-action on $h^{-1}(0)$ to a lattice $\Gamma(y)$ associated with the $\mathbb{R}^n$-action on $h^{-1}(y)$ where $y \in E$. Let us first compute the 'period shift' of the transition from $\Gamma = \Gamma(0)$ to $\Gamma(y)$.

Lemma 2.22. *Let $\psi : V \to U$ be Liouville coordinates associated to the completely integrable system $h : (M, \omega) \to \mathbb{R}^n$. Let $\Gamma =: \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \ldots, \gamma_n\}$ be the lattice associated with the $\mathbb{R}^n$-action on the regular fiber $h^{-1}(0)$. Then for all $1 \leq i \leq n$, the concatenation $\psi^{-1} \circ \Phi^h_{\gamma_i} \circ \psi$ defined on V is a local symplectic change of coordinates $(x, y) \mapsto (\xi(x, y), \eta(x, y))$ of the form*

$$\begin{cases} \xi(x, y) = x - \partial_y u_i(y), \\ \eta(x, y) = y \end{cases}$$

where $u_i : V \to \mathbb{R}$ is smooth and $\partial_y u_i(0) = 0$. To simplify notation, we abbreviate

$$\partial_y u_i(y) =: v_i(y),$$

getting in particular $v_i(0) = 0$.

Proof. Since ψ^{-1}, $\Phi^h_{\gamma_i}$, ψ each are symplectic, so is their concatenation. By definition, we have $(\Phi^h_{\gamma_i} \circ \psi)(x, y) = \psi(\xi(x, y), \eta(x, y))$ and the invariance of h under its flow implies

$$y = (h \circ \psi)(x, y) = (h \circ \Phi^h_{\gamma_i} \circ \psi)(x, y) = (h \circ \psi)(\xi(x, y), \eta(x, y)) = \eta(x, y).$$

Now we extend the relation $y = \eta(x, y)$ by means of the generating function in Example 2.13 locally to a symplectomorphism, i.e., there are smooth functions u_i such that

$$\xi(x, y) = x - \partial_y u_i(y) =: x - v_i(y).$$

We compute

$$(\xi, \eta)(0, 0) = (\psi^{-1} \circ \Phi^h_{\gamma_i} \circ \psi)(0) = \psi^{-1}(\Phi^h_{\gamma_i}(z)) = \psi^{-1}(z) = (0, 0)$$

implying $v_i(0) = 0$. □

Now we can give an explicit formula for the lattice $\Gamma(y)$ that extends Γ.

Lemma 2.23. *Let $h : (M, \omega) \to \mathbb{R}^n$ be a completely integrable system and $h^{-1}(0)$ a regular, connected, compact fiber. Then the rank n lattice $\Gamma =: \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \ldots, \gamma_n\}$ associated with the $\mathbb{R}^n$-action on $h^{-1}(0)$ extends to rank n lattices*

$$\Gamma(y) = \{\gamma_1 + v_1(y), \ldots, \gamma_n + v_n(y)\}$$

associated with the $\mathbb{R}^n$-action on $h^{-1}(y)$ with v_i defined as in Lemma 2.22 and $y \in \mathbb{R}^n$ sufficiently close to $0 \in \mathbb{R}^n$. The lattice satisfies $\Gamma(0) = \Gamma$ and induces a diffeomorphism $h^{-1}(y) \simeq \mathbb{R}^n/\Gamma(y)$.

PROOF. Let ψ be Liouville coordinates associated with h. Close to $(0,0) \in \mathbb{R}^{2n}$, we compute

$$\Phi^h_{\gamma_i}(\psi(x,y)) = \psi(\xi,\eta) \overset{2.22}{=} \psi(x - v_i(y), y) = \Phi^h_{-v_i(y)}(\psi(x,y))$$

where the smooth functions v_i stem from Lemma 2.22. This identity is equivalent to $\Phi^h_{\gamma_i+v_i(y)}(\psi(x,y)) = \psi(x,y)$ for all (x,y) sufficiently close to $(0,0) \in \mathbb{R}^{2n}$. Therefore we get as isotropy group

$$\{t \in \mathbb{R}^n \mid \Phi^h_t(\psi(x,y)) = \psi(x,y)\} = \mathrm{Span}_{\mathbb{Z}}\{\gamma_i + v_i(y) \mid 1 \le i \le n\} =: \Gamma(y).$$

Because of $v_i(0) = 0$ we recover $\Gamma(0) = \Gamma$. Since the $\gamma_1, \ldots, \gamma_n$ are linearly independent, the vectors $\gamma_i(y) := \gamma_i + v_i(y)$ with $1 \le i \le n$ are also linearly independent for (x,y) sufficiently close to $(0,0)$. Thus $\Gamma(y)$ is a rank n lattice. Arguing as in Proposition 2.19, we obtain $h^{-1}(y) \simeq \mathbb{R}^n/\Gamma(y)$. □

Now we are ready to extend θ.

LEMMA 2.24. *Let $h : (M,\omega) \to \mathbb{R}^n$ be a completely integrable system with Liouville coordinates ψ and let $h^{-1}(0)$ be a regular, connected, compact fiber. Then there exists a neighbourhood $E \subseteq \mathbb{R}^n$ of the origin such that*

$$\theta : (\mathbb{R}^n \times E, \omega_{st}) \to (M,\omega), \quad \theta(x,y) := \Phi^h_x(\psi(0,y))$$

is welldefined and satisfies

(i) $\theta^*\omega = \omega_{st}$,
(ii) $\theta(x + \gamma(y), y) = \theta(x,y)$ for all $\gamma(y) \in \Gamma(y)$.

Moreover, for all $y \in E$, the map θ induces diffeomorphisms

$$\bar\theta : \mathbb{R}^n/\Gamma(y) \to h^{-1}(y), \quad \bar\theta(x + \Gamma(y)) := \theta(x,y).$$

PROOF. Lemma 2.21 assures the existence of some open neighbourhood $E \subseteq \mathbb{R}^n$ of the origin and of some compact $K \subseteq \mathbb{R}^n$ containing a fundamental domain of $\Gamma(0)$ so that θ is welldefined on $K \times E$. By definition of a basis vector $\gamma_i(y) \in \Gamma(y)$, we find

$$\begin{aligned} \theta(x + \gamma_i(y), y) &= \Phi^h_{x+\gamma_i(y)}(\psi(0,y)) = (\Phi^h_x \circ \Phi^h_{\gamma_i(y)} \circ \psi)(0,y) \\ &= (\Phi^h_x \circ \psi)(0,y) = \theta(x,y). \end{aligned} \tag{2.25}$$

The flow lines are defined up to the boundary of $K \times E$ where they will be patched together with flow lines defined on 'shifted' domains $(K + \gamma(y)) \times E$ for suitable $\gamma(y) \in \Gamma(y)$ according to (2.25). Since h is smooth all initial value problems of the associated ordinary differential equations have a unique maximal solution. Thus θ is in fact defined on $\mathbb{R}^n \times E$.

In order to show $\theta^*\omega = \omega_{st}$, we will write θ as concatenation of suitable symplectic maps. Let $(x, y) \in \mathbb{R}^n \times E$ and consider the Liouville coordinates $\psi : V \to U$. Choose $\tilde{x} \in \mathbb{R}^n$ sufficiently close to x such that $(x - \tilde{x}, y) \in V$.

$$f : \mathbb{R}^n \times E \to \mathbb{R}^n \times E, \quad f(x, y) := (x - \tilde{x}, y),$$

satisfies $f^*\omega_{st} = \omega_{st}$. We compute

$$\begin{aligned}\theta(x, y) &= \Phi^h_x(\psi(0, y)) = \Phi^h_{\tilde{x}+(x-\tilde{x})}(\psi(0, y)) = (\Phi^h_{\tilde{x}} \circ \Phi^h_{x-\tilde{x}} \circ \psi)(0, y)\\ &= (\Phi^h_{\tilde{x}} \circ \psi)(x - \tilde{x}, y) = (\Phi^h_{\tilde{x}} \circ \psi \circ f)(x, y)\end{aligned}$$

and get

$$\theta^*\omega = f^* \circ \psi^* \circ (\Phi^h_{\tilde{x}})^*\omega = f^* \circ \psi^*\omega = f^*\omega_{st} = \omega_{st}$$

by definition of $\Phi^h_{\tilde{x}}$ and ψ. Since $\theta : \mathbb{R}^n \times \{y\} \to h^{-1}(y)$ is surjective, so is $\bar{\theta}$. Moreover, the map $\bar{\theta}$ is injective since $\bar{\theta}(u) = \bar{\theta}(v)$ implies $\theta(u, y) = \theta(v, y)$ which in turn implies $(u - v) \in \Gamma(y)$. □

To conclude the proof of Theorem 2.4 (Arnold-Liouville), we now normalize the lattices $\Gamma(y)$ to the standard lattice $\mathbb{Z}^n$.

Lemma 2.26. *Let $h : (M, \omega) \to \mathbb{R}^n$ be a completely integrable system and let $h^{-1}(0)$ be a regular, connected, compact fiber. Then there exist*

- *open neighbourhoods $D, E \subseteq \mathbb{R}^n$ of the origin,*
- *an open neighbourhood $U \subseteq M$,*
- *a symplectomorphism $\bar{\psi} : (\mathbb{R}^n/\mathbb{Z}^n \times D, \omega_{st}) \to U \subseteq (M, \omega)$,*
- *a diffeomorphism $\mu : E \to D$*

such that $\mu \circ h \circ \bar{\psi}(x, y) = y$ and $\bar{\psi}^*\omega = \omega_{st}$, i.e., they satisfy the claim of Theorem 2.4 (Arnold-Liouville).

Proof. The lattice $\Gamma(y) = \{\gamma_1(y), \ldots, \gamma_n(y)\}$ was introduced in Lemma 2.22 by means of generating functions. Let us call them $W_1(y), \ldots, W_n(y)$ and set $V(\xi, y) := \sum_{k=1}^n \xi_k W_k(y)$. We obtain

$$\begin{cases} \eta_j := \partial_{\xi_j} V(\xi, y) = W_j(y), \\ x_j := \partial_{y_j} V(\xi, y) = \displaystyle\sum_{k=1}^n \xi_k \partial_{y_j} W_k(y). \end{cases}$$

By construction, the identity $\eta = W(y)$ satisfies $\partial_y W(y) = (\gamma_1(y), \ldots, \gamma_n(y))$ which has rank n. Thus $\eta = W(y)$ is a local diffeomorphism near $y = 0$ which we denote by

$$\mu : E \to D, \qquad y \mapsto \mu(y) := \eta$$

where D, E are suitable open neighbourhoods of the origin. Therefore $\partial_\xi V = W$ has full rank and $\det(\partial_\xi V) \neq 0$. Now reverse the construction of generating functions to get a symplectomorphism

$$\zeta : \mathbb{R}^n \times D \to \mathbb{R}^n \times E, \qquad (\xi, \eta) \mapsto (x, y)$$

with $\zeta(e_j, \eta) = (\gamma_j(y), y)$ where $e_1, \ldots, e_n$ are the standard unit vectors of $\mathbb{R}^n$. Moreover, we find $\zeta : \mathbb{Z}^n \times \{\eta\} \to \Gamma(y) \times \{y\}$ if $\eta = \mu(y)$. We define a symplectomorphism $\tilde{\psi}$ by recalling θ from Lemma 2.24 and setting

$$\tilde{\psi} := \theta \circ \zeta : (\mathbb{R}^n \times D, \omega_{st}) \to U \subseteq (M, \omega)$$

for a suitable open set U. The map $\tilde{\psi}$ satisfies $\tilde{\psi}(\xi + e_j, \eta) = \tilde{\psi}(\xi, \eta)$ for all $1 \leq j \leq n$. Therefore we may pass to the quotient and define

$$\bar{\psi} : \mathbb{R}^n/\mathbb{Z}^n \times D \to U \subseteq M, \quad \bar{\psi}(\xi, \eta) := \bar{\psi}(\xi + \mathbb{Z}^n, \eta) := \tilde{\psi}(\xi, \eta)$$

and we compute

$$(\mu \circ h \circ \bar{\psi})(\xi, \eta) = (\mu \circ h \circ \theta \circ \zeta)(\xi, \eta) = (\mu \circ h \circ \theta(x, y) = \mu(y) = \eta.$$

□

This finishes the proof of Theorem 2.4 (Arnold-Liouville).

2.4 Local normal form for nondegenerate singular points

By Theorem A.29 (Sard), the set of critical points of a smooth function has generically Lebesgue measure zero. Nevertheless, it is precisely the set of critical points and fibers and their induced dynamical behaviour that distinguishes one dynamical system from another. The reason is that, concerning local behaviour, regular fibers 'all look the same' due to Theorem 2.4 (Arnold-Liouville) whereas singular fibers offer more variety.

REMARK 2.27. *Note that the rank of a point stays invariant under the flow, i.e., we may speak of the* **rank of an orbit** *or* **of a flow line**. *In other words, apart from fixed points, singular points always come in a family whose dimension equals the rank of these points.*

Since the action induced by the flow 'carries singularities around', being generic means being generic within the class of maps inducing a (maybe local) $\mathbb{R}^n$-action. The same goes for notions of degeneracy.

We will see in this section, that, under certain assumptions, the image of the momentum map of a completely integrable system carries a lot of information about the system. Of particular interest is hereby the bifurcation diagram: We will see that not all types of critical values can appear everywhere in the image of the momentum map. Usually vertices and edges are critical values, but there can be also critical values in the interior of the image of the momentum map, see the 'switching' of a vertex into a 'critical point in the interior' and back in Figure 1.1.

Moreover, we will study how singular fibers look like and if all points in a singular fiber are necessarily singular. Furthermore, we will consider a local normal form for a whole neighbourhood of a singular point.

Lemma 2.28. *Fixed points are precisely the rank* 0 *points.*

Proof. In order to have rank 0, the Jacobian of the momentum map has to vanish completely in this point in local Darboux coordinates. This is equivalent with all Hamiltonian vector fields vanishing, i.e., the point is a fixed point of the induced integrable system. □

Let us assume for the rest of this chapter that all fibres are compact and connected.

Nondegeneracy of singular points on $2n$-dimensional symplectic manifolds is defined and discussed for example in [**Bolsinov** & **Fomenko**, Section 1.8.3]. To keep notation at a minimum, we will discuss nondegeneracy of singular points only on four dimensional manifolds. Possible references therefore are [**Bolsinov** & **Fomenko**, Sections 1.8.1 and 1.8.2] and [**Hohloch** & **Palmer**]. Nondegeneracy is often defined via Lie group theory using so-called Cartan subalgebras, but the reformulation in terms of linear algebra is much more useful for explicit calculations of examples. So we opted to present the linear algebra approach.

Definition 2.29. *Let* (M, ω) *be a four dimensional symplectic manifold,* $h = (h_1, h_2) : (M, \omega) \to \mathbb{R}^2$ *a completely integrable system, and* $z \in \mathrm{Fix}(h)$. *Choose a basis of* T_zM *and let* Ω_z *be the matrix of* ω_z *and* $d^2h_1|_z$

and $d^2h_2|_z$ the matrices representing the Hessians of h_1 and h_2 w.r.t. this basis. z is said to be **nondegenerate** *if*

(a) $d^2h_1|_z$ and $d^2h_2|_z$ are linearly independent.

(b) There exist $c_1, c_2 \in \mathbb{R}$ such that the matrix

$$c_1\Omega_z^{-1}d^2h_1|_z + c_2\Omega_z^{-1}d^2h_2|_z$$

has four distinct eigenvalues.

Nondegeneracy for rank 1 singular points needs some preparations. Let (M, ω) be a four dimensional symplectic manifold, $h = (h_1, h_2) : (M, \omega) \to \mathbb{R}^2$ a completely integrable system, and $z \in M$ a singular point of rank one, i.e. $1 = \mathrm{rk}(z) = \mathrm{rk}(Dh|_z) = \mathrm{rk}(Dh_1|_z, Dh_2|_z)$, i.e., $Dh_1|_z$ and $Dh_2|_z$ are linearly dependent. Thus there exist some $c_1, c_2 \in \mathbb{R}$ such that $c_1Dh_1|_z + c_2Dh_2|_z = 0$. In particular, the orbit of z given by $\{\Phi_t^h(z) \mid t \in \mathbb{R}^2\}$ is 1-dimensional. Denote by $L := \mathrm{Span}_{\mathbb{R}}\{X^{h_1}(z), X^{h_2}(z)\} \in T_zM$ the tangent line to the orbit through z. The **symplectic complement** of L is defined as

$$L^\omega := \{u \in T_zM \mid \omega_z(u, v) = 0 \ \forall\ v \in L\}.$$

We remember $0 = \{h_1, h_2\} = \omega(X^{h_1}, X^{h_2})$ which together with $L = \mathrm{Span}_{\mathbb{R}}\{X^{h_1}(z), X^{h_2}(z)\}$ yields $L \subseteq L^\omega$. Since h_1 and h_2 Poisson commute, they are invariant under the flow of h. Thus the operator $c_1d^2h_1|_z + c_2d^2h_2|_z$ descends to the quotient L^ω/L and we define

Definition 2.30. *Let (M, ω) be a four dimensional symplectic manifold and $h = (h_1, h_2) : (M, \omega) \to \mathbb{R}^2$ a completely integrable system. Let $z \in M$ be a singular point of rank one and $c_1, c_2 \in \mathbb{R}$ such that $c_1Dh_1|_z + c_2Dh_2|_z = 0$. The point z is* **nondegenerate** *if $c_1d^2h_1|_z + c_2d^2h_2|_z$ is invertible on L^ω/L.*

An explicit example in dimension four with proofs for the nondegeneracy of fixed points and singular points of rank one can be found in [**Hohloch** & **Palmer**].

The following theorem provides a local normal form for nondegenerate singular points of completely integrable systems. The definition of nondegeneracy of higher rank singular points in dimension greater than four can be found in [**Bolsinov** & **Fomenko**, Section 1.8.3].

Theorem 2.31 (**Local normal form**). *Let (M, ω) be a symplectic manifold of dimension $2n$ and $h = (h_1, \ldots, h_n) : (M, \omega) \to \mathbb{R}^n$ a completely integrable system and $z \in M$ a nondegenerate singular point. Then there exist local symplectic Darboux coordinates $(x, y) := (x_1, \ldots, x_n, y_1, \ldots, y_n)$ in a neighbourhood $U \subset M$ of z such that there exists a function $f := (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ with $\{h_k, f_{ij}\} = 0$ for all $1 \leq k, j \leq n$ whose component functions f_i stem from the following list:*

1) **elliptic component:**
$$f_i(x, y) = \tfrac{1}{2}(x_i^2 + y_i^2),$$
2) **hyperbolic component:**
$$f_i(x, y) = x_i y_i,$$
3) **focus-focus component**, *comes always as a pair (f_i, f_{i+1}):*
$$\begin{cases} f_i(x, y) & = x_i y_{i+1} - x_{i+1} y_i, \\ f_{i+1}(x, y) & = x_i y_i + x_{i+1} y_{i+1}, \end{cases}$$
4) **nonsingular component** *(also called* **regular component***):*
$$f_i(x, y) = y_i.$$

Proof. The proof of this theorem is spread throughout the literature and there is, up to my knowledge, no comprehensive presentation of it. The local normal form was announced by [**Eliasson 1984**], but the proof appears to have some gaps. Altogether, there are at least the following contributions:

- The C^∞ case in two dimensions is described by the *Lemme de Morse isochore* in [**Colin de Verdière** & **Vey**].
- The two dimensional analytic case appears in [**Rüssmann**].
- The analytic case in dimension $2n$ was done by [**Vey**].
- C^∞ for the elliptic case in dimension $2n$ was done by [**Eliasson 1990**].
- Another proof for C^∞ in the elliptic case in dimension $2n$ was provided by [**Dufour** & **Molino**].
- Low dimensional hyperbolic cases have been dealt with in [**Miranda**].
- The focus-focus case in dimension four has been dealt with by [**Vũ Ngọc** & **Wacheux**] and [**Chaperon**].
- The infinitesimal case was proven by [**Miranda** & **Vũ Ngọc**].
- A completely different approach was presented by [**Wang**].
- The equivariant case with an action of a compact group was treated in [**Miranda** & **Zung**].

□

Following [**Bolsinov** & **Fomenko**] or [**Vũ Ngọc 2006**], there is an interpretation of the components of Theorem 2.31 (Local normal form) in terms of eigenvalues. On 4-dimensional manifolds, this boils down to

REMARK 2.32. *Let (M, ω) be a 4-dimensional manifold and $h = (h_1, h_2) : (M, \omega) \to \mathbb{R}^2$ a completely integrable system with nondegenerate fixed point $z \in M$. Pick $c_1, c_2 \in \mathbb{R}$ such that $A_{c_1,c_2} := c_1 Dh_1|_z + c_2 Dh_2|_z$ has four distinct eigenvalues. Then*

(i) An elliptic component corresponds to a pair of imaginary eigenvalues $\pm i\beta$ of A_{c_1,c_2} where $\beta \in \mathbb{R}^{\neq 0}$.
(ii) A hyperbolic component corresponds to a pair of real eigenvalues $\pm\alpha \in \mathbb{R}^{\neq 0}$ of A_{c_1,c_2}.
(iii) A focus-focus component corresponds to a quadruple of complex eigenvalues $\pm\alpha \pm i\beta$ of A_{c_1,c_2} where $\alpha, \beta \in \mathbb{R}^{\neq 0}$.

The type of eigenvalue does not depend on the chosen $c_1, c_2 \in \mathbb{R}$.

COROLLARY 2.33. *On a 4-dimensional manifold, the rank more or less determines the type of a point of a completely integrable system:*

(i) If rank = 2, we have a regular point.
(ii) If rank = 1, we have an elliptic-regular point.
(iii) If rank = 0, we have a fixed point of elliptic-elliptic or focus-focus type.

Let us now get some geometric intuition for some of these types of singular points.

LEMMA 2.34. *Let (M, ω) be 4-dimensional and $h : (M, \omega) \to \mathbb{R}^2$ a completely integrable system with a singular point $z \in M$. If z is elliptic-elliptic then the system looks locally near z like the uncoupled oscillator in Example 1.49 and its fibers near the origin.*

PROOF. Use Example 1.49 and Theorem 2.31 (Local normal form). □

Now let us have a look at focus-focus points.

Lemma 2.35. *A focus-focus point can be modelled as transverse intersection of complex planes.*

Proof. Let us restrict ourselves to the four dimensional situation. By setting $\zeta_1 := x_1 + ix_2$ and $\zeta_2 := y_1 + iy_2$, identify $\mathbb{R}^2 \simeq \mathbb{C}$ and consider the map

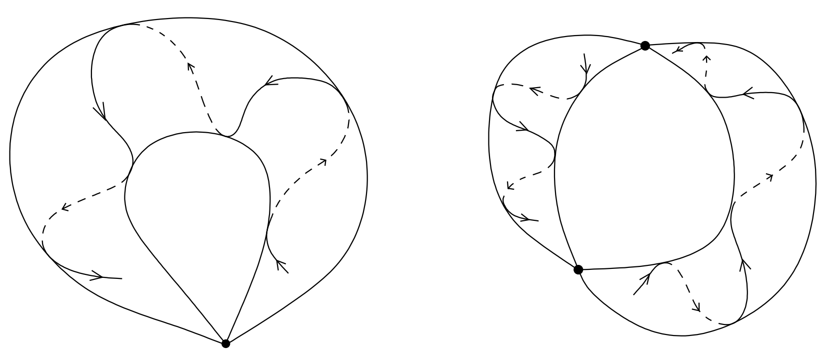
$$g : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad g(\zeta_1, \zeta_2) := \overline{\zeta}_1 \zeta_2 = (x_1 y_1 + x_2 y_2) + i(x_1 y_2 - x_2 y_1).$$

With the notations of Theorem 2.31 (Local normal form), we obtain

$$g(\zeta_1, \zeta_2) = f_2(x, y) + i f_1(x, y).$$

Then $g^{-1}(0) = (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$ is the fiber above the singular value $0 \in \mathbb{R}^2$, consisting of two transversely intersecting complex planes. The nearby fibers $g^{-1}(c)$ with $c \in \mathbb{C}^{\neq 0}$ are regular and can be seen as cylinders or hyperboloids, see Example 3.36. □

Intuitively, a fiber over a focus-focus singular value can be seen as a (maybe multiply) 'pinched torus', i.e., a torus where (at least) one circle has been contracted to a point (which is precisely the focus-focus point). Note that a focus-focus fiber with one focus-focus points consists of *two* orbits, namely the singular focus-focus point and the rest which consists of regular points. A focus-focus point can be seen as isolated fixed point with hyperbolic expansion and contraction behaviour while admitting a local $\mathbb{S}^1$-action, see for instance [**Chaperon**] and [**Vũ Ngọc** & **Wacheux**]. Intuitively, the flow on a focus-focus fiber behaves as in Figure 2.1.



(a) (b)

Figure 2.1. (a) A fiber with one focus-focus fixed point. The flow spirals away from the focus-focus points and is again attracted to it. (b) A fiber with two focus-focus points and flow lines on the regular parts in between.

3. Global properties of integrable systems

So far, we studied the local behaviour of integrable systems. Hereby we encountered the local normal form around a regular point in terms of Theorem 2.4 (Arnold-Liouville) and the local normal form around nondegenerate singular points in terms of Theorem 2.31 (Local normal form).

Now we would like to 'patch all local pictures together' and classify global behaviour of integrable systems. The problem is that, if we do not limit ourselves to systems with certain properties, no classification is possible since more or less anything could happen.

We restrict ourselves to classifications that take symplectic properties into account. There are also classifications that consider purely topological properties of integrable systems, see for example [**Fomenko**] and [**Bolsinov** & **Fomenko**].

We will focus on two types of systems, namely *toric* integrable systems and *semitoric* integrable systems:

Toric systems are (more or less) the 'easiest possible' type of integrable systems and are usually introduced via so-called Hamiltonian torus actions. Toric systems are distinguished by their flow having the property $\Phi_t = \Phi_{t+T}$ for all $t \in \mathbb{R}^n$ and a certain $T = (T_1, \dots, T_n) \in \mathbb{R}^n$, i.e., the flow is periodic, i.e., the flow parameter t lives in fact in the n-torus $\mathbb{R}^n/(T_1\mathbb{Z} \oplus \cdots \oplus T_n\mathbb{Z})$. To have the same period in all coordinates, often the request $T_1 = \cdots = T_n$ is made. Nondegenerate singular points of toric systems will turn out to have only components of elliptic and regular type.

The moment we want to admit focus-focus components, the flow cannot be periodic any more in all coordinates. At least in one coordinate, the action on the focus-focus fibre is nonperiodic, i.e., an $\mathbb{R}$-action, see for example Figure 2.1 when the flow converges to and diverges from the focus-focus point. Roughly, semitoric systems are systems that are almost toric in the sense that they admit in addition to regular and elliptic components also focus-focus components for nondegenerate singular points but, apart from that, share most of the properties of toric systems.

3.1 Group action on manifolds

This section recalls a few notions from the theory of group actions on manifolds.

DEFINITION 3.1. *Let G be a group with neutral element e and M a manifold. G* **acts on** *M if the map $G \times M \to M$, $(g, z) \mapsto g.z$ satisfies*
1) $e.z = z \quad \forall\, z \in M$.
2) $h.(g.z) = (hg).z \quad \forall\, g, h \in G,\ \forall\, z \in M$.

Note that

LEMMA 3.2. *Let M be a manifold and G a group and set*

$$\mathrm{Aut}(M) := \{\varphi : M \to M \mid \varphi \text{ bijective}\}.$$

Then G acting on M can be seen as map

$$\varphi^G : G \to \mathrm{Aut}(M), \qquad g \mapsto \varphi^G_g \quad \text{with} \quad \varphi^G_g(z) := g.z$$

where the mapping $g \mapsto \varphi^G_g$ satisfies

$$\varphi^G_e = \mathrm{Id}_M \quad \text{and} \quad \varphi^G_h \circ \varphi^G_g = \varphi^G_{hg} \quad \forall\, g, h \in G.$$

PROOF. Left to the reader. □

Usually, not all points on a manifold are moved by a group action in the same way:

DEFINITION 3.3. *Let G be a group acting on a manifold M and let $z \in M$.*
1) The **stabilizer** *or* **isotropy group** *of $z \in M$ is defined as*

$$G_z := \{g \in G \mid g.z = z\}.$$

2) $z \in M$ is a **fixed point** *of the action of G on M if $G_z = G$.*

Thus fixed points are indeed points that 'do not move at all' under the action. Note that for $g \in G$ and $z \in M$ and $\tilde{z} := g.z$, the stabilizers satisfy $G_{\tilde{z}} = gG_z g^{-1}$, i.e., stabilizers along an orbit are conjugate to each other.

Definition 3.4. *Let G be a group acting on a manifold M and denote by e the neutral element of G.*

1) *The action of G on M is* **faithful** *or* **effective** *if only $e \in G$ acts trivially, i.e., $\varphi^G_g \neq \mathrm{Id}_M$ for all $g \in G \setminus \{e\}$, i.e., $\{e\} = \bigcap_{z \in M} G_z$.*
2) *The action of G on M is* **transitive** *if, for all $z, \tilde{z} \in M$, there exists $g \in G$ such that $g.z = \tilde{z}$.*
3) *The action of G on M is* **free** *if it has no fixed points, i.e., for all $z \in M$, the only solution of the equation $g.z = z$ is $g = e$.*

Note that a free action on a nonempty set is faithful.

3.2 Hamiltonian torus actions and toric integrable systems

In this section, we are interested in groups G and their actions where $g \mapsto \varphi^G_g$ can be seen as a Hamiltonian flow. The motivation is the following: If G is a group acting on a manifold M via $\varphi^G : G \to \mathrm{Aut}(M)$ then $\varphi^G_e = \mathrm{Id}_M$ and $\varphi^G_h \circ \varphi^G_g = \varphi^G_{hg}$ for all $g, h \in G$. If $G = \mathbb{R}$, this property coincides with the properties of a flow of an autonomous ODE.

Definition 3.5. *An action of an m-torus $\mathbb{T}^m$ on a $2n$-dimensional, symplectic manifold (M, ω) is called a* **Hamiltonian m-torus action** *if there exists a smooth function $h = (h_1, \dots, h_m) : (M, \omega) \to \mathbb{R}^m$ whose flow Φ^h satisfies $\Phi^h_t(z) = \varphi^{\mathbb{T}^n}_t(z)$ for all $t \in \mathbb{T}^n$ and $z \in M$. Such a map h is often called* **momentum map** *of the Hamiltonian torus action.*

In particular, the flow induced by the momentum map of a Hamiltonian torus action is periodic.

More details on (Hamiltonian) torus actions can be found for example in [**Cannas da Silva**], [**Audin 1991**], and [**McDuff** & **Salamon**]. Before we consider examples of Hamiltonian torus actions, let us introduce the following notation.

DEFINITION 3.6. *The* **convex hull** *of m points $x_1, \dots, x_m \in \mathbb{R}^n$ is given by*

$$\mathrm{Conv}(x_1, \dots, x_m) := \left\{ \sum_{k=1}^{n} s_k x_x \,\middle|\, 0 \leq s_1, \dots, s_m \leq 1,\ \sum_{k=1}^{m} s_k = 1 \right\} \subset \mathbb{R}^n.$$

Now we are ready to consider a few examples of Hamiltonian torus actions.

EXAMPLE 3.7. *Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be equipped with its standard symplectic form $\omega_{\mathbb{S}^2}$, see Example 1.15. Then the rotation of $\mathbb{S}^2$ around the z-axis can be seen as $\mathbb{S}^1 = \mathbb{T}^1$-action with momentum map*

$$h : \mathbb{S}^2 \to \mathbb{R}, \qquad h(x, y, z) = z.$$

The north pole $(1, 0, 0) \in \mathbb{S}^2$ and south pole $(0, 0, 1) \in \mathbb{S}^2$ are the only fixed points of the action. The image of the momentum map $h(\mathbb{S}^2) = [h(0, 0, 1), h(1, 0, 0)] \subset \mathbb{R}$ can be seen as convex hull of the images of the fixed points:

$$h(\mathbb{S}^2) = \{s\, h(0, 0, 1) + (1 - s)\, h(1, 0, 0) \mid 0 \leq s \leq 1\} \subset \mathbb{R}.$$

Recall the construction of the complex projective space from Example 1.16.

EXAMPLE 3.8. *The $\mathbb{T}^n$-action on $\mathbb{CP}^n$ given by*

$$(t_1, \dots, t_n).[z_0, \dots, z_n] := [z_o, e^{it_1} z_1, \dots, e^{it_n} z_n]$$

is Hamiltonian with momentum map $h = (h_1, \dots, h_n) : \mathbb{CP}^n \to \mathbb{R}^n$,

$$h([z_0, \dots, z_n]) := -\frac{1}{2} \left(\frac{|z_1|^2}{\sum_{j=0}^{n} \left|z_j\right|^2}, \dots, \frac{|z_n|^2}{\sum_{j=0}^{n} \left|z_j\right|^2} \right).$$

There are precisely $(n+1)$ fixed points, namely $[1, 0, \dots, 0]$, $\dots$, $[0, \dots, 0, 1]$, *i.e., points with stabilizer* $\mathbb{T}^n$*. The image* $h(\mathbb{CP}^n)$ *is an* n*-simplex whose vertices are the images of the fixed points*

$$\begin{aligned}
h([1, 0, \dots, 0]) &= (0, \dots, 0),\\
h([0, 1, 0 \dots, 0]) &= \left(-\frac{1}{2}, 0, \dots, 0\right),\\
h([0, 0, 1, 0 \dots, 0]) &= \left(0, -\frac{1}{2}, 0, \dots, 0\right),\\
\vdots \qquad & \qquad \vdots\\
h([0 \dots, 0, 1]) &= \left(0, \dots, 0, -\frac{1}{2}\right).
\end{aligned}$$

This simplex is in fact the convex hull of the images of the fixed points:

$$h(\mathbb{CP}^n) = \left\{ \left(-\frac{1}{2}s_1, \dots, -\frac{1}{2}s_n\right) \middle| \, 0 \le s_1, \dots, s_n \le 1, \ \sum_{k=1}^{n} s_k = 1 \right\}.$$

The strata of dimension k *of this simplex are precisely the images of points with stabilizer isomorphic to* $\mathbb{T}^{n-k}$*.*

Let us have a look at yet another torus action:

Example 3.9. *The* 2*-torus action on* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *given by*

$$\begin{aligned}
&\mathbb{T}^2 \times (\mathbb{CP}^1 \times \mathbb{CP}^1) \to \mathbb{CP}^1 \times \mathbb{CP}^1,\\
&(s, t).([u_0, u_1], [v_0, v_1]) := ([u_0, e^{is}u_1], [v_0, e^{it}v_1])
\end{aligned}$$

is Hamiltonian with momentum map $h : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{R}^2$*,*

$$h([u_0, u_1], [v_0, v_1]) := -\frac{1}{2}\left(\frac{|u_1|^2}{|u_0|^2 + |u_1|^2}, \ \frac{|v_1|^2}{|v_0|^2 + |v_1|^2}\right).$$

This action has precisely four fixed points, namely

$$([0, 1], [0, 1]), \ ([1, 0], [0, 1]), \ ([0, 1], [1, 0]), \ ([1, 0], [1, 0])$$

whose images under h *are*

$$\left(-\frac{1}{2}, -\frac{1}{2}\right), \ \left(0, -\frac{1}{2}\right), \ \left(-\frac{1}{2}, 0\right), \ (0, 0).$$

We find $h(\mathbb{CP}^1 \times \mathbb{CP}^1) = \left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, 0\right]$ *which can be seen as the convex hull of the images of the fixed points.*

In all the examples above, the image of the momentum map coincides with the convex hull of the images of the fixed points. We will now see if this is just a fluke or if there is more to it.

THEOREM 3.10 (**Convexity theorem of** [**Atiyah**] **and** [**Guillemin** & **Sternberg**]). *Let (M, ω) be a compact, connected, symplectic manifold of dimension $2n$ and let $h : M \to \mathbb{R}^m$ be the momentum map of a Hamiltonian $\mathbb{T}^m$-action on M. Then*

1) All level sets of h are connected.
2) $h(M)$ is convex.
3) $h(M)$ conincides in fact with the convex hull of the images under h of the fixed points of the $\mathbb{T}^m$-action. Therefore $h(M)$ is also called the **momentum polytope** *of the $\mathbb{T}^m$-action.*

PROOF. See [**McDuff** & **Salamon**]. Parts of the proof are also sketched in [**Cannas da Silva**]. □

On compact, symplectic manifolds of dimension four, [**Karshon**] classified all possible effective Hamiltonian $\mathbb{S}^1$-actions. In a series of papers, [**Karshon** & **Tolman 2001**], [**Karshon** & **Tolman 2003**], [**Karshon** & **Tolman 2014**] characterized effective Hamiltonian $\mathbb{T}^{n-1}$-actions on $2n$-dimensional compact symplectic manifolds.

Effectiveness of a Hamitonian m-torus action has the following geometric consequences.

LEMMA 3.11. *Let (M, ω) be a symplectic manifold carrying an effective Hamiltonian $\mathbb{T}^m$-action with momentum map $h = (h_1, \dots, h_m) : M \to \mathbb{R}^m$. Then there exists $z \in M$ with* $\mathrm{rk}(Dh|_z) = m$.

PROOF. ...still to be written... □

This is crucial for

PROPOSITION 3.12. *Let (M, ω) be a compact, connected, symplectic manifold of dimension $2n$ carrying an effective Hamiltonian $\mathbb{T}^m$-action. Then*

1) $m \leq n$.
2) The T^m-action has at least $(m+1)$ fixed points.

Proof. **1)** According to Lemma 3.11, there exists $z \in M$ with $\mathrm{rk}(Dh|_z) = m$. In addition, Corollary 2.16 implies that there are maximally n independent integrals. Since $Dh|_z$ is an $(m \times 2n)$-matrix, this forces $m \leq n$.
2) There exists, according to Lemma 3.11, a point $z \in M$ with $\mathrm{rk}(Dh|_z) = m$. This implies that $h(z)$ lies in the m-dimensional interior of the momentum polytope $h(M)$. The momentum polytope is according to Theorem 3.10 (Convexity) the convex hull of the images of the fixed points. Since a convex set that includes an open set of dimension m must be the convex hull of at least $(m+1)$ distinct points, the claim follows. □

We note

Corollary 3.13. *An effective $\mathbb{T}^n$-action on a $2n$-dimensional symplectic manifold gives rise to a completely integrable Hamiltonian system.*

Proof. ...still to be written... □

Theorem 3.10 (Convexity) states that a Hamiltonian $\mathbb{T}^m$-action gives rise to a convex polytope, namely its momentum polytope. Does there also exist a converse, i.e., given a convex polytope, can we construct a torus action that has this polytope as momentum polytope? With certain restrictions, the answer will turn out to be affirmative.

Definition 3.14. *Let $\triangle \subset \mathbb{R}^n$ be a convex polytope.*
1) $\triangle$ is **simple** *if there are precisely n edges meeting at each vertex.*
2) $\triangle$ is **rational** *if the slope of every edge is rational, i.e., for all vertices z of $\triangle$, every edge $E(z)$ emanating from z can be parametrised by $z + s\vec{E}(z)$ for some $\vec{E}(z) \in \mathbb{Z}^n$ and $s \geq 0$.*
3) $\triangle$ is **smooth** *if, for all vertices z of $\triangle$, the n edge vectors $\vec{E}_1(z), \ldots, \vec{E}_n(z) \in \mathbb{Z}^n$ can be chosen to be a basis of $\mathbb{Z}^n$.*
4) $\triangle$ is called **Delzant** *if $\triangle$ is simple, rational, and smooth.*

These polytopes are named after the French mathematician Thomas Delzant (currently working at the University of Strasbourg, France).

DEFINITION 3.15. *Let G be a group acting on a manifold M. A diffeomorphism $f : M \to M$ is* **equivariant** *if $f(g.z) = g.(f(z))$ for all $g \in G$ and all $z \in M$.*

There is the following classification:

THEOREM 3.16 (**Classification by** [**Delzant**]). *There exists a one to one correspondence between compact connected $2n$-dimensional symplectic manifolds (M, ω) equipped with a Hamiltonian $\mathbb{T}^n$-action (up to equivariant symplectomorphism) and Delzant polytopes. It is given by*

$$(M, \omega, \mathbb{T}^n, h) \mapsto h(M).$$

PROOF. We will go through the construction behind the surjectivity of the map $(M, \omega, \mathbb{T}^n, h) \mapsto h(M)$ for an explicitly given Delzant polytope in Example **??**. The general proof of Delzant's classification is beyond the scope of this course. We refer the interested reader to [**Cannas da Silva**] for the surjectivity of the map $(M, \omega, \mathbb{T}^n, h) \mapsto h(M)$ and to [**Kaufman**] and partially [**Guillemin**] for the rest. □

Thus a Hamiltonian $\mathbb{T}^n$-action on an $2n$-dimensional, compact, connected symplectic manifold is completely characterized by its momentum polytope.

REMARK 3.17. *The momentum polytopes $h(M)$ in Theorem 3.16 (Delzant) are according to Theorem 3.10 (Convexity) the convex hull of the images of the fixed points of the actions. Therefore the continuous and smooth object $(M, \omega, \mathbb{T}^n, h)$ in Theorem 3.16 (Delzant) is in fact completely characterized by a finite set of discrete data.*

If we drop the compactness condition, Theorem 3.16 (Delzant) is not true any more. The situation becomes more complicated and was solved in 2015 by [**Karshon** & **Lerman**]. The question when an effective Hamiltonian $\mathbb{S}^1$-action on a compact, symplectic 4-dimensional manifold can be extended to an $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$-action was answered in [**Karshon**].

3.3 Lie group actions on manifolds

If we are not only interested in topological features of group actions on manifolds but also want to involve the smooth structure of an underlying smooth manifold, the following type of groups are natural candidates.

DEFINITION 3.18. *A group G is a* **Lie group** *if it is a finite dimensional C^2-manifold and if the following two group operations of G are C^2-mappings:*

$$\begin{aligned} G \times G &\to G, \\ (g,h) &\mapsto gh \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\to G, \\ g &\mapsto g^{-1}. \end{aligned}$$

Many of the 'natural' groups are Lie groups:

EXAMPLE 3.19. *The following groups are Lie groups:*
$U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $GL(n)$, $(\mathbb{R},+)$, $\mathbb{S}^1 = U(1) = SO(2)$ *seen as rotations,* $(\mathbb{R}^n,+)$, $\mathbb{T}^n = (\mathbb{S}^1)^n$.

Now we want to generalize Hamiltonian torus actions on manifolds to Hamiltonian Lie group actions. Therefore we need the following notions.

DEFINITION 3.20. *Let G be a Lie group and $g \in G$. The* **left multiplication by** *g is the map*

$$L_g : G \to G, \qquad \tilde{g} \mapsto g\tilde{g}$$

and a vector field X on G is **left-invariant** *if $(L_g)_*X = X$ where $((L_g)_*X)|_{L_g(\tilde{g})} := D(L_g)|_{\tilde{g}}.X|_{\tilde{g}}$.*

One usually works with these *left* notions since they lead to homomorphisms whereas the analogous *right* notions give rise to antihomomorphisms.

Definition and Proposition 3.21. *Let G be a Lie group and denote by*

$$\mathrm{Lie}(G) =: \mathfrak{g}$$

the vector space of left-invariant vector fields on G. Equipped with the Lie bracket $[\cdot,\cdot]$ for vector fields, $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra and is called **Lie algebra of the Lie group** G. *As a vector space, $\mathfrak{g}$ is isomorphic to the tangent space of G at the identity T_eG via $X \mapsto X|_e$.*

Proof. The Lie bracket is compatible with the pushforward of vector fields such that $[(L_g)_*X, (L_g)_*Y] = (L_g)_*[X, Y]$ for all $g \in G$ and $X, Y \in \mathfrak{g}$. Therefore $\mathfrak{g}$ is closed under the Lie bracket such that the restriction of the Lie bracket to $\mathfrak{g}$ gives rise to a Lie subalgebra. The left-invariance allows to identify a vector field X with its value $X|_e$ at the identity $e \in G$ and vice versa. □

Definition 3.22. *Let G be a Lie group acting on a smooth manifold M via $\varphi^G : G \to \mathrm{Diff}(M)$. Consider $X \in \mathfrak{g}$ as (leftinvariant) vector field and denote the flow of the ODE $g' = X(g)$ on G by Φ_t^X. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \varphi^G_{\Phi_t^X(g)}(z) =: X^\#(z) \in T_zM$$

defines a vector field $X^\#$ on M, often called the **infinitesimal generator** *of the* **1-parameter group** *of the action induced by $X \in \mathfrak{g}$.*

There are several operations of Lie groups on itself that carry geometric and algebraic meanings in various contexts.

Definition 3.23. *1) Let $g \in G$. The action by conjugation*

$$\varphi_g^G : G \to G, \qquad \varphi_g^G(f) := gfg^{-1}$$

gives rise to the invertible linear map

$$Ad_g := D(\varphi_g^G)|_e : T_eG \simeq \mathfrak{g} \to T_eG \simeq \mathfrak{g}$$

which in turn gives rise to the **adjoint action** *or* **adjoint representation**

$$Ad : G \to GL(\mathfrak{g}), \qquad g \mapsto Ad_g$$

where $GL(\mathfrak{g})$ is the group of linear isomorphisms of $\mathfrak{g}$.

Note that we have $Ad_f \circ Ad_g = Ad_{fg}$. Since the Lie algebra is a vector space, we can pass to its dual:

DEFINITION 3.24. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and denote by*

$$\mathfrak{g}^* := \{\alpha : \mathfrak{g} \to \mathbb{R} \mid \alpha \text{ linear and continuous}\}$$

its dual (as a vector space). The dual paring

$$\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}, \qquad \langle \alpha, X \rangle := \alpha(X)$$

gives for each $g \in G$ rise to the dual map $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ via

$$\langle Ad_g^* \alpha, X \rangle = \langle \alpha, Ad_{g^{-1}}(X) \rangle$$

which in turn gives rise to the **coadjoint action** *or* **coadjoint representation**

$$Ad^* : G \to GL(\mathfrak{g}^*), \qquad g \mapsto Ad_g^*.$$

Note that we dualized with $Ad_{g^{-1}}$ in order to get the homomorphism property $Ad_f^* \circ Ad_g^* = Ad_{fg}^*$. The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ carries the structure of a so-called Lie coalgebra, see for example [**Michaelis**]. If $\mathfrak{g}$ is finite dimensional then $\mathfrak{g} \simeq \mathfrak{g}^*$ are isomorphic. Now we are ready to define a Hamiltonian Lie group action.

DEFINITION 3.25. *Let G be a Lie group acting on a smooth manifold M. The action is* **Hamiltonian** *if there exists a map*

$$h : M \to \mathfrak{g}^*$$

with

1) For all $X \in \mathfrak{g}$, we have

$$\omega(X^\#, \cdot) = dh^X$$

where the function $h^X : M \to \mathbb{R}$, $h^X(p) := \langle h(p), X \rangle$ is the 'component' of h in direction of X.

2) We have for all $g \in G$

$$h \circ \varphi_g^G = Ad_g^* \circ h,$$

i.e., h is equivariant w.r.t. the action φ^G of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$.

For $G = \mathbb{R}^n$ and $G = \mathbb{T}^n$, we have $\mathrm{Lie}(\mathbb{R}^n) \simeq \mathbb{R}^n$ and $\mathfrak{t}^n := \mathrm{Lie}(\mathbb{T}^n) = \mathbb{R}^n$. Therefore

COROLLARY 3.26. *In case of $G = \mathbb{R}^n$ and $G = \mathbb{T}^n$, we recover the notion of integrable Hamiltonian system and Hamiltonian $\mathbb{T}^n$-action.*

PROOF. Let us first consider the case $G = \mathbb{R}$ and denote the action of the Lie group $\mathbb{R}$ on a symplectic manifold (M, ω) by $\varphi^{\mathbb{R}} : \mathbb{R} \to \mathrm{Aut}(M)$. Since $\mathbb{R}$ is commutative, the action by conjugation is always the identity. Thus $Ad_g = \mathrm{Id}_{\mathrm{Lie}(\mathbb{R})}$ for all $g \in \mathbb{R}$. We find $\mathrm{Lie}(\mathbb{R})^* = \mathbb{R}^* \simeq \mathbb{R}$ and

$$\langle Ad_g^* \alpha, X \rangle = \langle \alpha, Ad_{g^{-1}}(X) \rangle = \langle \alpha, X \rangle$$

for all $g \in \mathbb{R}$, $X \in \mathrm{Lie}(\mathbb{R}) = \mathbb{R}$ and $\alpha \in \mathrm{Lie}(\mathbb{R})^* \simeq \mathbb{R}$ which implies $Ad_g^* = \mathrm{Id}_{\mathrm{Lie}(\mathbb{R})^*}$ for all $g \in \mathbb{R}$. Now let $h : M \to \mathrm{Lie}(\mathbb{R})^* \simeq \mathbb{R}$ be a momentum map of the Hamiltonian action of the Lie group $\mathbb{R}$ on (M, ω). We have $\mathrm{Lie}(\mathbb{R}) = \mathbb{R} = \mathrm{Span}_{\mathbb{R}}\{\mathbf{1}\}$ such that any $X \in \mathrm{Lie}(\mathbb{R})$ is of the form $X = X\mathbf{1}$. Thus we compute for $h^{\mathbf{1}} : M \to \mathbb{R}$ and $h^X : M \to \mathbb{R}$

$$\langle h^{\mathbf{1}}(p), \mathbf{1} \rangle = h^{\mathbf{1}}(p) \quad \text{and} \quad \langle h^X(p), X \rangle = X \langle h^X(p), \mathbf{1} \rangle = X h^X(p)$$

We consider $g' = X(g) = Xg$ which has the solution $g(t) = \exp(Xt)$ and therefore the flow $\Phi_t^X(g) = \exp(tX)g$. We obtain for $z \in M$

$$X^{\#}(z) = \frac{d}{dt}\bigg|_{t=0} \varphi^G_{\Phi_t^X(g)}(z) = \frac{d}{dt}\bigg|_{t=0} \varphi^G_{\exp(tX)g}.$$

The $\mathbb{R}$-action induced by a Hamiltonian system $\mathfrak{h} : (M, \omega) \to \mathbb{R}$ has the flow

`falls` $\mathfrak{h}$ `die klassische Ham fct sein soll, dann setze einfach` $\mathfrak{h}(p) := h^{\mathbf{1}}(p)$`, denn` $X^{c\mathfrak{h}} = cX^{\mathfrak{h}}$ `fuer` $c \in \mathbb{R}$ `(??) das sollte ok sein...?` Q?

□

We recall

DEFINITION 3.27. *The action of a Lie group G on a manifold M is said to be* **proper** *if the map $G \times M \to M \times M$ given by $(g, z) \mapsto (g.z, z)$ is proper.*

Recall that the quotient of a manifold under a Lie group action is under certain conditions again a manifold:

Theorem 3.28 (**Manifold quotient theorem**). *Let G be a Lie group and M a C^k-manifold with $k \geq 1$. Let the action of G on M be C^k-smooth, free and proper. Then the quotient by the group action M/G has a unique structure of a C^k-manifold of dimension* $\dim M - \dim G$.

Proof. See [**Duistermaat** & **Kolk**, Theorem 1.11.4]. □

Given a symplectic manifold with a free and proper Lie group action, we naturally would like the quotient to be symplectic and its symplectic structure to be compatible under the quotient map with the one on the original manifold. But unfortunately a 'symplectic version' of Theorem 3.28 (Manifold quotient theorem) cannot be true: the dimension of the quotient of a $2n$-dimensional symplectic manifold M under a Hamiltonian $\mathbb{S}^1$-action is $\dim(M/\mathbb{S}^1) = 2n - 1$ which is odd such that $M/\mathbb{S}^1$ certainly cannot be symplectic. Nevertheless, [**Marsden** & **Weinstein**] and [**Meyer**] came up with a welldefined notion, denoted by $M /\!\!/ G$. We formulate their result for Hamiltonian torus actions.

Theorem 3.29 ([**Marsden** & **Weinstein**] **and** [**Meyer**]). *Let $(M, \omega, \mathbb{T}^m, h)$ be a symplectic manifold of dimension $2n$ with a Hamiltonian $\mathbb{T}^m$-action with momentum map $h : M \to \mathbb{R}^m$. Let $j : h^{-1}(0) \hookrightarrow M$ be the inclusion and assume that $\mathbb{T}^m$ acts freely on $h^{-1}(0)$. Then*

1) $M_{red} := M /\!\!/ \mathbb{T}^m := h^{-1}(0)/\mathbb{T}^m$ is a manifold.
2) The quotient map $q : h^{-1}(0) \to h^{-1}(0)/\mathbb{T}^m$ is a principal $\mathbb{T}^m$-bundle.
3) There exists a symplectic form ω_{red} on M_{red} with $j^*\omega = q^*\omega_{red}$.

The symplectic manifold (M_{red}, ω_{red}) has various names in the literature, for instance **reduction**, **symplectic reduction**, **reduced space**, **symplectic quotient**, **Marsden-Weinstein quotient**, **Marsden-Weinstein-Meyer quotient**.

Proof. See [**Cannas da Silva**] or [**McDuff** & **Salamon**]. □

3.4 An explicit example for Delzant's construction

First, we have to write polytopes in a way that is accessible to algebraic and analytic methods.

LEMMA 3.30. *Let $\triangle \subset \mathbb{R}^n$ be a convex polytope. Then $\triangle$ can be written as intersection of n-dimensional halfspaces, i.e., there exists $d \in \mathbb{N}$ and $v_1, \ldots, v_d \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ such that*

$$\triangle = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle \leq \lambda_k \ \forall\ 1 \leq k \leq d\}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product.

PROOF. Denote by d the number of $(n-1)$-dimensional strata of $\triangle$. Choose $v_1, \ldots, v_d \in \mathbb{R}^n$ to be the outward pointing normal vectors to the d strata $S_1, \ldots, S_d$ of dimension $(n-1)$. Then consider v_k as vector with 'foot point' at the origine and define ℓ_k to satisfy $\ell_k \|v_k\| = dist(S_k, \{0\})$. Then $\ell_k v_k \in S_k$ and $x - \ell_k v_k$ is perpendicular to v_k for all $x \in S_k$. Thus

$$S_k \subset \{x \in \mathbb{R}^n \mid \langle x - \ell_k v_k, v_k \rangle = 0\}$$

and we calculate $0 = \langle x - \ell_k v_k, v_k \rangle = \langle x, v_k \rangle - \ell_k \|v_k\|^2$. Set $\lambda_k := \ell_k \|v_k\|^2$ and conclude

$$\triangle = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle \leq \lambda_k \ \forall\ 1 \leq k \leq d\}.$$

□

Since Delzant polytopes have rational slopes and other nice properties we have to 'fine tune' the shape of the normal vectors.

DEFINITION 3.31. *A vector $v \in \mathbb{Z}^n$ is called* **primitive** *if there exists no $l \in \mathbb{Z}$ with $|l| > 1$ and no $u \in \mathbb{Z}^n$ such that $v = lu$.*

We conclude

COROLLARY 3.32. *Let $\triangle \subset \mathbb{R}^n$ be a Delzant polytope. Then there exists $d \in \mathbb{N}$ and primitive vectors $v_1, \ldots, v_d \in \mathbb{Z}^n$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ such that*

$$\triangle = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle \leq \lambda_k \ \forall\ 1 \leq k \leq d\}.$$

For Delzant polytopes in $\mathbb{R}^n$ with d strata of dimension $(n-1)$, we always have $d \geq n + 1 > n$. We give the difference $d - n$ now an algebraic meaning.

LEMMA 3.33. *Let $\triangle = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle \leq \lambda_k \ \forall\ 1 \leq k \leq d\}$ be a Delzant polytope and $e_1, \dots, e_d$ the standard basis of $\mathbb{R}^d$. Then the definition of $\tau(e_k) := v_k$ for all $1 \leq k \leq d$ extends by linearity to surjective maps*

$$\tau : \mathbb{Z}^d \to \mathbb{Z}^n \quad \text{and} \quad \tau : \mathbb{R}^d \to \mathbb{R}^n.$$

Moreover, we obtain a welldefined map $\tau : \mathbb{R}^d/\mathbb{Z}^d =: \mathbb{T}^d \to \mathbb{R}^n/\mathbb{Z}^n =: \mathbb{T}^n$ which is also surjective.

PROOF. Let p be a vertex of $\triangle$ and let $E_1(p), \dots, E_n(p) \in \mathbb{Z}^n$ be n vectors spanning the n edges emanating from p. Since p is smooth, we have $\mathrm{Span}_{\mathbb{Z}}\{E_1(p), \dots, E_n(p)\} = \mathbb{Z}^n$ and therefore also $\mathrm{Span}_{\mathbb{R}}\{E_1(p), \dots, E_n(p)\} = \mathbb{R}^n$.
Always $(n-1)$ of the edge vectors span the n strata of dimensional $(n-1)$ meeting at p. Denote the primitive outer normal vector of the $(n-1)$-stratum spanned by $E_1(p), \dots, E_{k-1}(p), E_{k+1}(p), \dots, E_n(p)$ by $v_k(p)$. By construction and since $v_k(p)$ is primitive, we find

$$\mathrm{Span}_{\mathbb{Z}}\{E_1(p), \dots, E_{k-1}(p), E_{k+1}(p), \dots, E_n(p), v_k(p)\} = \mathbb{Z}^n.$$

Since these $(n-1)$-strata always have $n-2$ edge vectors in common, we can successively replace the edge vectors by normal vectors and still span the space $\mathbb{Z}^n$. □

Now we consider an explicit example for Delzant's construction:

3.5 Focus-focus points and monodromy

Delzant's classification for toric systems is 'easy and simple' in the sense that there is precisely one invariant, namely the image of the momentum map. A glance at Theorem 3.46 shows that this is not true any more once we 'leave the toric world'.

In this section, we investigate some phenomena that may happen in non-toric integrable systems. To this end, first note:

Lemma 3.34. *Let (M, ω) be a symplectic manifold of dimension $2n$ and $h : (M, \omega) \to \mathbb{R}^n$ a completely integrable system inducing an effective Hamiltonian $\mathbb{T}^n$-action. Then the only possible types of points are points with k elliptic and $(n-k)$ regular components with $0 \leq k \leq n$. More precisely, rank k points form the facets of dimension k of the momentum polytope. Fixed points are thus automatically of purely elliptic type.*

Proof. According to Theorem 3.16 (Delzant), completely integrable toric systems are classified by the image of their momentum map which is a Delzant polytope. By construction, these polygons are stratified in the sense that points of rank k are mapped precisely to facets of dimension k of the Delzant polytope. Given a point p in a facet of dimension k then, from the point of view of Theorem 2.31 (Local normal form), the point is the image of k regular and $(n-k)$ elliptic component. This local neighbourhoods are compatibel with each other whereever they overlap.

We show it for the case of an neighbourhood containing ??? points: take explicit computation from [**Pelayo** & **Vũ Ngọc 2011**]... check also what daniele said... Q?

□

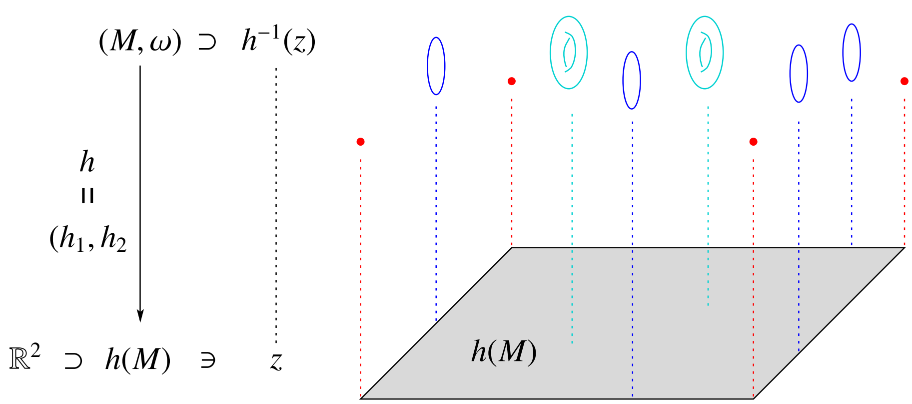


Figure 3.1. Example for the fibration of a toric integrable system $h : (M, \omega) \to \mathbb{R}^2$ with $\dim M = 4$. There are regular fibers (light blue), elliptic-regular fibers (dark blue), and elliptic-elliptic fibers (red).

Intuitively, this means that we have action angle coordinates over the *whole interior* of the momentum polytope of a toric system, but over action levels on the boundary of the momentum polytope we only 'keep' as many angle coordinates as the dimension of the fiber over this action level.

[**Duistermaat**] described in detail under which conditions a Lagrangian fibration admits global action angle coordinates. More precisely, he identified the topological obstructions (in terms of so-called monodromy and the Chern class) preventing global existence of action angle coordinates:

Theorem 3.35 ([**Duistermaat**]). *Let (M, ω) be a connected symplectic manifold of dimension $2n$ and B a smooth manifold B of dimension n and let $\pi : M \to B$ be a fibration such that the fibers $\pi^{-1}(b) \subset M$ are compact, connected Lagrangian submanifolds for all $b \in B$.*
Then the fibration is topologically trivial if and only if the monodromy and the Chern class are trivial. Furthermore, the following two statements are equivalent:

(1) There exists a smooth map $(\mathfrak{a}, \alpha) : M \to \mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n$ such that
  - $\omega = \sum_{k=1}^n d\alpha_k \wedge d\mathfrak{a}_k$,
  - $\mathfrak{a}_k$ *is constant on the fibers of* $\pi : M \to B$ *for all* $1 \leq k \leq n$.
  - α *is injective on each fibre of* $\pi : M \to B$.

(2) The fibration $\pi : M \to B$ is topologically trivial and ω is exact.

In the following, we want to gain some intuition about one of the obstructions to global action angle coordinates, namely monodromy. For the precise details, see for example [**Duistermaat**] or [**Cushman** & **Bates**]. Moreover, the monography [**Martynchuk**] gives an overview over the various appearances of monodromy in geometry and dynamical systems. An more geometric-algebraic approach via complex Morse functions and vanishing cycles can be found in [**Arnold 1990**].

Now have a look at the fibration in Figure 3.2. Due to the existence of focus-focus points, the system is not toric according to Lemma 3.34.
Now let us have once again a look at the local model of focus-focus points described in Lemma 2.35: There we had set $\zeta_1 := x_1 + ix_2$ and $\zeta_2 := y_1 + iy_2$, had identified $\mathbb{R}^2 \simeq \mathbb{C}$, and eventually had considered the map

$$g : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad g(\zeta_1, \zeta_2) := \overline{\zeta}_1 \zeta_2 = (x_1 y_1 + x_2 y_2) + i(x_1 y_2 - x_2 y_1).$$

Now we want to study this map in polar coordinates. Thus we set $\zeta_1 =: r_1 e^{i\varphi_1}$ and $\zeta_2 =: r_2 e^{i\varphi_2}$ and obtain

$$\begin{aligned} \hat{g} : (\mathbb{R}^{\geq 0} \times \mathbb{R}/\mathbb{Z}) \times (\mathbb{R}^{\geq 0} \times \mathbb{R}/\mathbb{Z}) &\to \mathbb{R}^{\geq 0} \times \mathbb{R}/\mathbb{Z}, \\ (r_1, \varphi_1, r_2, \varphi_2) &\mapsto (r_1 r_2, -\varphi_1 + \varphi_2). \end{aligned}$$

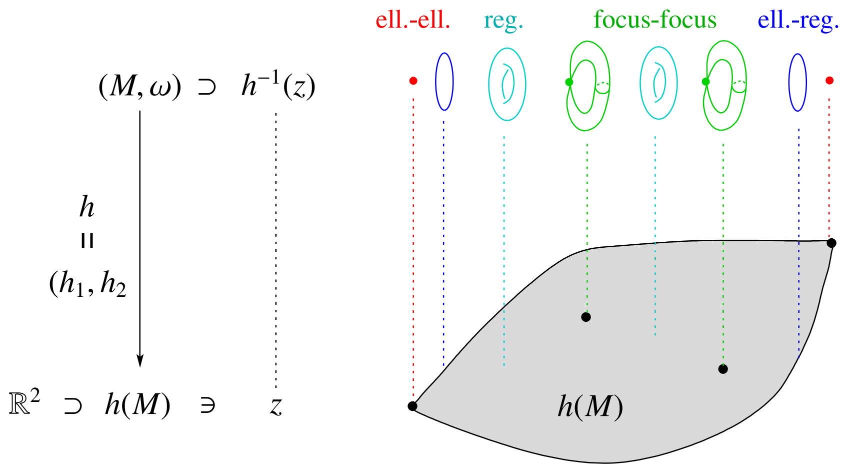


FIGURE 3.2. Example for the fibration of a non-toric integrable system $h : (M,\omega) \to \mathbb{R}^2$. There are regular fibers (light blue), elliptic-regular fibers (dark blue), elliptic-elliptic fibers (red) and two focus-focus fibers (green).

We are interested in the preimage nonsingular values $\zeta = re^{i\varphi}$, i.e., we have $\zeta \neq 0$ and therefore $r > 0$ in polar coordinates. The solutions of the equation $g^{-1}(\zeta)$ with $\zeta \neq 0$ are in polar coordinates given by

$$\hat{g}^{-1}(r,\varphi) = \{(r_1,\varphi_1,r_2,\varphi_2) \mid r = r_1 r_2,\ \varphi = -\varphi_1 + \varphi_2\}$$

and we can parametrise $\hat{g}^{-1}(r,\varphi)$ by the map

$$f_{r,\varphi} : \mathbb{R}^{>0} \times \mathbb{R}/\mathbb{Z} \to \hat{g}^{-1}(r,\varphi), \quad (s,t) \mapsto \left(s,\ t,\ \frac{r}{s},\ t+\varphi\right).$$

Now consider the path that parametrises a small circle around the origin and track the preimages of g along it. This means that we fix a small $\rho \in \mathbb{R}^{>0}$ and let φ travel once through $\mathbb{R}/\mathbb{Z}$:

$$\varphi \mapsto f_{\rho,\varphi}.$$

Now look what happens to the image of $s = 1$ and $t = 0$ when φ changes: We observe

$$\varphi \mapsto f_{\rho,\varphi}(1,0) = (1,0,\rho,\varphi)$$

i.e., the point walks once around the image of the cylinder $\mathbb{R}^{>0} \times \mathbb{R}/\mathbb{Z}$. This phenomenon is intuitively described in Figure 3.3. This local model with the cylinders describes the ‘cylindrical part’ of a torus near the cycle that gets pinched to a point when approaching the focus-focus fiber. On the tori above the path around the focus-focus value, this phenomenon can be described as follows: Still think of the focus-focus value as located at the origin. Recall the lattice from Lemma 2.23 and denote the lattice above the path $\varphi \mapsto (\rho, \varphi)$ with $\varphi \in [0, 1 - \varepsilon] \subset \mathbb{R}/\mathbb{Z}$ and $\varepsilon > 0$ small by $\Gamma(\rho, \varphi)$. The lattice is of the form

$$\Gamma(\rho, \varphi) = \{\gamma_1 + \nu_1(\rho, \varphi), \ldots, \gamma_n + \nu_n(\rho, \varphi)\}$$

with $\Gamma(\rho, 0) = \{\gamma_1, \ldots, \gamma_n\}$. The phenomenon from Figure 3.3 looks in terms of lattices as follows: The basis changes along the path $\varphi \mapsto (\rho, \varphi)$ and differs at the points $\varphi = 0$ and $\varphi = 1$, but we have $\Gamma(0) = \gamma(1)$, i.e., just by looking at the lattice, we find nothing amiss, but we cannot define action angle variables along the loop since we did a full jump in period such that the loop of basis is not closing continuously, see Figure 3.4.This effect is called monodromy and, in dimension two, is usually described by the so-called **monodromy matrix** which is the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

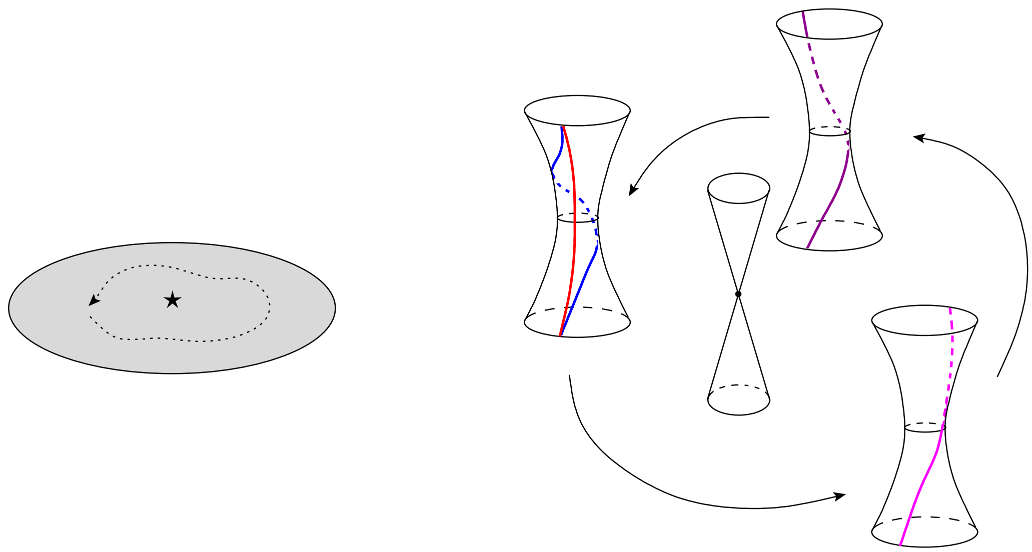
realising the transition from $\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$ to $\{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$

EXAMPLE 3.36. *hier monodromy via polar coordinates on cyminder...*

DEFINITION 3.37. *This twisting phenomenon in the fibers when travelling once around a focus-focus fiber is called* **monodromy**. *When considered as a map of the associated lattices of the tori then the change in basis after one loop is given by the matrix* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

COROLLARY 3.38. *Monodromy prevents the existence of well-defined action angle variables in neighbourhood of a focus-focus singular point since the periods of the lattices are not closing continuously along a closed path around the focus-focus point.*

3.6 Semitoric systems and their classification



(a) (b)

FIGURE 3.3. Geometric interpretation of monodromy:
(a) A path (dotted) looping once around a focus-focus value.
(b) Observe in the local model the deformation of the vertical (red) line in a regular fiber into a (purple) line wrapping more and more around the cylinder when following the fibers over the path in (a). After walking precisely once around the singular fiber, the red line is transformed into the blue one that wraps precisely once around the cylinder.

In order to leave the rigid 'toric world' within the set of integrable systems while keeping enough structure to admit a 'nice' classification, [**Pelayo & Vũ Ngọc 2009**], [**Pelayo & Vũ Ngọc 2011**] studied the following class of completely integrable systems on (not necessarily compact) 4-dimensional manifolds.

DEFINITION 3.39. *Let* (M, ω) *be a connected symplectic* 4*-dimensional manifold. A completely integrable Hamiltonian system* $h = (h_1, h_2) : (M, \omega) \to \mathbb{R}^2$ *is* **semitoric** *if*

1) h_1 *is proper.*

2) h_1 *induces an effective Hamiltonian* $\mathbb{S}^1$*-action,*

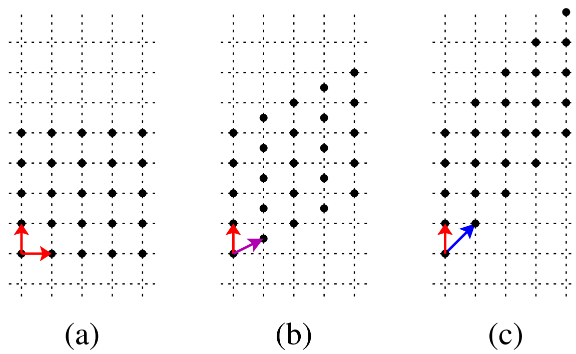


FIGURE 3.4. Geometric interpretation of monodromy via lattices: The vectors $\gamma_1(\tau) := (1,\tau)^T$ and $\gamma_2(\tau) := (0,1)^T$ and their spanned lattice $\Gamma(\tau) := \mathrm{Span}_{\mathbb{Z}}\{\gamma_1(\tau), \gamma_2(\tau)\}$ satisfy $\tau = 0$ in (a), $\tau = \frac{1}{2}$ in (b), and $\tau = 1$ in (c). We have $\Gamma(0) = \Gamma(1)$ although their bases differ.

3) h has only nondegenerate singularities.
4) Focus-focus fibers (if any) of h contain at most one focus-focus point.
5) h admits no hyperbolic components in the sense of Theorem 2.31 (Local normal form).

This definition restricts the variety of singular points:

LEMMA 3.40. *Semitoric systems may only have points of the following types:*

- *Rank 2: regular points.*
- *Rank 1: elliptic-regular points.*
- *Rank 0: fixed poinst of focus-focus or elliptic-elliptic type.*

PROOF. Since semitoric systems live on 4-dimensional manifolds, the differential of the momentum map can only have rank zero, one or two. Since two is the maximal rank, these points are regular. By definition, semitoric systems only admit nondegenerate singular points. Moreover, singular points may not have hyperbolic components. Thus Theorem 2.31 (Local normal form) implies that the only possibility for rank one points is the combination of a regular and an elliptic component. Rank zero can only be attained at fixed points with two elliptic components or one pair of focus-focus components. □

The system in Figure 3.2 displays all these types of points. Many physically relevant systems are in fact semitoric:

EXAMPLE 3.41. *The coupled angular momenta system in Example 1.50 is semitoric.*

PROOF. Using numerics, the appearance of a focus-focus point was first noticed by [**Sadovskií** & **Ẑhilinskií**]. The semitoric properties were proven in [**Le Floch** & **Pelayo**] and [**Alonso** & **Dullin** & **Hohloch 2018**]. □

The coupled angular momenta system sometimes also goes by the name **Jaynes-Cummings model** or **Gaudin model**.

EXAMPLE 3.42. *Consider* $(\mathbb{S}^2 \times \mathbb{R}^2, \lambda\omega_{\mathbb{S}^2} \oplus \mu\omega_{\mathbb{R}^2})$ *with* $\lambda, \mu > 0$. *The system* $(L, H) : \mathbb{S}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$L(x, y, z, u, v) := (z - 1) + \mu \frac{u^2 + v^2}{2},$$
$$H(x, y, z, u, v) := \frac{1}{2}(xu + yv)$$

is semitoric and usually called **coupled spin oscillator**.

PROOF. [**Pelayo** & **Vũ Ngọc 2012**] and [**Alonso** & **Dullin** & **Hohloch 2017**]. □

The coupled angular momenta system depends on an 'interpolating' parameter. It is in fact a special case of much more general systems:

EXAMPLE 3.43. *Whole families of semitoric systems are studied in* [**Hohloch** & **Palmer**] *and* [**Le Floch** & **Palmer**].

If we want to study classifications of semitoric systems, we have to specify which systems we consider to be isomorphic.

DEFINITION 3.44. *Two semitoric systems* (M, ω, h) *and* $(\hat{M}, \hat{\omega}, \hat{h})$ *are* **isomorphic** *if there exists a pair* (Ψ, ψ), *where* $\Psi : (M, \omega) \to (\hat{M}, \hat{\omega})$ *is a symplectomorphism and* $\psi : h(M) \subset \mathbb{R}^2 \to \hat{h}(\hat{M}) \subset \mathbb{R}^2$ *is a locally defined (orientation preserving) diffeomorphism of the form*

$\psi(x, y) = (\psi_1, \psi_2)(x, y) = (x, \psi_2(x, y))$ *making the following diagram commute*

$$\begin{tikzcd} (M, \omega) \arrow[r, "\Psi"] \arrow[d, "h"'] & (\hat{M}, \hat{\omega}) \arrow[d, "\hat{h}"] \\ \mathbb{R}^2 \arrow[r, "\psi"'] & \mathbb{R}^2. \end{tikzcd}$$

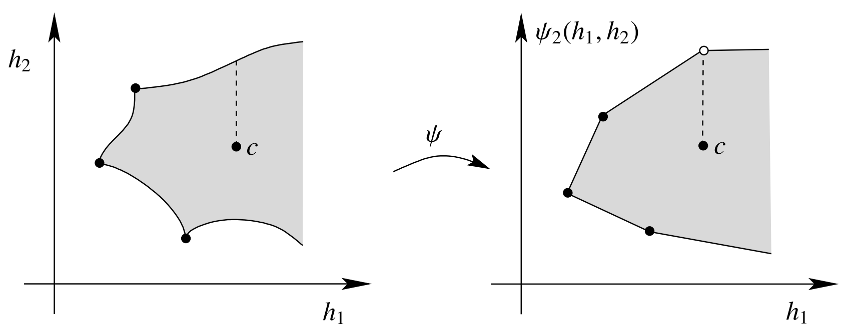


FIGURE 3.5. Isomorphic systems: The 'straightening procedure' turns a semitoric system into a toric system on the whole manifold *minus the preimage of the vertical cuts including the focus-focus values*.

A semitoric system gives rise to an $\mathbb{S}^1 \times \mathbb{R}$-action which places them 'between' general integrable systems (have an $\mathbb{R} \times \mathbb{R}$-action) and toric systems (admitting an $\mathbb{S}^1 \times \mathbb{S}^1$-action). Semitoric systems differ from toric systems in particular by the possible existence of focus-focus singularities.

REMARK 3.45 ([**Vũ Ngọc 2007**]). *A semitoric system has (if at all) only a finite number of focus-focus critical values.*

[**Pelayo** & **Vũ Ngọc 2009**] use the following five invariants to classify semitoric systems. Let $(M, \omega, \Phi = (J, H))$ be a (not necessarily compact) semitoric system. Its list of *semitoric invariants* consists of:

(1) The number m_{FF} of focus-focus points $\mathfrak{c}_1, \ldots, \mathfrak{c}_{m_{FF}} \in M$.
(2) Taylor series $S_i := S_i(\mathfrak{c}_i) \in \mathbb{R}[[X, Y]]$ for $1 \leq i \leq m_{FF}$ associated to the focus-focus points $\mathfrak{c}_1, \ldots, \mathfrak{c}_{m_{FF}} \in M$.

    The Taylor series S_i of the focus-focus point $\mathfrak{c}_i$ is a *semilocal* invariant, i.e., it describes the whole focus-focus fiber. Roughly, [**Vũ Ngọc 2003**] constructs a 'generating function' $\mathscr{S}_i$ for the Lagrangian fibration near the focus-focus point $\mathfrak{c}_i$ which extends smoothly over the focus-focus singularity. Its Taylor series is S_i. One can see these Taylor series also as some kind of 'modified' **Birkhoff normal forms**.

(3) The (equivalence class of a) polygon obtained by a straightening procedure from the image of the momentum map (see Figure 3.5) and equipped with (a choice of) vertical cuts to the focus-focus points. These polygons are usually called **semitoric polygons**.
(4) The heights of the focus-focus points in the semitoric polygon.
(5) The **twisting index** is (roughly) a tuple $(k_1, \dots, k_{m_{FF}})$ of integers where k_i is the rotation vector of a certain periodic flow near the ith focus-focus fiber measured w.r.t. the toric 'background basis' obtained from the straightening procedure.

An important observation is the fact that the underlying manifold is compact if and only if the semitoric polygon is compact.

Theorem 3.46 ([**Pelayo** & **Vũ Ngọc 2009**, **Pelayo** & **Vũ Ngọc 2011**]). *Two semitoric integrable systems are isomorphic if and only if they have the same semitoric invariants.*

The semitoric invariants of the coupled spin oscillator are calculated in [**Pelayo** & **Vũ Ngọc 2012**] and [**Alonso** & **Dullin** & **Hohloch 2017**]. The semitoric invariants of the coupled angular momenta are calculated in [**Le Floch** & **Pelayo**] and [**Alonso** & **Dullin** & **Hohloch 2018**].

[**Hohloch** & **Sabatini** & **Sepe**] shows which minimal choice of semitoric invariants is needed in order to recover the Karshon graph (see [**Karshon**]) of the underlying Hamiltonian $\mathbb{S}^1$-action of a semitoric system.

4. Infinite dimensional integrable systems

So far, we studied integrability of Hamiltonian ODEs that live on $2n$-dimensional symplectic manifolds. In this chapter, we will investigate integrability notions for PDEs. Since, in this case, the underlying spaces are usually infinite dimensional, not all results from finite dimensional integrability theory hold true, for example, ‘at most n linearly independent integrals’ is not true any more. Furthermore, there are new phenomena unknown to integrable Hamiltonian ODEs in finite dimensions.

The literature is vast and, when crossing over to physics, often lacks mathematical rigor. Nevertheless, there are some accessible text books and lecture notes like [**Batlle**], [**Dunajski**], and [**Miwa** & **Jimbo** & **Date**]. A short summary of the most important facts can be found in [**Abraham** & **Marsden**, Section 5.5].

4.1 From the wave equation to the Korteweg-de Vries (KdV) equation

The formulation and discovery of the so-called *Korteweg-de Vries equation* was motivated by the following observation: In 1834, J. Scott Russell (1808 – 1882, Scottish civil engineer) noticed a 'single wave not changing shape' travelling down a narrow channel in front of a boat, see [**Scott Russell**, page 319]. Such waves are called **solitons**. A nice simulation of such a wave can be found on YouTube, see

`https://www.youtube.com/watch?v=D14QuUL8x60`

Although [**Boussinesq**, Footnote on page 360] already mentioned briefly an equation describing such travelling waves in 1877, the breakthrough came in 1895 when Korteweg and de Vries gave an explicit solution of such a soliton in their work [**Korteweg** & **de Vries**]. A historical overview over the research around the Korteweg-de Vries equation can be found in [**de Jager**].

To explain the form of the Korteweg-de Vries equation, let us introduce it as generalization of the **1-dimensional wave equation**

$$u_{tt} = c^2 u_{xx}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with coordinates $(x, t) \in \mathbb{R} \times \mathbb{R}$, (constant) **phase velocity** $c \in \mathbb{R}$ and partial derivatives $u_x := \partial_x u$ and $u_{xx} := \partial_x u_x$ and $u_t := \partial_t u$ and $u_{tt} := \partial_t u_t$.

In 1750, the French mathematician Jean-Baptiste le Rond d'Alembert (1717 – 1783) solved the 1-dimensional wave equation:

Theorem 4.1 (**D'Alembert**). *Let $c \in \mathbb{R}$.*

1) The general solution of the 1-dimensional wave equation $u_{tt} = c^2 u_{xx}$ is of the form

$$u(x, t) = F(x - ct) + G(x + ct)$$

where $F, G : \mathbb{R} \to \mathbb{R}$ are arbitrary smooth functions.

2) Given smooth functions $f, g : \mathbb{R} \to \mathbb{R}$, then the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = g(x) & \text{for } x \in \mathbb{R} \end{cases}$$

has the unique solution

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Proof. **1)** We consider the change of coordinates

$$h : \mathbb{R}^2 \to \mathbb{R}^2, \quad h(x,t) := (\xi(x,t), \eta(x,t)) := (x+ct, x-ct)$$

and set $\tilde{u} := u \circ h^{-1}$. We calculate $Dh = \begin{pmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}$ and compute

$$\begin{aligned} u_{xx} &= (\tilde{u} \circ h)_{xx} = (D\tilde{u}.h_x)_x = (\tilde{u}_\xi \xi_x + \tilde{u}_\eta \eta_x)_x = (\tilde{u}_\xi + \tilde{u}_\eta)_x \\ &= D(\tilde{u}_\xi).h_x + D(\tilde{u}_\eta).h_x = \tilde{u}_{\xi\xi}\xi_x + \tilde{u}_{\xi\eta}\eta_x + \tilde{u}_{\eta\xi}\xi_x + \tilde{u}_{\eta\eta}\eta_x \\ &= \tilde{u}_{\xi\xi} + 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta} \end{aligned}$$

and

$$\begin{aligned} u_{tt} &= (\tilde{u} \circ h)_{tt} = (D\tilde{u}.h_t)_t = (\tilde{u}_\xi \xi_t + \tilde{u}_\eta \eta_t)_t = (c\tilde{u}_\xi - c\tilde{u}_\eta)_t = c(\tilde{u}_\xi - \tilde{u}_\eta)_t \\ &= c(D(\tilde{u}_\xi).h_t - D(\tilde{u}_\eta).h_t) = c(\tilde{u}_{\xi\xi}\xi_t + \tilde{u}_{\xi_\eta}\eta_t - \tilde{u}_{\eta\xi}\xi_t - \tilde{u}_{\eta\eta}\eta_t) \\ &= c^2(\tilde{u}_{\xi\xi} - 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}). \end{aligned}$$

Putting this together, we obtain

$$0 = u_{tt} - c^2 u_{xx} = c^2(\tilde{u}_{\xi\xi} - 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}) - c^2(\tilde{u}_{\xi\xi} + 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}) = -4c^2\tilde{u}_{\xi\eta}.$$

Thus solving $u_{tt} = c^2 u_{xx}$ is equivalent to solving $\tilde{u}_{\xi\eta} = 0$ and transforming back. Integrating $\tilde{u}_{\xi\eta} = 0$ w.r.t. ξ yields $\tilde{u}_\eta(\xi,\eta) = H(\eta)$ with an arbitrary smooth function $H : \mathbb{R} \to \mathbb{R}$. Integrating now w.r.t. η yields

$$\tilde{u}(\xi,\eta) = F(\eta) + G(\xi)$$

for an arbitrary smooth function $G : \mathbb{R} \to \mathbb{R}$ and a function $F : \mathbb{R} \to \mathbb{R}$ with $F' = H$. Therefore we get

$$\begin{aligned} u(x,t) = (\tilde{u} \circ h)(x,t) &= \tilde{u}(\xi(x,t), \eta(x,t)) = F(\eta(x,t)) + G(\xi(x,t)) \\ &= F(x-ct) + G(x+ct). \end{aligned}$$

2) Left to the reader. □

The general solution $u(x,t) = F(x-ct) + G(x+ct)$ from Theorem 4.1 (D'Alembert) consists, for $c > 0$ and $t \to \infty$, of a 'wave travelling to the right' with 'shape' $x \mapsto F(x-ct)$ and a 'wave travelling to the left' with 'shape' $x \mapsto G(x+ct)$.

EXAMPLE 4.2. *Let $c > 0$ and let $u(x,t) = F(x-ct) + G(x+ct)$ be the general solution of the 1-dimensional wave equation $u_{tt} = c^2 u_{xx}$ from Theorem 4.1 (D'Alembert).*

1) *For $F(s) := e^{-s^2}$ and $G(s) := 0$, the solution $u(x,t) = e^{-(x-ct)^2}$ consists of one wave with the shape of the Gauss curve travelling to the right, i.e., $x \mapsto u(x,0)$ is the standard Gauss curve and $x \mapsto u(x,t_0)$ for $t_0 \in \mathbb{R}$ is the Gauss curve with maximum at $x_{max} = ct_0$, i.e., translated to the right by ct_0.*
2) *For $F(s) := e^{-s^2} =: G(s)$, the solution $u(x,t) = e^{-(x-ct)^2} + e^{-(x+ct)^2}$ consists of two 'somewhat overlapping' Gauss curve shaped waves. At $t = 0$, the waves overlap completely, forming* one *big wave of shape $x \mapsto u(x,0) = 2e^{-x^2}$. The larger $|t|$ becomes, the more the two waves separate, making their two wave crests more and more distinguishable.*

The wave equation is formulated under several simplifying assumption like for example

- *No dissipation*, i.e., the equation is invariant under 'time reversal' $t \mapsto -t$.
- The amplitude of oscillations is small, i.e., there are no nonlinear terms like for instance u^2.
- *No dispersion* (**dispersion** means that waves with different wave length travel with different phase velocity), i.e., the **group velocity** (definition see below) is constant.

Now we follow [**Dunajski**], relaxing these assumptions by allowing for dispersion and nonlinearity. To this end, consider the two waves $v(x,t) := F(x-ct)$ and $w(x,t) := G(x+ct)$ in the general solution of the 1-dimensional wave equation and compute

$$\begin{aligned} v_x(x,t) &= F'(x-ct), \quad & v_t(x,t) &= -cF'(x-ct), \\ w_x(x,t) &= G'(x+ct), \quad & w_t(x,t) &= cG'(x+ct). \end{aligned}$$

Thus the two waves satisfies the PDEs

$$v_x + \frac{1}{c} v_t = 0 \qquad \text{and} \qquad w_x - \frac{1}{c} w_t = 0.$$

Now consider the 'complex' wave given by $z(x,t) := e^{i(kx-\omega(k)t)}$ where $k \in \mathbb{Z}$ and $\omega : \mathbb{R} \to \mathbb{R}$. The derivative ω' of ω is called **group velocity**. We calculate

$$\begin{aligned}
z_x(x,t) &= ike^{i(kx-\omega(k)t)} = ikz(x,t),\\
z_{xx}(x,t) &= (ik)^2 e^{i(kx-\omega(k)t)} = -k^2 z(x,t),\\
z_t(x,t) &= -i\omega(k)e^{i(kx-\omega(k)t)} = -i\omega(k)z(x,t),\\
z_{tt}(x,t) &= (-i\omega(k))^2 e^{i(kx-\omega(k)t)} = -\omega^2(k)z(x,t).
\end{aligned}$$

Thus $z(x,t) = e^{i(kx-\omega(k)t)}$ satisfies the wave equation $z_{tt} = \vartheta^2 z_{xx}$ for $\vartheta = \frac{\omega(k)}{k}$. If we assume $\omega(k) = ck$ then $\omega'(k) = c$, i.e., the group velocity equals the phase velocity and $\vartheta = c$. Then we obtain again the PDE

$$z_x - \frac{1}{c} z_t = 0.$$

But if we work for instance with $\omega(k) := c(k - \beta k^3)$ with $\beta \in \mathbb{R}^{\neq 0}$, we will get dispersion, changing the underlying PDEs: We now get

$$z_t = -i\omega(k)e^{i(kx-\omega(k)t)} = -ic(k-\beta k^3)z(x,t)$$

and compute in addition

$$z_{xxx}(x,t) = (ik)^3 e^{i(kx-\omega(k)t)} = -ikz(x,t).$$

Therefore we obtain the PDE

$$z_x + \beta z_{xxx} + \frac{1}{c} z_t = 0.$$

Setting $\rho(x,t) := \frac{1}{c} z(x.t)$ and $j(x,t) := z(x,t) + \beta z_{xx}(x,t)$, this PDE becomes

$$j_x + \rho_t = 0.$$

In this context, ρ is usually called **density** and j is said to be the **current**. In physics, an identity of the form $j_x + \rho_t = 0$ is often called **conservation law** or **continuity equation**. If we modify the current by adding a nonlinear term, i.e., if we consider

$$j(x,t) := z(x,t) + \beta z_{xx}(x,t) + \frac{\alpha}{2} z^2(x,t)$$

with $\alpha \in \mathbb{R}^{\neq 0}$, we obtain a PDE of the form

$$\frac{1}{c} z_t + z_x + \beta z_{xxx} + \alpha z z_x = 0$$

Applying the variable change $x \mapsto x - ct$ and rescaling leads to

$$u_t - 6uu_x + u_{xxx} = 0.$$

This equation is called **Korteweg-de Vries equation**, short **KdV**, and is named after the Dutch mathematicians Diederik Johannes Korteweg (1848 – 1941) and Gustav de Vries (1866 – 1934) who found in 1895 an explicit solution describing a ‘soliton wave’.

Theorem 4.3 ([**Korteweg** & **de Vries**]).

$$u(x,t) := -\frac{2\lambda^2}{\cosh^2(\lambda(x - 4\lambda^2 t - \tau_0))}$$

with parameter $\lambda \in \mathbb{R}$ and position shift $\tau_0 \in \mathbb{R}$ is an explicit soliton solution of the KdV equation $u_t - 6uu_x + u_{xxx} = 0$.

Proof. Calculate the derivatives of u and verify the equation. A constructive proof, i.e., how to come up with this formula, can be found in [**Dunajski**, Section 2.3.1]. □

We really need *all* terms in $u_t - 6uu_x + u_{xxx} = 0$ to obtain solutions with soliton behaviour:

Remark 4.4. *1) Solutions of $u_t - 6uu_x = 0$ that form a 'nice' soliton at time $t = 0$ will break ('shock') at some time $t_0 > 0$ due to discontinuities of the first derivatives.*
2) $u_t + u_{xxx} = 0$ leads to dispersion, i.e., a 'nice' soliton at time $t = 0$ will dilute at some time $t_0 > 0$ into a bigger wave with preceeding and following smaller waves.

The next breakthrough came at the end of the 1960s when, in a series of papers, [**Miura**], [**Miura** & **Gardner** & **Kruskal**], [**Su** & **Gardner**], [**Gardner**], [**Kruskal** & **Miura** & **Gardner** & **Zabusky**], [**Gardner** & **Greene** & **Kruskal** & **Miura**] found more soliton solutions and conservation laws, in particular solitons of different size and speed either passing through each other from opposite directions or overtaking each other – without changing the shape except while 'overlapping'. There are several videos on YouTube demonstrating this phenomenon, see for example

`https://www.youtube.com/watch?v=v5MGNcCnuE4`
`https://www.youtube.com/watch?v=H4rN3Wr4ctw`

There are also other PDEs that admit conservation laws and integrability properties similar to the KdV equation:

Example 4.5. *The* **Sine-Gordon equation**

$$u_{tt} - u_{xx} + \sin(u) = 0$$

is used when studying surfaces of constant negative curvature.

A YouTube video describing an 'artistic' soliton solution of the Sine-Gordon equation can be found here:

`https://www.youtube.com/watch?v=SAbQ4MvDqEE`

EXAMPLE 4.6. *The* **nonlinear Schrödinger equation**

$$iu_t = -\frac{1}{2}u_{xx} + \kappa |u|^2 u$$

describes propagation of light.

Moreover

EXAMPLE 4.7. *The* **Camassa-Holm equation**

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

describes waves in shallow water.

PDEs defined over a discrete space are often called **lattices**. An important example was found by the Japanese physicist Morikazu Toda (1917 – 2010) in 1967:

EXAMPLE 4.8. *The* **Toda lattice** *is given by*

$$\begin{cases} q_t(n,t) = p(n,t), \\ p_t(n,t) = e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))} \end{cases}$$

where $n \in \mathbb{Z}$. It describes a discrete chain of particles, parametrized by $n \in \mathbb{Z}$, that interact with their adjacent left and right 'neighbours'.

Videos of soliton solutions of the Toda lattice can be found on Gerald Teschl's webpage

`https://www.mat.univie.ac.at/~gerald/ftp/book-jac/toda.html`

There are various techniques to study these equations. The most popular are

- Hamiltonian formalism,
- Integrals of motion as in the finite dimensional case,
- Hierarchies of equations,

- Scattering,
- Lax pairs and Lax equation.

In the next section, we will get a taste of the first three items. For the last two, we refer the reader to the literature, for example to [**Batlle**], [**Dunajski**], and [**Miwa** & **Jimbo** & **Date**].

4.2 Hamiltonian formalism and first integrals of the KdV equation

In this section, we follow [**Abraham** & **Marsden**, Example 5.5.7] and drop almost all mathematical rigor and compute purely formally to get a quick impression of the Hamiltonian formalism in infinite dimension. Let us just remark that the underlying space of real-valued functions in one variable $\mathcal{E}$ needs to admit at least three weak derivaties to accommodate for $u_t - 6uu_x + u_{xxx} = 0$. Moreover the boundary term when integrating by parts need to vanish, i.e., $\int u_x(x)\, v(x)\, dx = \int u(x)\, v_x(x)\, dx$ should always hold true.

We refer to [**Batlle**], [**Dunajski**], and [**Miwa** & **Jimbo** & **Date**] for the necessary theorie of (pseudo)differential operators and suitable Hilbert and/or Banach spaces.

We equip the function space $\mathcal{E}$ with the symplectic form

$$\omega_g(u, v) := \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} v(x)u(y) - u(x)v(y)\, dy \right) dx$$

where $g \in \mathcal{E}$ is a function $g : \mathbb{R} \to \mathbb{R}$ and $u, v \in T_g\mathcal{E}$, i.e., $x \mapsto u(x) \in T_{g(x)}\mathbb{R} \simeq \mathbb{R}$ and $x \mapsto v(x) \in T_{g(x)}\mathbb{R} \simeq \mathbb{R}$ are vector fields along $x \mapsto g(x)$. Neglecting the foot point g, we consider u and v in the following as functions $u, v : \mathbb{R} \to \mathbb{R}$. Note that ω is a so-called *weak* symplectic form such that we normally should be very careful when switching between the form and Hamiltonian vector fields. Consider the function

$$H : \mathcal{E} \to \mathbb{R}, \qquad H(g) := \int_{-\infty}^{\infty} g^3(x) + \frac{1}{2}(g_x)^2(x)\, dx \tag{4.9}$$

whose Hamiltonian vector field X^H is indirectly defined via (attention with the sign convention!)

$$\omega(X^H, v) = dH(v).$$

Lemma 4.10. *The Hamiltonian vector field X^H of $H : (\mathcal{E}, \omega) \to \mathbb{R}$ defined in* (4.9) *is given by*

$$X^H(g) = \partial_x(3g^2 - g_{xx}) = 6gg_x - g_{xxx}.$$

Proof. We calculate first

$$\begin{aligned} dH|_g(v) &= \frac{d}{ds}\Big|_{s=0} H(g + sv) \\ &= \frac{d}{ds}\Big|_{s=0} \int_{-\infty}^{\infty} (g + sv)^3(x) + \frac{1}{2}((g + sv)_x)^2(x)\, dx \end{aligned}$$

This is a parameter depending integral where we – assuming the integrand to be sufficiently well-behaved – may switch the order of integration and differentiation:

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{d}{ds}\Big|_{s=0} \left((g + sv)^3(x) + \frac{1}{2}((g + sv)_x)^2(x)\right) dx \\ &= \int_{-\infty}^{\infty} \frac{d}{ds}\Big|_{s=0} \Big(g^3(x) + 3g^2(x)sv(x) + 3g(x)s^2v^2(x) \\ &\qquad + \frac{1}{2}(g_x)^2(x) + g_x(x)sv_x(x) + \frac{1}{2}s^2(v_x)^2(x)\Big)\, dx \\ &= \int_{-\infty}^{\infty} 3g^2(x)v(x) + g_x(x)v_x(x) \end{aligned}$$

Finally conclude by means of integration by parts

$$= \int_{-\infty}^{\infty} \left(3g^2(x) - g_{xx}(x)\right) v(x)\, dx.$$

Now abbreviate $(X^H(g))(z) =: X^H(z) \in T_{g(z)}\mathbb{R} \simeq \mathbb{R}$ and consider

$$\begin{aligned}\omega_g(X^H(g), v) &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{x} v(x)X^H(y) - X^H(x)v(y)\, dy\, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} v(x)\left(\int_{-\infty}^{x} X^H(y)\, dy\right) - X^H(x)\left(\int_{-\infty}^{x} v(y)\, dy\right) dx\end{aligned}$$

and obtain with integration by parts

$$\begin{aligned}&= \frac{1}{2}\cdot 2\int_{-\infty}^{\infty} v(x)\left(\int_{-\infty}^{x} X^H(y)\, dy\right) dx \\ &= \int_{-\infty}^{\infty} v(x)\left(\int_{-\infty}^{x} X^H(y)\, dy\right) dx.\end{aligned}$$

Comparison of both sides of $\omega_g(X^H(g), v) = dH|_g(v)$ yields

$$\int_{-\infty}^{x} X^H(y)\, dy = 3g^2(x) - g_{xx}(x)$$

such that we obtain

$$(X^H(g))(x) = X^H(x) = \partial_x(3g^2(x) - g_{xx}(x)).$$

□

We conclude

Corollary 4.11. *The function H in* (4.9) *gives rise to the Hamiltonian equation*

$$\partial_t u = X^H(u) = 6uu_x - u_{xx}$$

where $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. This equation is equivalent to the standard Korteweg-de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

Proof. Since we want to observe the change in time we have to 'move' $g \in \mathcal{E}$ by means of a time variable $t \in \mathbb{R}$. Therefore replace $g : \mathbb{R} \to \mathbb{R}$ by $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ seen as $u(x,t) = (u(x))(t)$. We find

$$\partial_t u = X^H(u) = 6uu_x - u_{xx}$$

which is equivalent to $u_t - 6uu_x + u_{xxx} = 0$, i.e., we regain the Korteweg-de Vries equation. □

Moreover

REMARK 4.12. *For $\rho(x,t) := u(x,t)$ and $j(x,t) := -3u^2(x,t) + u_{xx}(x,t)$ we obtain the conservation law*

$$\rho_t + j_x = 0.$$

More generally

REMARK 4.13. *Let f be a suitably differentiable function in several variables. The Hamiltonian $H : (\mathcal{E}, \omega) \to \mathbb{R}$ given by*

$$H(g) := \int_{-\infty}^{\infty} f(g(x), g_x(x), g_{xx}(x), \dots)\, dx$$

has as Hamiltonian vector field

$$X^H(g) = \partial_x \left(\frac{\delta f}{\delta g} \right)$$

where $\frac{\delta f}{\delta g} := \partial_g f - \partial_x \left(\partial_{g_x} f \right) - \partial_x^2 \left(\partial_{g_{xx}} f \right) - \dots$.

PROOF. Verify the identity

$$\omega_g(X^H(g), v) = \int_{-\infty}^{\infty} \frac{\delta f}{\delta g}(x)\, v(x)\, dx = dH|_g(v).$$

□

Now we want to study integrability notions of the Korteweg-de Vries equation. Define the **Poisson bracket** of two functions $K, L : \mathcal{E} \to \mathbb{R}$ as

$$\{K, L\} := \omega(X^K, X^L).$$

We will see that the KdV equation is in fact only one item within a whole family of similarly generated partial differential equations. This was discovered and studied in a series of papers by [**Miura**], [**Miura** & **Gardner** & **Kruskal**], [**Su** & **Gardner**], [**Gardner**], [**Kruskal** & **Miura** & **Gardner** & **Zabusky**], [**Gardner** & **Greene** & **Kruskal** & **Miura**] resulting in:

THEOREM 4.14. *For $g \in \mathcal{E}$, set $X_1(g) := g_x$ and $f_1(g) := \frac{1}{2}g^2$ and let $a, b \in \mathbb{R}$. Abbreviate $\mathscr{D} := \partial_x$ which has (on a suitable Hilbert space) an inverse integration operator denoted by $\mathscr{D}^{-1}$. For $j \geq 2$, define*

$$\begin{aligned} X_j(g) &:= (ag + a\mathscr{D}g\mathscr{D}^{-1} + b\mathscr{D}^2)X_{j-1} \\ &= (ag\mathscr{D} + a\mathscr{D}g + b\mathscr{D}^2)\frac{\delta f_{j-1}}{\delta g}, \end{aligned}$$

i.e., we have $X_j(g) = \partial_x\left(\frac{\delta f_j}{\delta g}\right)$.

1) *Then the family of equations*

$$u_t = X_j(u), \quad \text{for } j \geq 1$$

is called **KdV hierarchy** *or* **higher order KdV equations** *and comprises for $a = 2$, $b = -1$ and $j = 2$ the standard KdV equation $u_t - 6uu_x + u_{xxx} = 0$.*

2) *The higher order KdV equations $u_t = X_j(u)$ are Hamiltonian with Hamiltonian functions $F_j(g) := \int_{-\infty}^{\infty} f_j(g(x))\, dx$ and Hamiltonian vector fields $X^{F_j} = X_j$.*

3) *The higher order KdV equations are integrable in the sense that $\{F_j, F_k\} = 0$ for all $j, k \geq 1$.*

4) *The Hamiltonian H from the standard KdV equation satisfies $H = F_2$ and therefore $\{H, F_k\} = 0$ holds true for all $k \geq 1$. Thus all F_k are integrals of the standard KdV equation.*

PROOF. **1)** $X_1(u) := u_x$ leads to $u_t = X_1(u) = u_x$. Now keep in mind that $u_x = \mathscr{D}u$ and calculate with $a = 2$ and $b = -1$

$$\begin{aligned} u_t &= X_2(u) = (2u + 2\mathscr{D}u\mathscr{D}^{-1} - \mathscr{D}^2)\mathscr{D}u \\ &= 2u\mathscr{D}u + 2\mathscr{D}(u\mathscr{D}^{-1}\mathscr{D}u) - \mathscr{D}^3u \\ &= 2uu_x + 2\partial_x(u^2) - u_{xxx} \\ &= 2uu_x + 4uu_x - u_{xxx} \\ &= 6uu_x - u_{xxx} \end{aligned}$$

which is the standard KdV equation $u_t - 6uu_x + u_{xxx} = 0$.
2) Using the identity $X_j(g) = \partial_x\left(\frac{\delta f_j}{\delta g}\right)$ and applying Remark 4.13 to the Hamiltonian $F_j(g) = \int_{-\infty}^{\infty} f_j(g(x))\, dx$, we obtain $X^{F_j} = X_j$.

3) In order to show integrability we first prove a recurrence relation for the Poisson bracket. We use the short notation $X_j(z) := X_j(g)(z)$ and compute

$$\begin{aligned}
\{F_j, F_k\} &= \omega(X^{F_j}, X^{F_k}) = \omega(X_j, X_k) \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{x} X_k(x)X_j(y) - X_j(x)X_k(y)\, dy\, dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} X_k(x)\left(\int_{-\infty}^{x} X_j(y)\, dy\right) - X_j(x)\left(\int_{-\infty}^{x} X_k(y)\, dy\right) dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} X_k(x)\left(\int_{-\infty}^{x} \partial_x\left(\frac{\delta f_j}{\delta g}(y)\right) dy\right) - X_j(x)\left(\int_{-\infty}^{x} \partial_x\left(\frac{\delta f_k}{\delta g}(y)\right) dy\right) dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} X_k(x)\left(\frac{\delta f_j}{\delta g}(x)\right) - X_j(x)\left(\frac{\delta f_k}{\delta g}(x)\right) dx.
\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}
&= \int_{-\infty}^{\infty} X_k(x)\left(\frac{\delta f_j}{\delta g}(x)\right) dx \qquad (4.15) \\
&= \int_{-\infty}^{\infty} \left((ag\mathscr{D} + a\mathscr{D}g + b\mathscr{D}^3)\frac{\delta f_{k-1}}{\delta g}\right)(x)\left(\frac{\delta f_j}{\delta g}(x)\right) dx
\end{aligned}$$

and another integration by parts yields

$$\begin{aligned}
&= -\int_{-\infty}^{\infty} \left(\frac{\delta f_{k-1}}{\delta g}(x)\right)\left((a\mathscr{D}g + ag\mathscr{D} + b\mathscr{D}^3)\frac{\delta f_j}{\delta g}\right)(x)\, dx \\
&= -\int_{-\infty}^{\infty} \left(\frac{\delta f_{k-1}}{\delta g}(x)\right) X_{j+1}(x)\, dx \qquad (4.16)
\end{aligned}$$

Reversing the steps that lead to (4.15), we can turn (4.16) into

$$\begin{aligned}
&= -\omega(X_{k-1}, X_{j+1}) = \omega(X_{j+1}, X_{k-1}) \\
&= \{F_{j+1}, F_{k-1}\}
\end{aligned}$$

thus obtaining the relation

$$\{F_j, F_k\} = \{F_{j+1}, F_{k-1}\}.$$

Now we show that in fact $\{F_j, F_k\} = 0$ holds true for all $j, k \geq 1$. First, consider the case $|j - k|$ even and assume w.l.o.g. that $j < k$. By increasing $j \mapsto j + 1$ and decreasing $k \mapsto k - 1$ simultaneously $\frac{|j-k|}{2}$ times, we obtain

$$\{F_j, F_k\} = \{F_{j+1}, F_{k-1}\} = \cdots = \left\{F_{\frac{j+k}{2}}, F_{\frac{j+k}{2}}\right\} = 0.$$

Second, consider the case $|j - k|$ odd and assume w.l.o.g. that $j < k$. By increasing $j \mapsto j + 1$ and decreasing $k \mapsto k - 1$ simultaneously $|j - k|$ times, we obtain

$$\{F_j, F_k\} = \{F_{j+1}, F_{k-1}\} = \cdots = \{F_{k-1}, F_{j+1}\} = \{F_k, F_j\}.$$

Since the Poisson bracket is skewsymmetric, we also have

$$\{F_j, F_k\} = -\{F_k, F_j\}.$$

Both identities together imply $\{F_j, F_k\} = 0$.

4) This follows immediately from **3)**. □

A. Appendix

This appendix recalls various necessary or helpful notions from ODE theory and differential geometry.

A.1 Manifolds and submanifolds

If one wants to work on, say, the sphere $\mathbb{S}^2$ one faces the problem that the sphere is described *implicitly* by the equation

$$\mathbb{S}^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \sum_{i=1}^{k} x_i^2 = 1 \right\}.$$

More precisely, this description of the sphere needs three coordinates although the sphere itself is only 2-dimensional. Therefore two coordinates should suffice to describe the sphere. Unfortunately, it is only possible to parametrise – by means of two coordinates – subsets of the sphere but never the whole sphere if one works with (open subsets of) $\mathbb{R}^2$ as domains of definition for the parametrization. Working with (partially) closed domains is not very practical since one then needs to define differentiability on the boundary of these sets (which is possible but cumbersome).

Let us now find 2-dimensional 'patches' on the sphere that can easily be parametrized by open sets of $\mathbb{R}^2$. Denote by $\mathbb{D}^2$ the open unit disk in $\mathbb{R}^2$.

Consider the upper and lower halfspheres

$$S_u := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 > 0\} \quad \text{and} \quad S_\ell := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 < 0\},$$

with the maps $\psi_u : S_u \to \mathbb{D}^2$ and $\psi_\ell : S_\ell \to \mathbb{D}^2$ given by

$$\psi_u(x_1, x_2, x_3) = (x_1, x_2), \qquad \psi_u^{-1}(y_1, y_2) = \left(y_1, y_2, \sqrt{1 - y_1^2 - y_2^2}\right),$$

$$\psi_\ell(x_1, x_2, x_3) = (x_1, x_2), \qquad \psi_\ell^{-1}(y_1, y_2) = \left(y_1, y_2, -\sqrt{1 - y_1^2 - y_2^2}\right).$$

Moreover, there are the right and left halfspheres

$$S_r := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_2 > 0\} \quad \text{and} \quad S_l := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_2 < 0\},$$

with the maps $\psi_r : S_r \to \mathbb{D}^2$ and $\psi_l : S_l \to \mathbb{D}^2$ given by

$$\psi_r(x_1, x_2, x_3) = (x_1, x_3), \qquad \psi_r^{-1}(y_1, y_3) = \left(y_1, \sqrt{1 - y_1^2 - y_3^2}, y_3\right),$$

$$\psi_l(x_1, x_2, x_3) = (x_1, x_3), \qquad \psi_l^{-1}(y_1, y_3) = \left(y_1, -\sqrt{1 - y_1^2 - y_3^2}, y_3\right).$$

Analogously, we get the front and back halfspheres

$$S_f := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_1 > 0\} \quad \text{and} \quad S_b := \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_1 < 0\}.$$

with the maps $\psi_f : S_f \to \mathbb{D}^2$ and $\psi_b : S_b \to \mathbb{D}^2$ given by

$$\psi_f(x_1, x_2, x_3) = (x_2, x_3), \qquad \psi_f^{-1}(y_2, y_3) = \left(\sqrt{1 - y_2^2 - y_3^2}, y_2, y_3\right),$$

$$\psi_b(x_1, x_2, x_3) = (x_2, x_3), \qquad \psi_b^{-1}(y_2, y_3) = \left(-\sqrt{1 - y_2^2 - y_3^2}, y_2, y_3\right).$$

The union

$$S_u \cup S_\ell \cup S_r \cup S_l \cup S_f \cup S_b = \mathbb{S}^2$$

covers the whole sphere and, on each 'patch' S_i, the sphere is described by the '2-dimensional coordinates' $\psi_i^{-1} : \mathbb{D}^2 \to S_i$ for all $i \in \{u, \ell, r, l, f, b\}$.

Let us now generalize this concept. Recall that a **homeomorphism** is a continuous, bijective map whose inverse is also continuous. A **C^k-diffeomorphism** is a homeomorphism that is C^k-differentiable and whose inverse is also C^k-differentiable.

Definition A.1. *1) An* **m-dimensional C^k-differentiable manifold** *is a topological space M together with open subsets $U_i \subseteq M$ and homeomorphisms $\psi_i : U_i \to \psi_i(U_i) \subseteq \mathbb{R}^m$ such that their composition*

$$\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \subseteq \mathbb{R}^m \to \psi_j(U_i \cap U_j) \subseteq \mathbb{R}^m$$

is a C^k-diffeomorphism for all i, j. In case of $k = 0$, we speak of **topological manifolds**, *in case of $k = \infty$ of* **smooth manifolds**.

2) *The pair* (U_i, ψ_i) *is called a* **(coordinate) chart** *of* M *and* $\psi_j \circ \psi_i^{-1}$ **change of charts** *or* **change of coordinates**. *The union of all charts* (U_i, ψ_i) *is called an* **atlas** *of* M.

The suitable notion for 'subset of manifolds' is the following.

Definition A.2. *Let* M *be an* m*-dimensional* C^k*-manifold and* $k \in \mathbb{N}_0$ *with* $k \leq m$. *A subset* $N \subseteq M$ *is an* **(embedded)** k**-dimensional submanifold of** M *if, for all* $x \in N$, *there exists a chart* (U, ψ) *of* M *with* $p \in U$ *such that* $\psi(U \cap M) = \psi(U) \cap (\mathbb{R}^k \times \{0\}^{m-k}) \subseteq \mathbb{R}^m$.

If $(U_i, \psi_i)_{i \in I}$ is an atlas of a manifold M and if N is a submanifold of M, then the restrictions $(N \cap U_i, \psi_i|_{N \cap U_i})$ form an atlas of N.

A.2 (Co)tangent bundle and differential forms in $\mathbb{R}^m$

If we want to measure the volume of an m-dimensional subset in $\mathbb{R}^m$, we may use the m-dimensional Lebesgue measure. But if the subset has dimension $\tilde{m} < m$, the m-dimensional Lebesgue mesure of this set is zero. It is a priori not clear how to use the $\tilde{m}$-dimensional Lebesgue measure in $\mathbb{R}^m$ to measure the volume of $\tilde{m}$-dimensional subsets since the sets can lie in a very complicated way in $\mathbb{R}^m$.

The idea is to come up with a notion that can handle 'intermediate' volumes. Let us see what kind of properties this notion must have. Given a parallelogram $P_{u,v}$ spanned by two vectors $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ in $\mathbb{R}^2$, its volume is given by

$$\mathrm{vol}(P_{u,v}) = \det(u, v) = u_1 v_2 - u_2 v_1.$$

Shearing the parallelogram by means of the map $(u, v) \mapsto (u + \lambda v, u)$ with $\lambda \in \mathbb{R}$ does not change its volume, algebraically expressed by

$$\det(u + \lambda v, v) = \det(u, v) + \lambda \det(v, v) = \det(u, v) + \lambda \cdot 0 = \det(u, v).$$

Scaling a vectors of the parallelogram by a scalars $\lambda \in \mathbb{R}$ corresponds to the volume transformation

$$\operatorname{vol}(P_{\lambda u,v}) = \det(\lambda u, v) = \lambda \det(u, v) = \lambda \operatorname{vol}(P_{u,v}).$$

This suggests that whatever notion we introduce should satisfy

- **Multilinearity**, i.e., linearity in each variable w.r.t. addition and scalar multiplication of vectors.
- **Skewsymmetry**, i.e., the property corresponding to $\det(u, u) = 0$ or, equivalently, $\det(u, v) = -\det(v, u)$.

Moreover, recall that the infinitesimal change of volume in the transformation formula of integrals

$$\int\limits_V f(y)\, dy = \int\limits_{\psi^{-1}(V)} (f \circ \psi)(x)\, |\det(D\psi|_x)|\; dx.$$

is given by the determinant of the Jacobian of the transformation, i.e., watching the change of volume of the *linearization* is enough to describe the change of volume under the (nonlinear) transformation. This suggests that our new notion should work on the level of functions and derivatives (and therefore tangent spaces).

A.2 Tangent bundle

We recall the definition of the tangent space for subsets of $\mathbb{R}^m$. For $V \subseteq \mathbb{R}^m$, we define the **tangent space of** V **in** $p \in V$ by

$$T_pV := \{p\} \times \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, \varepsilon > 0,\ \exists\, \gamma :]-\varepsilon, \varepsilon[\to V \text{ differentiable}, \\ \gamma(0) = p,\ \gamma'(0) = v \end{array} \right\}$$

which is, by neglecting the **foot point** p, often seen as

$$T_pV \simeq \{v \in \mathbb{R}^n \mid \exists\, \varepsilon > 0,\ \exists\, \gamma :]-\varepsilon, \varepsilon[\to V \text{ diff.}, \gamma(0) = p,\ \gamma'(0) = v\}.$$

Remark A.3. *1) Let $V \subseteq \mathbb{R}^m$ be open. Then $T_pV \simeq \{p\} \times \mathbb{R}^m$ for all $p \in V$.*
2) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and let $r \in \mathbb{R}^n$ be a regular value of f. Then the level set $f^{-1}(r)$ has dimension $m - n$ and $T_p(f^{-1}(r)) = \ker Df|_p$ for all $p \in f^{-1}(r)$.

The **tangent space** of V is given by

$$TV := \bigcup_{p \in V} T_p V$$

and comes with a natural projection

$$\pi : TV \to V,$$

namely sending an element to its foot point. The tangent space TV together with its projection π to the **base space** V is usually called **tangent bundle**. A map $\sigma : V \to TV$ satisfying $\pi(\sigma(p)) = p$ for all $p \in V$ is called a **section** of TV. It is a map that assigns to each $p \in V$ precisely one vector in $T_p V$, i.e., a map of the form

$$v : V \to TV, \quad p \mapsto v_p \in T_p V.$$

REMARK A.4. *The sections of the tangent bundle $TV \to V$ are preciesely the vector fields on V.*

A.2 Cotangent bundle

Within this subsection, we assume that $V \subseteq \mathbb{R}^m$ is open. Let us use the coordinates $(x_1, \ldots, x_m)$ on V and consider a point $p \in V$. The curves

$$t \mapsto p + t\,(0, \ldots, 0, 1, 0, \ldots, 0),$$

with the 1 at the ith position, all equal p at time $t = 0$ and have linearly independent tangent vectors

$$\partial_{x_i}|_p := (0, \ldots, 0, 1, 0, \ldots, 0)^T \in T_p V.$$

Here we keep track of the foot point by writing $|_p$ ('at the foot point p'). Since $\dim(T_p V) = m$, the vectors $\partial_{x_1}|_p, \ldots, \partial_{x_m}|_p$ form a basis of $T_p V$. An arbitrary vector $v_p \in T_p V$ can thus be written as $v_p = \sum_{i=1}^m v_i|_p\, \partial_{x_i}|_p$. The dual vector space

$$(T_p V)^* := \{\alpha_p : T_p V \to \mathbb{R} \mid \alpha_p \text{ linear}\}$$

also has dimension m and we endow it with the 'dual' basis $dx_1|_p, \ldots, dx_m|_p$ by requiring

$$dx_i|_p\,(\partial_{x_j}|_p) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

A functional $\alpha_p \in (T_pV)^*$ can thus be written as $\alpha_p = \sum_{i=1}^m \alpha_i|_p \, dx_i|_p$. Evaluating α_p on v_p using linearity and duality gives

$$\alpha_p(v_p) = \left(\sum_{i=1}^m \alpha_i|_p \, dx_i|_p\right)\left(\sum_{i=1}^m v_i|_p \, \partial_{x_i}|_p\right) = \sum_{i=1}^m \alpha_i|_p v_i|_p \quad \in \mathbb{R}.$$

The union

$$(TV)^* := \bigcup_{p\in V} (T_pV)^*$$

is the **cotangent space** and it also comes with a projection $\pi : (TV)^* \to V$ by sending all elements to their foot points. The cotangent space $(TV)^*$ together with its projection π to the **base space** V is usually called **cotangent bundle**. Maps $\sigma : V \to (TV)^*$ satisfying $\pi(\sigma(p)) = p$ are called **sections**. This are maps that assigns to each $p \in V$ precisely one functional in T_pV, i.e., maps of the form

$$\alpha : V \to (TV)^*, \quad p \mapsto \alpha_p \in (T_pV)^*,$$

meaning $\alpha_p : T_pV \to \mathbb{R}$ is linear for all $p \in V$.

A.2 Differential forms in $\mathbb{R}^m$

Now we construct maps from the $k$fold product of the tangent space to $\mathbb{R}$ that are multilinear and skewsymmetric (often called **alternating** instead of skewsymmetric).

Let $V \subseteq \mathbb{R}^m$ be open with coordinates $(x_1, \dots, x_m)$ and, for all $p \in V$, endow T_pV with the basis $\partial_{x_1}|_p, \dots, \partial_{x_m}|_p$ and $(T_pV)^*$ with the basis $dx_1|_p, \dots, dx_m|_p$.

Notation A.5. *1) Functions $f : V \to \mathbb{R}$ are called* 0**-forms** *on V. Evaluated at a point $p \in V$, a 0-form is a scalar $f(p) \in \mathbb{R}$.*
2) Sections $\alpha : V \to (TV)^*$ are called 1**-forms** *on V. Evaluated at a point $p \in V$, a 1-form is a functional $\alpha_p \in (T_pV)^*$, meaning, a linear map $\alpha_p : T_pV \to \mathbb{R}$.*

Now we introduce an operation that will produce forms of higher order.

DEFINITION A.6. *1) The* **exterior product** *or* **wedge product** *of a* 0-*form f and a* 1-*form α on V is given by the* 1-*form*

$$f \wedge \alpha := f\alpha$$

defined by $(f \wedge \alpha)_p := (f\alpha)_p := f(p)\alpha_p \in (T_pV)^*$ for all $p \in V$.

2) The **exterior product** *or* **wedge product** *of two* 1-*forms α, β on V is given by the* 2-**form** *$\alpha \wedge \beta$ on V that is defined via*

$$(\alpha \wedge \beta)_p(u_p, v_p) := \alpha_p(u_p)\beta_p(v_p) - \alpha_p(v_p)\beta_p(u_p)$$

for all $p \in V$ and all $u_p, v_p \in T_pV$.

The wedge product for 0- and 1-forms satisfies the following properties:

- $f \wedge \alpha = \alpha \wedge f$ for all 0-forms f and all 1-forms α.
- $\alpha \wedge (f\beta) = f(\alpha \wedge \beta)$ for all 0-forms f and all 1-forms α and β.
- $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ for all 1-forms α, β, γ.
- $\alpha \wedge \beta = -\beta \wedge \alpha$ for all 1-forms α, β.

These properties lead to the representation of a 2-form ω as

$$\omega = \sum_{1 \le i_1 < i_2 \le m} \omega_{i_1 i_2} dx_{i_1} \wedge dx_{i_2}$$

with $\omega_p = \sum_{1 \le i_1 < i_2 \le m} \omega_{i_1 i_2}(p)(dx_{i_1} \wedge dx_{i_2})|_p$ for all $p \in V$, i.e., $\omega_{i_1 i_2} : V \to \mathbb{R}$ is a function for all indices $1 \le i_1 < i_2 \le m$. In particular, we find (neglecting the foot point notation for the moment)

$$(dx_i \wedge dx_j)(u, v) = u_i v_j - u_j v_i$$

which recovers the determinant of the vectors $u = (u_1, \ldots, u_m)^T$ and $v = (v_1, \ldots, v_m)^T$ in case $i = 1$, $j = 2$, and $m = 2$.

Iterating the wedge product with 0-, 1-, and 2-forms leads to arbitrary k-forms. More precisely

DEFINITION AND PROPOSITION A.7. *Let $k, \ell \in \mathbb{N}_0$. The* **exterior product** *or* **wedge product** *of a k*-**form** *α and a ℓ*-**form** *β on V is defined as the $(k + \ell)$-form $\alpha \wedge \beta$ given by*

$$(\alpha \wedge \beta)_p(\mathbf{u}_p, \mathbf{v}_p) := \alpha_p(\mathbf{u}_p)\beta_p(\mathbf{v}_p) - \alpha_p(\mathbf{v}_p)\beta_p(\mathbf{u}_p)$$

for all $p \in V$ and all $\mathbf{u}_p \in (T_pM)^k$ and all $\mathbf{v}_p \in (T_pV)^\ell$. We have

- *$f \wedge \alpha = \alpha \wedge f = f\alpha$ for all 0-forms f and all k-forms α.*
- *$\alpha \wedge (f\beta) = f(\alpha \wedge \beta)$ for all 0-forms f, all k-forms α, and all ℓ-forms β.*

- *$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ for all k-forms α and all ℓ-forms β, γ.*
- *$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$ for all k-forms α and all ℓ-forms β.*

These properties lead to the representation of a k-form α as

$$\alpha = \sum_{1 \le i_1 < \cdots < i_k \le m} \alpha_{i_1 \ldots i_2} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

with $\alpha_p = \sum_{1 \le i_1 < \cdots < i_k \le m} \alpha_{i_1 \ldots i_k}(p) \big(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big)|_p$ for all $p \in V$, meaning, $\alpha_{i_1 \ldots i_k} : V \to \mathbb{R}$ is a function for all indices $1 \le i_1 < \cdots < i_k \le m$.

Definition A.8. *Let $V \subseteq \mathbb{R}^m$ be open, $p \in V$, let $k \in \mathbb{N}^{\ge 2}$, and let $\mu_p : (T_p V)^k \to \mathbb{R}$ be a map.*
1) μ_p is **multilinear** *if μ_p is linear in each variable.*
2) μ_p **alternates** *or is* **alternating** *if for all $u_p, v_p \in T_p V$*
$$\mu_p(\ldots, u_p, v_p, \ldots) = -\mu_p(\ldots, v_p, u_p, \ldots).$$

We set

$$\Lambda^k((T_p V)^*) := \Big\{ \mu_p : (T_p V)^k \to \mathbb{R} \,\Big|\, \mu_p \text{ multilinear and alternating} \Big\}$$

and

$$\Lambda^k(V) := \Lambda^k((TV)^*) := \bigcup_{p \in V} \Lambda^k((T_p V)^*).$$

This space also carries a projection $\pi : \Lambda^k(V) \to V$ by sending multilinear, alternating maps μ_p to their footpoint p. Sections of the bundle $\Lambda^k(V) \to V$ are maps $\sigma : V \to \Lambda^k(V)$ satisfying $\pi(\sigma(p)) = p$.

Remark A.9. *1) k-forms can be seen as multilinear, alternating maps. More precisely, a k-form α is a section*
$$\alpha : V \to \Lambda^k(V), \quad p \mapsto \alpha_p \in \Lambda^k((T_p V)^*).$$
2) Alternating implies that, on spaces of dimension m, all k-forms with $k > m$ vanish.

We will need the following definition for the definition of symplectic forms.

Definition A.10. *A 2-form ω on $V \subseteq R^m$ is* **nondegenerate** *if, for all $p \in V$, $\omega_p(u_p, v_p) = 0$ for all $v_p \in T_p V$ implies $u_p = 0$.*

In local coordinates, a 2-form ω on an open $V \subseteq \mathbb{R}^m$ can be represented by a skewsymmetric $(m \times m)$-matrix Ω defined by the equation

$$\omega_p(u_p, v_p) = (u_p)^T \Omega_p v_p \qquad \forall\, p \in V,\ \forall\, u_p, v_p \in T_p V. \tag{A.11}$$

Being nondegenerate means in terms of linear algebra that $\det(\Omega_p) \neq 0$ for all $p \in V$.

Lemma A.12. *Nondegenerate 2-forms only exist on even dimensional spaces.*

Proof. Let ω be a nondegenerate 2-form on $V \subseteq \mathbb{R}^m$ and represent it, for all $p \in V$, by its skewsymmetric $(m \times m)$-matrix as in (A.11). Then the skewsymmetry implies $\det(\Omega_p) = (-1)^m \det(\Omega_p)$ for all $p \in V$. Thus $\det(\Omega_p) \neq 0$ is only possible if m is even. □

The following map is of high importance in (co)homology theory since it represents the boundary operator of De Rham cohomology.

Definition and Proposition A.13.
Let $k \in \mathbb{N}_0$. The operator $d : \Lambda^k(V) \to \Lambda^{k+1}(V)$ given on 0-forms f by

$$(df)_p := \sum_{i=1}^{m} \partial_{x_i} f(p) dx_i|_p$$

for all $p \in V$ and on k-forms $\alpha = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \alpha_{i_1 \ldots i_2} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ by

$$\sum_{1 \leq i_1 < \cdots < i_k \leq m} (d\alpha_{i_1 \ldots i_2}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

is called **exterior derivative** *and satisfies $d \circ d = 0$.*

The following types of forms are the 'building blocks' of so-called chain complexes in cohomology theory.

Definition A.14. *A k-form α is* **closed** *if $d\alpha = 0$. A k-form is* **exact** *if there exists a $(k-1)$-form β with $d\beta = \alpha$.*

Note that $d \circ d = 0$ implies that exact forms are closed.

LEMMA A.15 (**Poincaré**). *Locally, all closed forms are exact.*

PROOF. See for example [**Warner**] or [**Petersen**] or [**Bott** & **Tu**]. □

Given a map, there is a way to construct new k-forms out of old ones. If the map is a diffeomorphism, we can invert the construction.

DEFINITION AND PROPOSITION A.16. *Let $U, V \subseteq \mathbb{R}^m$ be open and $\psi : U \to V$ surjective and differentiable. Let α be a k-form on V. The* **pullback** *of α under ψ defines the k-form $\psi^*\alpha$ on U via*

$$(\psi^*\alpha)_p((u_1)_p, \ldots, (u_k)_p) := \alpha_{\psi(p)}(D\psi|_p.(u_1)_p, \ldots, D\psi|_p.(u_k)_p)$$

for all $p \in U$ and all $(u_1)_p, \ldots, (u_k)_p \in T_pU$. The pullback satisfies the following properties:

- $\psi^* f = f \circ \psi$ *for all* 0*-forms $f : V \to \mathbb{R}$.*
- $\psi^*(c\alpha) = c(\psi^*\alpha)$ *for all constants $c \in \mathbb{R}$ and all k-forms α.*
- $\psi^*(\alpha + \beta) = \psi^*\alpha + \psi^*\beta$ *for all k-forms α, β.*
- $d(\psi^*\alpha) = \psi^*(d\alpha)$ *for all surjective, differentiable $\psi : U \to V$ and all k-forms α on V.*

A.3 Differential forms on manifolds

This section still needs to be written... Roughly we can already say that one tries to 'pull back' all definitions from $\mathbb{R}^m$ to the manifold M by means of the charts (U_i, ψ_i). Since there may be several charts covering one point in the manifold, one has to show that this is welldefined... ...which is a bit tedious...

A.4 Flows of autonomous ODEs

Let M be an m-dimensional smooth manifold, $\zeta \in M$ and X a vector field on M that is locally Lipschitz. Then the initial value problem

$$z' = X(z), \qquad z(0) = \zeta$$

has a unique maximal solution defined on an interval I_ζ containing 0. We denote this solution by $z_\zeta : I_\zeta \to M$. Instead of only tracking the 'time variable' $t \in I_\zeta$, we can also consider the dependence of z_ζ on the 'space variable' ζ, meaning, we may study the mapping $(t, \zeta) \mapsto z_\zeta(t)$. It satisfies $z_\zeta(0) = \zeta$ and $\partial_t z_\zeta(t) = X(z_\zeta(t))$. This motivates

Definition and Proposition A.17. *Let M be a smooth manifold and X a C^k-vector field on M with $k \geq 1$. Set $V := \bigcup_{\zeta \in M} I_\zeta \times \{\zeta\} \subseteq \mathbb{R} \times M$. Then V is open and the* **(local) flow of** $z' = X(z)$ *is given by the C^k-mapping*

$$\Phi : V \to M, \quad \Phi_t(\zeta) := \Phi(t, \zeta) := z_\zeta(t).$$

It satisfies

(i) $\Phi_0(\zeta) = \zeta \quad \forall\, \zeta \in M,$

(ii) $\Phi_{t+s}(\zeta) = \Phi_t(\Phi_s(\zeta)) \quad \forall\, \zeta \in M,\ \forall\, s, (t+s) \in I_\zeta.$

Switching from the $\Phi(t, \zeta)$ to the $\Phi_t(\zeta)$ notation is motivated by the otherwise rather cumbersome formulation of properties (i) and (ii). The property $\Phi_t \circ \Phi_s = \Phi_{t+s}$ is often called **flow property**. It means that flowing first for time s and then additionally for time t is the same as flowing directly for time $(t+s)$. The property $\Phi_0 = \mathrm{Id}$ says that *flowing for time $t = 0$ simply* means *stay where you are*.

$\Phi_t(\zeta) = z_\zeta(t)$ has the following geometric meaning: consider $\zeta \in M$ and follow the solution z_ζ from $\zeta = z_\zeta(0)$ for time t. The point reached by the solution z_ζ at time t is $\Phi_t(\zeta)$. We often use the following short notation for properties (i) and (ii):

$$\Phi_0 = \mathrm{Id} : M \to M \qquad \text{en} \qquad \Phi_t \circ \Phi_s = \Phi_{t+s}.$$

Moreover, it is often useful to fix t and consider the induced map

$$\Phi_t : M \to M$$

that shows the 'evolution' of the differential equation when 'jumping' directly from time 0 to time t. In this notation, compare the meaning of the (partial) derivatives:

$$D\Phi_t|_p = \partial_p \Phi|_{(t,p)} \qquad \text{en} \qquad \left.\frac{d}{dt}\right|_{t=0} \Phi_t(p) = \partial_t \Phi|_{(t,p)}.$$

The flow property implies

COROLLARY A.18. *$\Phi_t : M \to M$ is invertible and $(\Phi_t)^{-1} = \Phi_{-t}$.*

An autonomous differential equation or its flow is called **complete** if $I_\zeta = \mathbb{R}$ for all $\zeta \in M$. For example, flows on compact (sub)manifolds without boundary like spheres or tori are always complete.

Around regular points, the flow can be 'ironed' into a nice normal form:

THEOREM A.19 (**Flow box theorem**). *Let $x' = f(x)$ be an autonomous ODE with flow Φ on a smooth manifold M and let $z \in M$ be a regular point, i.e., $f(z) \neq 0$. Then there exists a neighbourhood $U \subseteq M$ of z and a neighbourhood $\tilde{U} \subseteq \mathbb{R}^n$ of the origin and a coordinate transformation $\psi : U \to \tilde{U}$ such that $\psi(z) = 0$ and $x' = f(x)$ is transformed on $\tilde{U}$ into*

$$\begin{cases} y_1' &= 1, \\ y_2' &= 0, \\ \vdots & \\ y_n' &= 0, \end{cases}$$

which has flow $\tilde{\Phi}_t(\tilde{z}) = \tilde{z} + te_1$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$, i.e., the transformed flow is parallel to the y_1-axis.

PROOF. See for example [**Hohloch2**] or [**Teschl**]. □

This means in particular that the flow of an ODE near regular points always looks the same up to a coordinate transformation.

DEFINITION A.20. *A fixed point z of an ODE $x' = f(x)$ is* **hyperbolic** *if $Df|_z$ has no eigenvalues of the form $i\sigma$ with $\sigma \in \mathbb{R}$.*

Near hyperbolic fixed points, the ODE is C^0-conjugate to its linearization:

THEOREM A.21 (**Hartman-Grobman**). *Let $x' = f(x)$ be an autonomous ODE with flow Φ on a smooth, n-dimensional manifold M. Let $z \in M$ be a hyperbolic fixed point. Denote the flow of $y' = Df|_z.y$ on $T_zM \simeq \mathbb{R}^n$*

by Ψ. Then there exist open neighbourhoods $U \subseteq M$ of z and $V \subseteq \mathbb{R}^n$ of the origin $\mathbf{0} \in \mathbb{R}^n$ and a homeomorphism $h : U \to V$ such that

$$h(\Phi_t(x)) = \Psi_t(h(x))$$

for all $t \in \mathbb{R}$ and $x \in U$ with $\Phi_t(x) \in U$.

Proof. See for example [**Palis** & **de Melo**]. □

A.5 Some results from (functional) analysis

We recall some useful notions from global and functional analysis. For more details, see for example [**Hohloch1**] and the references therein.

Definition A.22. *Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be normed vector spaces over a field $\mathbb{F}$ and $T : X \to Y$ a map.*

1) T is a **linear operator** *if*

$$T(\lambda x + \tilde{x}) = \lambda T(x) + T(\tilde{x}) \quad \forall\, x, \tilde{x} \in X, \ \forall\, \lambda \in \mathbb{F}.$$

2) A linear operator T is **bounded** *if*

$$\exists\, C > 0 : \quad \|T(x)\|_Y \le C\, \|x\|_X \quad \forall\, x \in X.$$

3) The set of linear bounded operators from X to Y is denoted by $\mathscr{B}(X, Y)$.

4) The **operator norm** *of $T \in \mathscr{B}(X, Y)$ is given by*

$$\|T\| := \sup\{\|T(x)\|_Y \mid x \in X, \|x\|_X \le 1\}.$$

5) The space $\mathscr{B}(X, \mathbb{R})$ and $\mathscr{B}(X, \mathbb{C})$ are the real and complex **dual space** *of X. Sometimes they are denoted by X^* or X' in the literature.*

Moreover

Definition A.23. *Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be Banach spaces and $U \subseteq X$ open. $f : U \to Y$ is* **(Fréchet) differentiable in** *$x \in U$ if there exists*

$T_x \in \mathscr{B}(X, Y)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - T_x(h)\|_Y}{\|h\|_X} = 0.$$

If such a T_x exists it is usually denoted by $Df|_x$ or $df(x)$ and called the **(Fréchet) derivative** *of f in $x \in U$. The map f is* **(Fréchet) differentiable** *if f is (Fréchet) differentiable in all $x \in U$.*

Higher regularity is defined as follows.

DEFINITION A.24. *Let X, Y be Banach spaces, $U \subseteq X$ open and $f : U \to Y$ Fréchet differentiable. f is C^k for $k \in \mathbb{N}^{\geq 1}$ if the following map is C^{k-1} :*

$$U \to \mathscr{B}(X, Y), \qquad x \mapsto Df|_x$$

The *Inverse function theorem* from the finite dimensional setting generalizes verbatim to Banach spaces:

THEOREM A.25 (**Inverse function theorem**). *Let X, Y be Banach spaces and $U \subseteq X$ open, and $f : U \to Y$ a C^k-map with $k \in \mathbb{N}^{\geq 1}$. Let $x_0 \in U$ with $Df|_{x_0} \in \mathscr{B}(X, Y)$ bijective. Then there exists an open neighbourhood $U_0 \subseteq U$ of x_0 such that the restriction $f|_{U_0} : U_0 \to Y$ is injective, $V_0 := f(U_0)$ is open in Y and*

$$(f|_{U_0})^{-1} : V_0 \to U_0 \text{ is } C^k \quad \text{and} \quad (Df^{-1})|_y = \left(Df|_{f^{-1}(y)}\right)^{-1} \quad \forall\, y \in V_0.$$

PROOF. Cf. appendix of [**McDuff** & **Salamon**]. □

When generalizing the *Implicite function theorem* from the finite dimensional setting, things get more interesting. First we define

DEFINITION A.26. *Let X, Y be Banach spaces, $U \subseteq X$ open and pathconnected, and $f : U \to Y$ a C^k-map.*
1) f is a **Fredholm map** *if $Df|_x$ is a Fredholm operator for all $x \in U$.*
2) If f is a Fredholm map we define its **Fredholm index** *via*

$$\mathrm{Ind}(f) := \mathrm{Ind}(Df|_x) \quad \forall\, x \in U.$$

Moreover, we need

DEFINITION A.27. *Let X, Y be Banach spaces, $U \subseteq X$ open and $f : U \to Y$ a C^k-map. $y \in Y$ is a* **regular value** *of f if $Df|_x \in \mathscr{B}(X, Y)$ is surjective for all $x \in f^{-1}(y) \subseteq U$.*

The following theorem displays the geometric implications Fredholm maps have. Note that $\mathbb{R}^n$ with its usual norms is a Banach space and that linear maps between finite dimensional vector spaces are always Banach. So the finite dimensional version of the *implicite function theorem* is well included in the following infinite dimensional version.

THEOREM A.28 (**Implicite function theorem**). *Let X, Y be Banach spaces, $U \subseteq X$ open, $f : U \to Y$ a C^k-map with $k \in \mathbb{N}^{\geq 1}$, and $y \in Y$ a regular value of f. Then $M := f^{-1}(y) \subseteq X$ is a C^k-Banach manifold whose tangent space satisfies*

$$T_x M = \ker Df|_x \quad \forall\, x \in M.$$

If f is Fredholm then M is a finite dimensional manifold where the dimension of the connected component of M containing x is given by

$$\dim(T_x M) = \dim(\ker(Df|_x)) < \infty.$$

PROOF. Cf. appendix of [**McDuff** & **Salamon**]. □

It remains to inquire how 'typical' it is for a value $y \in Y$ to be regular.

THEOREM A.29 (**Sard's Theorem**). *Let X, Y be separable Banach spaces, $U \subseteq X$ open, $f : U \to Y$ a C^k-map with $k \geq \max\{1, \mathrm{Ind}(f) + 1\}$. Then*

$$Y_{reg}(f) := \{y \in Y \mid y \text{ regular value of } f\}$$

is of second Baire category, i.e., it is a countable intersection of open and dense sets.

PROOF. Cf. appendix of [**McDuff** & **Salamon**]. □

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