Integrable Systems: from Poisson to Nijenhuis
Lecture 1

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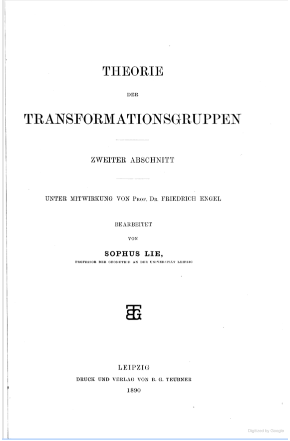
Plan

- Historical remarks: three portraits
- What is Geometry?
- Poisson and Nijenhuis: main definition and simplest examples
- Poisson and Nijenhuis: splitting theorem
- Poisson and Nijenhuis: the most regular case
- Liouville integrability and integrability in quadratures

Siméon Denis Poisson (1781 – 1840)

"Mémoire sur la variation des constantes arbitraires dans les questions de mécanique" presented to the *Institut de France* on October 16, 1809 and published in the *Journal de l'Ecole Polytechnique*, 15e cahier, **8**: 266–344.

Sophus Lie (1842 – 1899)



THEORIE

DER

TRANSFORMATIONSGRUPPEN

ZWEITER ABSCHNITT

UNTER MITWIRKUNG VON PROF. DR. FRIEDRICH ENGEL

BEARBEITET

VON

SOPHUS LIE,

PROFESSOR DER GEOMETRIE AN DER UNIVERSITÄT LEIPZIG

LEIPZIG

DRUCK UND VERLAG VON B. G. TEUBNER

1890

Lie, S. *Theorie der Transformationsgruppen*
Vol. 1 (1888), Vol. 2 (1890), Vol. 3 (1893)

https://en.wikipedia.org/wiki/List_of_important_publications_in_mathematics

Albert Nijenhuis (1926 – 2015)

Dutch-American mathematician who specialised in differential geometry and the theory of deformations in algebra and geometry, and later worked in combinatorics.

Alma mater: University of Amsterdam

Doctoral advisor: Prof. Jan Arnoldus Schouten

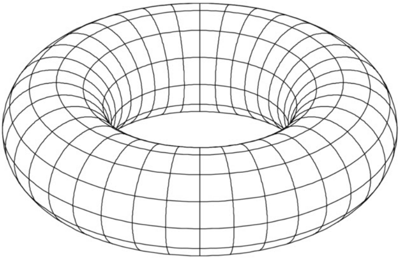
A. Nijenhuis, X_{n-1}-forming sets of eigenvectors, Proc. Kon. Ned. Akad. Amsterdam 54 (1951) 200–212.

References (BKM = AB, A. Konyaev, V. Matveev)

- BKM, Nijenhuis geometry. *Adv. in Math.*, 394: 108001, 2022.
- A. Konyaev, Nijenhuis geometry II: Left-symmetric algebras and linearization problem for Nijenhuis operators. *Diff. Geom. and Appl.*, 74: 101706, 2021.
- BKM, Nijenhuis Geometry III: $\mathfrak{gl}$-regular Nijenhuis operators. *Rev. Mat. Iberoam.*, 40 (2024), no. 1, pp. 155–188.
- BKM, Nijenhuis Geometry IV: conservation laws, symmetries and integration of certain non-diagonalisable systems of hydrodynamic type in quadratures, *Nonlinearity*, 37: 105003, 2024.
- BKM, Stäckel problem for non-diagonal Killing tensors: Yano-Patterson lifts, algebra of strong symmetries and quadratic in momenta integrals, *arXiv:2512.17609*.
- BKM, Duality of operator Frobenius algebras and solution of Eisenhart-Stäckel problem in the non-diagonal case, *arXiv:2604.03195*.

What is GEOMETRY?

Space, manifold M^n $+$ Structure



Structure is usually defined by means of a tensor, like, g_{ij}, ω_{ij}, or P^{ij}

Naively, in coordinates, the geometric structure is defined by means of a matrix $A = \big(a_{ij}(x)\big)$ whose entries depend on coordinates $x = (x^1, \ldots, x^n)$ and satisfy some algebraic and differential conditions.

Definition

A Poisson structure P is a bi-vector field $P = \left(P^{ij}(x)\right)$, $P^{ij} = -P^{ij}$, such that the operation (Poisson bracket) $\{\ ,\ \}$ on the space of smooth functions $C^\infty(\mathrm{M})$:

$$\{f, g\} := P(\mathrm{d}\, f, \mathrm{d}\, g) = \sum_{i,j} P^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{f, \{g, h\}\} + \{f, \{g, h\}\} = 0, \quad \text{for all } f, g, h \in C^\infty(\mathrm{M})$$

or, equivalently,

$$P^{i\alpha} \frac{\partial P^{jk}}{\partial x^\alpha} + P^{j\alpha} \frac{\partial P^{ki}}{\partial x^\alpha} + P^{k\alpha} \frac{\partial P^{ij}}{\partial x^\alpha} = 0.$$

Nijenhuis geometry

Definition

A Nijenhuis structure L is a field of endomorphisms $L = \left(L^i_j(x)\right)$ such that the Nijenhuis identity holds

$$L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta] = 0 \quad \text{for all vector fields } \xi,\eta$$

or, equivalently,

$$L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k} = 0$$

(summation over repeated indices s is assumed).
The vector-values 2-form

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] + [L\xi,L\eta] - L[L\xi,\eta] - L[\xi,L\eta]$$

is known as the Nijenhuis torsion of an operator L.
Alternatively, as a map from "1-forms" to "2-forms": $\mathcal{N}_L : \ \alpha \mapsto \beta$, where

$$\beta(\cdot\,,\cdot) = \mathrm{d}\,({L^*}^2\alpha)(\cdot\,,\cdot) + \mathrm{d}\,\alpha(L\cdot\,,L\cdot) - \mathrm{d}\,(L^*\alpha)(L\cdot\,,\cdot) - \mathrm{d}\,(L^*\alpha)(\cdot\,,L\cdot).$$

Elementary examples: Poisson

- Constant Poisson structure $P = \left(P^{ij}\right)$, $P^{ij} \in \mathbb{R}$.
- Canonical bracket on T^*M: in the canonical coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$

$$\{p_i, x_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = \{x_i, x_j\} = 0, \quad i, j = 1, \ldots, n.$$

- Lie-Poisson bracket

$$\{x_i, x_j\} = \sum_k c_{ij}^k x_k,$$

  where c_{ij}^k is the structure tensor of a certain finite-dimensional Lie algebra.
- Quadratic bracket (just a particular example):

$$\begin{array}{lll} \{a, b\} = ab, & \{a, c\} = ac, & \{a, d\} = 2bc, \\ \{b, c\} = 0, & \{b, d\} = bd, & \{c, d\} = cd. \end{array}$$

- Poisson brackets related to resonances (superintegrability) in Hamiltonian dynamics.

Elementary examples: Nijenhuis

- Constant operator:

$$L(x) = \left(L^i_j \right), \qquad L^i_j \in \mathbb{R}.$$

- Scalar operator:

$$L(x) = f(x) \cdot \mathrm{Id},$$

  where $f(x)$ is an arbitrary smooth function

- Complex structure

- $L_1 = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & x_n \end{pmatrix}, \quad L_2 = \begin{pmatrix} x_1 & 1 & 0 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 \\ x_3 & 0 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 0 & 1 \\ x_5 & 0 & 0 & 0 & 0 \end{pmatrix}$

- $L_3 = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 & 0 \\ x_3 & x_2 & x_1 & 0 & 0 \\ x_4 & x_3 & x_2 & x_1 & 0 \\ x_5 & x_4 & x_3 & x_2 & x_1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 & 0 \\ x_3 & 0 & x_1 & 0 & 0 \\ x_4 & 0 & 0 & x_1 & 0 \\ x_5 & x_4 & x_3 & x_2 & x_1 \end{pmatrix}$

Poisson Geometry: Splitting theorem

Theorem (A. Weinstein, S. Lie)

Every Poisson structure P locally splits into direct product

$$P = P_1 \times P_2$$

where P_1 is non-degenerate and $P_2(p_0) = 0$. More precisely, there exists a coordinate system $p_1, q_1, \ldots, q_k, p_k$ and $z_1, \ldots, z_s$, $2k + s = n$, such that

$$P = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ & & & \boxed{\quad P_2(z) \quad} \end{pmatrix}$$

Sketch of proof

Fact 1. Let $\xi \neq 0$ be a vector field. Then (locally) there exist independent functions $h, f_1, \dots, f_{n-1}$ such that

$$\xi(h) = 1, \quad \xi(f_i) = 0.$$

Equivalently, if we take $h, f_1, \dots, f_n$ as coordinates, then $\xi = (1, 0, \dots, 0)$.

Fact 2. Let ξ_1 and ξ_2 be lin. ind. vector field such that $[\xi_1, \xi_2] = 0$. Then (locally) there exist independent functions $h, g, f_1, \dots, f_{n-2}$ such that

$$\begin{aligned} \xi_1(h) = 1,\ \xi_2(h) &= 0, \\ \xi_1(g) = 0,\ \xi_2(g) &= 1, \\ \xi_1(f_i) = \xi_2(f_i) &= 0. \end{aligned}$$

Equivalently, if we take $g, h, f_1, \dots, f_{n-2}$ as coordinates, then $\xi_1 = (1, 0, \dots, 0)$ and $\xi_2 = (0, 1, 0, \dots, 0)$.

Let $P(p_0) \neq 0$ (otherwise everything is done). Take g such that $\mathcal{X}_g \neq 0$. Then there exists h such that $\mathcal{X}_g(h) = 1$, i.e., $\{g, h\} = 1$.
Next consider $\mathcal{X}_g$ and $\mathcal{X}_h$. We have $[\mathcal{X}_g, \mathcal{X}_h] = \mathcal{X}_{\{g,h\}} = \mathcal{X}_1 = 0$. Then there exist $f_1, \dots, f_{n-2}$ such that

$$\mathcal{X}_h(f_i) = 0, \quad \mathcal{X}_g(f_i) = 0,$$

Equivalently,

$$\{h, f_i\} = \{g, f_i\} = 0.$$

Now take $g, h, f_1, \dots, f_{n-2}$ to be new coordinates. What is the matrix of P in this coordinate system? Answer:

$$\begin{pmatrix} 0 & 1 & 0 \ \dots \ 0 \\ -1 & 0 & 0 \ \dots \ 0 \\ & & \boxed{\{f_i, f_j\}} \end{pmatrix}$$

The Jacobi identity immediately implies that $\{f_i, f_j\} = \phi_{ij}(f_1, \dots, f_{n-2})$. Induction completes the proof.

Splitting theorem

Let $\chi_{L(x)}(t) = \det\big(t\,\mathrm{Id} - L(x)\big) = t^n - \sigma_1(x) t^{n-1} - \sigma_2(x) t^{n-2} - \ldots - \sigma_n(x)$ be the characteristic polynomial of L and

$$\chi_{L(p_0)}(t) = \chi_1(t)\,\chi_2(t)$$

be its factorisation at a point $p_0 \in \mathrm{M}^n$ into two factors with no common roots (over $\mathbb{R}$), $\deg \chi_1 = r$, $\deg \chi_2 = n - r$.

Theorem

There exist local coordinates $(x_1, \ldots, x_r, y_{r+1}, \ldots, y_n)$ *such that*

$$L(x, y) = \begin{pmatrix} \boxed{L_1(x)} & 0 \\ 0 & \boxed{L_2(y)} \end{pmatrix}, \qquad (1)$$

with $\chi_1 = \chi_{L_1}$ *and* $\chi_2 = \chi_{L_2}$.

In particular the distributions $\mathcal{D}_i = \mathrm{Ker}\,\chi_i(L)$ *(*$i = 1, 2$*) are integrable.*

In other words, L *splits into a direct sum of two Nijenhuis operators:* $L(x, y) = L_1(x) \oplus L_2(y)$.

Proof of Splitting Theorem

Step 1. If L is Nijenhuis, then L^2 is Nijenhuis too.

Step 2. If L is Nijenhuis, then $p(L)$ is Nijenhuis, where $p(\cdot)$ is a polynomial (with constant coefficients).

Step 3. If L is Nijenhuis, then any convergent power series $\sum a_k L^k$, $a_k \in \mathbb{R}$, i.e. any real analytic function $f(L)$ is Nijenhuis, e.g. $\exp L$, $\sin L$, ...

Step 4. Now let $T_x\mathrm{M} = \mathcal{D}_1 \oplus \mathcal{D}_2$ be the decomposition of the tangent space into the subspaces related to the factorisation $\chi_L = \chi_1 \cdot \chi_2$ and $P_i : T_x\mathrm{M} \to \mathcal{D}_i$ denote the projector onto $\mathcal{D}_i$.
Fact from Matrix Analysis: $P_i = \sum a_k L^k = f_i(L)$ real analytic function. Therefore P_i is Nijenhuis.

Step 5. P_i is a very simple operator with two constant eigenvalues, 0 and 1. We prove the Splitting Theorem for P_i instead, to find coordinates $(x_1, \ldots, x_r, y_{r+1}, \ldots, y_n)$.

Step 6. Then $L = LP_1 + LP_2 = \begin{pmatrix} L_1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ & L_2 \end{pmatrix}$, where LP_1 and LP_2 are both Nijenhuis.

Step 7. Verification that $\partial_{y_\alpha} L_1 = 0$ and $\partial_{x_\beta} L_2 = 0$ is straightforward from the definition.

Corollaries: Haantjes theorem

Corollary

Every Nienhuis operator L locally splits into a direct sum of Nijenhuis operators $L = L_1 \oplus L_2 \oplus \dots$ each of which at the point $p \in \mathrm{M}$ has either a single real eigenvalue or a single pair of complex eigenvalues.

Theorem (Haantjes)

Let L be a Nijenhuis operator which is $\mathbb{R}$-diagonalisable at a point p and, all of its eigenvalues are different. Then there exist local coordinates $(x_1, \dots, x_n)$ such that

$$L(x) = \begin{pmatrix} \lambda_1(x_1) & & & \\ & \lambda_2(x_2) & & \\ & & \ddots & \\ & & & \lambda_n(x_n) \end{pmatrix}$$

Moreover, if $\lambda_i' \neq 0$, $i = 1, \dots, n$, then we can take the eigenvalues $u_i = \lambda_i(x_i)$ as new local coordinates to simplify L even further:

$$L(u) = \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix}$$

Non-degenarate Poisson structures: symplectic case

Let P be a Poisson structure such that $\det P \neq 0$ everywhere.
In some sense, this is the best and most generic case (in even dimension).

Theorem

If $\det P \neq 0$*, i.e.,* P *is invertible, then its inverse* $\Omega = P^{-1}$ *is a symplectic form. More precisely, if* $\Omega = \left(\omega_{ij}\right)$*, then the differential form*

$$\omega = \sum_{i<j} \omega_{ij} \, \mathrm{d}\, x^i \wedge \mathrm{d}\, x^j$$

is closed (and, of course, non-degenerate in the sense that $\det \Omega \neq 0$*). Moreover, there exists a local coordinate system* $q_1, \ldots, q_n, p_1, \ldots, p_n$ *such that*

$$\omega = \mathrm{d}\, p_1 \wedge \mathrm{d}\, q_1 + \mathrm{d}\, p_2 \wedge \mathrm{d}\, q_2 + \cdots + \mathrm{d}\, p_n \wedge \mathrm{d}\, q_n,$$

or, equivalently

$$P \simeq \partial_{q_1} \wedge \partial_{p_1} + \partial_{q_2} \wedge \partial_{p_2} + \cdots + \partial_{q_2} \wedge \partial_{p_2}.$$

$\mathfrak{gl}$-regular Nijenhuis operators

Question. What is the best, most generic (Nijenhuis) operator?

Answer. Since L is defined up to similarity transformation $L \mapsto CLC^{-1}$, the answer must be given in terms of its orbit $\mathcal{O}_L = \{CLC^{-1} \mid C \in \mathrm{GL}(n)\}$: this orbit must be regular, i.e., have maximal dimension. We call such operators $\mathfrak{gl}$-*regular*.

Theorem (Real analytic case)

Let L be a $\mathfrak{gl}$-regular Nijenhuis operator. Then there exist local coordinate systems $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in which L reduces to the first and second companion forms:

$$L(u) = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1} & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad \textit{and} \quad L(v) = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

where σ_i are the coefficients of the characteristic polynomial of L in the corresponding coordinate system.

Open problem. Is it still true in the C^∞-smooth case?

Integrable systems and Poisson commuting integrals

Let $(\mathrm{M}^{2n}, \omega)$ be a symplectic manifold (equivalently, Poisson manifold with a non-degenerate Poisson structure $P = \Omega^{-1}$)
A (Liouville) integrable Hamiltonian system on M^{2n} can be understood as a collection of n functions $F_1 = H, F_2, \dots, F_n$ satisfying the following two conditions:

- $F_1, \dots, F_n$ are functionally independent, i.e., their differentials $\mathrm{d}\, F_1, \dots, \mathrm{d}\, F_n$ are linearly independent on an open dense subset $U \subset \mathrm{M}$;
- $F_1, \dots, F_n$ Poisson commute, i.e., $\{F_i, F_j\} = 0$, $i, j = 1, \dots, n$.

Theorem

An integrable Hamiltonian system can (locally) be integrated in quadratures.

"In quadratures" means that two (local) operations are allowed:

- solving systems of functional equations, like $F(x, y, z) = 0 \Rightarrow z = g(x, y)$;
- integrating closed 1-forms, i.e., if $\mathrm{d}\, \alpha = 0$, then we can find f such that $\mathrm{d}\, f = \alpha$.

Sketch of proof

More general result. Consider a system of ODEs

$$\frac{\mathrm{d}\, x}{\mathrm{d}\, t} = \xi(x), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \tag{2}$$

which admits $n-k$ independent first integrals $F_1, \dots, F_{n-k}$ and k linearly independent commuting symmetries $\xi_1, \dots, \xi_k$ (vector fields such that $[\xi, \xi_i] = 0$). Let $\xi_i(F_j) = 0$, then (2) can be integrated in quadratures.

Step 1. Resolve $F_j(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = u_j$ and take the new coordinate system $x_1, \dots, x_k$, $u_1 = F_1(x), \dots, u_{n-k} = F_{n-k}(x)$. Since $\xi_i(F_j) = \xi_i(u_j) = 0$, then in the new coordinates $\xi = (*, ..., *, 0, ..., 0)$ and $\xi_i = (*, ..., *, 0, ..., 0)$ so that we can ignore $u_1, \dots, u_{n-k}$ thinking of them as parameters.

Step 2. Now we have k independent vector fields ξ_i given in coordinates $x_1, \dots, x_k$. Consider the dual basis $\alpha_1, \dots, \alpha_k$ of differential 1-forms. Since ξ_i pairwise commute, then α_i are all closed, and by integrating them we construct new coordinates $y_1, \dots, y_k$ such that $\mathrm{d}\, y_i = \alpha_i$ or equivalently $\xi_i = \partial_{y_i}$.

Step 3. Since $[\xi, \xi_i] = 0$, then (2) becomes $\frac{\mathrm{d}\, y_i}{\mathrm{d}\, t} = c_i$, $c_i \in \mathbb{R}$. The job is done.