Introduction to polynomial Poisson brackets, I

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Lecture 1 (90 minutes):

- A (very brief) introduction to Poisson geometry;
- An approach to polynomial Poisson brackets;
- Poisson cohomology of log-canonical Poisson structures.

Lecture 2 (90 minites):

- Deforming log-symplectic log-canonical Poisson structures;
- Examples!

Main reference: (M. Matviichuk and J.-H. Lu, arXiv:2503.05644),

- Deformations of $\mathbb{T}$-log-symplectic log-canonical Poisson structures and symmetric Poisson CGL extensions.

Poisson geometry, named after the French mathematician and physicist S. D. Poisson (1781 - 1840), has its origin in 19th century Hamiltonian mechanics. (Picture from Wikipedia).

Poisson distribution; Poisson processes; Poisson kernel, ...

Modern Poisson geometry started with

1. A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées. *J. Diff. Geom.* 12 (2), 253 - 300 (1977);
2. A. Weinstein, The local structure of Poisson manifolds, *J. Diff. Geom.* 18 (3), 523 - 557 (1983).

Today, Poisson geometry has close connections with

- mechanics; integrable systems;
- symplectic geometry;
- non-commutative geometry;
- cluster algebras;
- representation theory; quantum groups;
- $\cdots\cdots$.

Definitions

- A Poisson algebra over a field $\mathbf{k}$ is a commutative $\mathbf{k}$-algebra A together with a compatible Lie bracket, i.e., a $\mathbf{k}$-bilinear skew-symmetric map

$$\{,\} : A \times A \longrightarrow A,$$

also called the *Poisson bracket*, such that

$$\text{Jacobi identity}: \quad \{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0,$$

$$\text{Leibniz rule}: \quad \{a,bc\} = b\{a,c\} + c\{a,b\} \quad (\textit{compatibility}).$$

- A Poisson space is a "space" P with a Poisson bracket on $A = \mathrm{Fun}(P)$, the "algebra of functions" of P.

- One can thus define real or complex Poisson manifolds; algebraic Poisson varieties, or Poisson schemes.

Poisson Structures on $\mathbf{k}^n = \mathbb{R}^n$ or $\mathbb{C}^n$ with coordinates $(x_1, \ldots, x_n)$.

Leibniz's rule $\Longrightarrow$ Poisson bracket $\{,\}$ is determined by

$$\pi_{ij} = \{x_i, x_j\} \in \mathrm{Fun}(\mathbf{k}^n), \qquad 1 \leq i < j \leq n,$$

via

$$\{f, g\} = \sum_{1 \leq i < j \leq n} \{x_i, x_j\} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right), \qquad f, g \in \mathrm{Fun}(\mathbf{k}^n).$$

- Smooth Poisson structure on $\mathbb{R}^n$: need $\pi_{ij} \in C^\infty(\mathbb{R}^n)$;
- Holomorphic Poisson structure on $\mathbb{C}^n$: need π_{ij} holomorphic;
- Algebraic Poisson structure on $\mathbf{k}^n$: need $\pi_{ij} \in \mathbf{k}[x_1, \ldots, x_n]$.

(!!!!!) Serious constraints on the $\binom{n}{2}$ functions collection

$$\{\pi_{ij} \in \mathrm{Fun}(\mathbf{k}^n) : \ 1 \leq i < j \leq n\} :$$

Jacobi's identity $\Longrightarrow$ the π_{ij}'s must satisfy the $\binom{n}{3}$ equations

$$\sum_{m=1}^{n} \pi_{m,k} \frac{\partial \pi_{i,j}}{\partial x_m} + \pi_{m,j} \frac{\partial \pi_{k,i}}{\partial x_m} + \pi_{m,i} \frac{\partial \pi_{j,k}}{\partial x_m} = 0, \quad 1 \leq i < j < k \leq n,$$

which is an overdetermined system of nonlinear PDEs. Thus

- Poisson structures are not easy to make up artificially.

First Examples of Poisson Structures on $\mathbf{k}^n$

1. Constant Poisson structures: $\{x_i, x_j\} = c_{ij}$ are all constants. First example by S. D. Poisson in 1809 on $\mathbb{R}^{2n}$:

$$\{x_i,\, x_{n+i}\} = 1, \quad 1 \le i \le n, \quad \{x_j, x_k\} = 0, \quad \text{otherwise.}$$

2. Linear Poisson brackets

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^k x_k, \;\; 1 \le i, j \le n,$$

$\{c_{ij}^k \in \mathbf{k}\}$ are structure constants of an n-dimensional Lie algebra.

3. Log-canonical (quadratic) Poisson brackets:

$$\{x_i, x_j\} = c_{ij} x_i x_j, \quad 1 \le i, j \le n, \;\; c_{ij} \;\; \text{constants},$$

as long as matrix (c_{ij}) is skew-symmetric.

4. Higher-order?

In geometric terms: the Poisson bivector field

- A Poisson structure on a manifold P is equivalently given by a Poisson bivector field $\pi \in \Gamma(\wedge^2 TP)$:

$$\pi = \sum_{i<j} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

  in local coordinates $(x_1, \ldots, x_n)$, where $\pi_{ij} \stackrel{\mathrm{def}}{=} \{x_i, x_j\}$, and

$$\{f, g\} = \pi(df \wedge dg), \qquad f,\, g \in C^\infty(P).$$

- The Jacobi identity becomes the condition

$$\boxed{[\pi,\, \pi] = 0},$$

  where [,] is the Schouten bracket on multi-vector fields on P.

What is Poisson Geometry, I?

- *Poisson geometry is the geometry of an integrable 2-vector field* $\pi \in \Gamma(\wedge^2 TP)$.

Compare with

- Riemannian geometry is the geometry of a symmetric positive-definite 2-tensor field $g \in \Gamma(S^2(T^*P))$;
- Symplectic geometry is the geometry of a non-degenerate closed 2-form $\omega \in \Gamma(\wedge^2 T^*P)$;
- Complex geometry is the geometry of an integrable tensor field $J \in \mathrm{End}(T_{\mathbb{C}}P)$;
- Kähler geometry is the geometry of a compatible triple $(g,\ \omega,\ J)$.

- A Poisson structure $\pi \in \Gamma(\wedge^2 TP)$ defines a bundle map

$$\pi^{\#} : \quad T^*P \longrightarrow TP$$

which turns an $f \in C^\infty(P)$ into its Hamiltonian vector field

$$H_f := \pi^{\#}(df).$$

- Symplectic 2-form $\omega \quad \Longleftrightarrow \quad$ Poisson 2-vector field π such that $\pi^{\#} : T^*P \to TP$ is invertible.

Fundamental Theorem of Poisson Geometry

- Every Poisson manifold is a disjoint union of symplectic leaves, namely the integral submanifolds of $\mathrm{Im}(\pi^{\#})$.

Examples of Poisson structures "in nature":

- Before the 1980s:
  1. Constant Poisson structures;
  2. Linear Poisson structures: dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$;
  3. Symplectic structures (as non-degenerate Poisson structures), e.g., T^*P for any manifold P;
  4. Quotients of symplectic structures by symmetry groups;
  5. Poisson submanifolds of a given Poisson manifold.
- After the 1980's:
  1. Classes of Poisson varieties in Lie theory from quantum groups;
  2. From algebraic geometry and geometric representation theory: Hitchin moduli spaces, Coulomb branches, ...

What is Poisson Geometry, II?

The geometry of a Poisson manifold (M, π) concerns

1. the symplectic-leaf decomposition of M defined by π;
2. normal forms and stability of symplectic leaves;
3. good local (such as Darboux) coordinates;
4. Hamiltonian vector fields and moment maps (T. Holm lectures);
5. integrable systems (A. Bolsinov lectures);
6. integration into symplectic groupoids (M. C. Ten lectures);
7. Deformation theory and Poisson (co)homology (my lectures)
8. Deformation and geometrical quantization (S. Gutt lctures);
9. Shifted Poisson structures (M. C. Ten lectures);
10. $\cdots\cdots$.

Poisson Geometry vs. Symplectic Geometry

1. Poisson manifolds are better settings for Hamiltonian mechanics;
2. Existence of symplectic structures on a manifold is a serious constraint (e.g, no symplectic structure on S^4), but not for Poisson structures (any manifold has the zero Poisson structure);
3. Symplectic structures on P may not extend, as symplectic structures, to (partial) compactifications of P, but may extend as Poisson structures.

Schouten bracket:

For an n-dimensional smooth manifold M, and with

$$\mathfrak{X}^p(M) := \Gamma(\wedge^p TM), \qquad \mathfrak{X}^\bullet(M) := \bigoplus_{p \geq 0} \mathfrak{X}^p(M),$$

the Schouten bracket is the graded bilinear map

$$[\ ,\ ] : \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \longrightarrow \mathfrak{X}^{p+q-1}(M)$$

characterized by the following properties:

1. $[f, g] = 0$ for $f, g \in C^\infty(M)$;
2. $[X, f] = X(f)$ for $X \in \mathfrak{X}^1(M)$ and $f \in C^\infty(M)$;
3. for $P \in \mathfrak{X}^p(M)$, $Q \in \mathfrak{X}^q(M)$, and $R \in \mathfrak{X}^r(M)$,
$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R];$$
4. $[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]$.

It satisfies the graded Jacobi identity

$$[P, [Q, R]] = [[P, Q], R] + (-1)^{(p-1)(q-1)}[Q, [P, R]].$$

Exercise . For $\pi, \pi_1, \pi_2, \pi_3 \in \mathfrak{X}^2(M)$, graded Jacobi gives

$$\begin{aligned}
[\pi_1, \pi_2] &= [\pi_2, \pi_1], \\
[\pi_1, [\pi_2, \pi_3]] &= [[\pi_1, \pi_2], \pi_3] - [\pi_2, [\pi_1, \pi_3]], \\
[\pi_1, [\pi, \pi]] &= 2[[\pi_1, \pi], \pi] = -2[\pi, [\pi_1, \pi]], \\
[\pi, [\pi, \pi]] &= 0.
\end{aligned}$$

Lemma–Definition.

1. If π is Poisson, then $\mathrm{d}_\pi = [\pi, -]$ on $\mathfrak{X}^\bullet(M)$ satisfies $\mathrm{d}_\pi^2 = 0$.
2. The cohomology of

$$0 \longrightarrow \mathfrak{X}^0(M) \xrightarrow{\mathrm{d}_\pi} \mathfrak{X}^1(M) \xrightarrow{\mathrm{d}_\pi} \mathfrak{X}^2(M) \xrightarrow{\mathrm{d}_\pi} \cdots \xrightarrow{\mathrm{d}_\pi} \mathfrak{X}^n(M) \longrightarrow 0$$

is the Poisson cohomology $\mathrm{H}^\bullet(M, \pi)$.

We now turn to our approach to polynomial Poisson brackets.

Laurent poly-vector fields on $\mathbb{C}^n$

- Let $x = (x_1, \ldots, x_n)$ be the standard coordinates on $\mathbb{C}^n$.
- Consider the space of Laurent poly-vector fields on $\mathbb{C}^n$

$$\mathfrak{X}^p_{\mathrm{Laur}}(\mathbb{C}^n) := \mathbb{C}[x^{\pm 1}] \otimes_{\mathbb{C}} \bigwedge^p \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle,$$

$$\mathfrak{X}^\bullet_{\mathrm{Laur}}(\mathbb{C}^n) = \bigoplus_{p=0}^{n} \mathfrak{X}^p_{\mathrm{Laur}}.$$

Our main tool:

$(\mathbb{C}^\times)^n$-weight space decomposition of $\mathfrak{X}^\bullet_{\mathrm{Laur}}(\mathbb{C}^n)$ under $(\mathbb{C}^\times)^n$-action

$$(\mu_1, \ldots, \mu_n) \cdot (x_1, \ldots, x_n) = (\mu_1 x_1, \ldots, \mu_n x_n).$$

- Identify character lattice of $(\mathbb{C}^\times)^n$ with $\mathbb{Z}^n = \mathbb{Z}^n_{\mathrm{col}}$;
- A $(\mathbb{C}^\times)^n$ weight is a vector $w = (w_1, \ldots, w_n)^t \in \mathbb{Z}^n$:

$$(\mathbb{C}^\times)^n \longrightarrow \mathbb{C}^\times, \;\; \mu \longmapsto \mu^w := \mu_1^{w_1} \cdots \mu_n^{w_n}.$$

- $\mathbb{Z}^n$ has standard basis

$$e_j := (0, \ldots, 0, \underset{j\text{-th}}{1}, 0, \ldots, 0)^t, \qquad j \in [1, n].$$

- x_j has weight e_j, $\frac{\partial}{\partial x_j}$ has weight $-e_j$, and

$$\partial_j := x_j \frac{\partial}{\partial x_j} \qquad \text{has weight } 0.$$

- For $w = (w_1, \ldots, w_n)^t \in \mathbb{Z}^n$ and $J = \{1 \leq j_1 < \cdots < j_p \leq n\}$,

$$x^w \partial_J := x_1^{w_1} \cdots x_n^{w_n} \partial_{j_1} \wedge \cdots \wedge \partial_{j_p} \qquad \text{has weight } \; w.$$

- Thus for any $p \in [1, n]$ and $w \in \mathbb{Z}^n$,

$$\mathfrak{X}^p_{\mathrm{Laur}}(\mathbb{C}^n)^w = \mathrm{Span}_{\mathbb{C}}\{x^w \partial_J : |J| = p\},$$
$$\dim \mathfrak{X}^p_{\mathrm{Laur}}(\mathbb{C}^n)^w = \binom{n}{p}.$$

Schouten bracket on Laurent poly-vectors on $\mathbb{C}^n$.

For $I = \{i_1 < \cdots < i_q\}$ and $w \in \mathbb{Z}^n$, set

$$w \cdot \partial_I = \sum_{l=1}^{q} (-1)^{l+1} w_{i_l} \partial_{i_1} \wedge \cdots \wedge \widehat{\partial_{i_l}} \wedge \cdots \wedge \partial_{i_q}.$$

Lemma. For $v, w \in \mathbb{Z}^n$ and $I, J \subset \{1, \ldots, n\}$,

$$[x^v \partial_I, x^w \partial_J] = x^{v+w} \left((-1)^{|I|-1} (w \cdot \partial_I) \wedge \partial_J - \partial_I \wedge (v \cdot \partial_J) \right).$$

Consider now algebraic poly-vector fields

$$\mathfrak{X}^{\bullet}_{\mathrm{Alg}}(\mathbb{C}^n) := \bigoplus_{p=0}^{n} \mathfrak{X}^{p}_{\mathrm{Alg}}(\mathbb{C}^n), \qquad \mathfrak{X}^{p}_{\mathrm{Alg}}(\mathbb{C}^n) := \mathbb{C}[x] \otimes_{\mathbb{C}} \bigwedge^{p} \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$$

and $(\mathbb{C}^{\times})^n$-weight decomposition

$$\mathfrak{X}^{\bullet}_{\mathrm{Alg}}(\mathbb{C}^n) = \bigoplus_{w \in \mathbb{Z}^n} \mathfrak{X}^{\bullet}_{\mathrm{Alg}}(\mathbb{C}^n)^w.$$

As $\frac{\partial}{\partial x_j}$ is weight $-e_j$ for $j \in [1, n]$, we have

Weight Constraint Lemma: For $w = (w_1, \ldots, w_n)^t \in \mathbb{Z}^n$,

$$\boxed{\mathfrak{X}^{\bullet}_{\mathrm{Alg}}(\mathbb{C}^n)^w \neq 0 \quad \Longleftrightarrow \quad w \in (\mathbb{Z}_{\geq -1})^n.}$$

For $w = (w_1, \ldots, w_n)^t \in (\mathbb{Z}_{\geq -1})^n$, let

$$J_w := \{j \in [1, n] : w_j = -1\}.$$

Notation:

- For $w = (w_1, \ldots, w_n)^t \in (\mathbb{Z}_{\geq -1})^n$ and $J_w := \{j \in [1, n] : w_j = -1\}$,

$$\mathfrak{X}^p_{\mathrm{Alg}}(\mathbb{C}^n)^w \neq 0 \quad \Longleftrightarrow \quad w \in (\mathbb{Z}_{\geq -1})^n \text{ and } |J_w| \leq p,$$
$$\mathfrak{X}^p_{\mathrm{Alg}}(\mathbb{C}^n)^w = x^w \mathrm{Span}_{\mathbb{C}}\{\partial_{J_w} \wedge \partial_K : K \subset [1, n],\ J_w \cap K = \emptyset\}.$$

- For $p \in [0, n]$, define

$$\mathcal{W}_{[p]} := \{w \in (\mathbb{Z}_{\geq -1})^n : |J_w| = p\},$$
$$\mathcal{W}_p := \{w \in (\mathbb{Z}_{\geq -1})^n : |J_w| \leq p\} = \bigsqcup_{q=0}^{p} \mathcal{W}_{[q]}.$$

Then $(\mathbb{Z}_{\geq 0})^n = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_n$, and

$$\mathfrak{X}^p_{\mathrm{Alg}}(\mathbb{C}^n) = \bigoplus_{w \in \mathcal{W}_p} \mathfrak{X}^p_{\mathrm{Alg}}(\mathbb{C}^n)^w.$$

Example: For $w \in \mathcal{W}_2$,

$$\mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^w = \begin{cases} x^w \operatorname{Span}_{\mathbb{C}}\{\partial_j \wedge \partial_k : 1 \leq j < k \leq n\}, & J_w = \varnothing, \\ x^w \operatorname{Span}_{\mathbb{C}}\{\partial_j \wedge \partial_k : k \in [1,n] \backslash \{j\}\}, & J_w = \{j\}, \\ x^w \operatorname{Span}_{\mathbb{C}} \partial_j \wedge \partial_k, & J_w = \{j < k\}. \end{cases}$$

Consequently,

$$\dim_{\mathbb{C}} \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^w = \begin{cases} \binom{n}{2}, & |J_w| = 0, \\ n-1, & |J_w| = 1, \\ 1, & |J_w| = 2. \end{cases}$$

Observation: For $\pi \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)$, the overdetermined PDEs arising from

$$[\pi, \pi] = 0$$

are in fact algebraic relations between the weights of monomial terms in π and their coefficients.

Our strategy: Write an algebraic Poisson bi-vector field as

$$\pi = \pi_0 + \pi_{\mathrm{tail}}, \qquad \pi_0 = \sum_{j<k} \lambda_{j,k} x_j x_k \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} = \sum_{j<k} \lambda_{j,k} \partial_j \wedge \partial_k,$$

- π_0 is the log-canonical term of weight-zero.
- $\pi_{\mathrm{tail}} = \sum_{w \in \mathrm{W}(\pi)} \pi_w$ has finitely many weight components
- $\mathrm{W}(\pi) \subset \mathcal{W}_2 \setminus \{0\}$ is the set of tail weights.

Study and construct a class of π by imposing conditions on $\mathrm{W}(\pi)$.

Notation: Fix standard coordinates $(x_1, \ldots, x_n)$ on $\mathbb{C}^n$ and denote by

$$\mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$$

the set of all algebraic Poisson structures on $\mathbb{C}^n$.

One-parameter test degenerations and Alekseev-Davydenkova (AD) cones

For $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\mathrm{row}}$, put

$$\varphi^*_{a,m} x_j = a^{m_j} x_j \qquad (a \in \mathbb{C}^\times).$$

If $\pi = \pi_0 + \sum_{w \in \mathsf{W}(\pi)} \pi_w \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$, then

$$(\varphi^{-1}_{a,m})_* \pi = \pi_0 + \sum_{w \in \mathsf{W}(\pi)} a^{mw} \pi_w.$$

Definition of AD cones of π:

$$\begin{aligned}
\mathcal{D}^{\mathrm{AD}}_\pi(\mathbb{Z}) &= \{m \in \mathbb{Z}^n_{\mathrm{row}} : \lim_{a \to 0} \pi_{a,m} \text{ exists}\} \\
&= \{m \in \mathbb{Z}^n_{\mathrm{row}} : mw \geq 0 \text{ for all } w \in \mathsf{W}(\pi)\}, \\
\mathcal{D}^{\mathrm{AD}}_\pi(\mathbb{Z})^o &= \{m \in \mathbb{Z}^n_{\mathrm{row}} : \lim_{a \to 0} \pi_{a,m} = \pi_0\} \\
&= \{m \in \mathbb{Z}^n_{\mathrm{row}} : mw > 0 \text{ for all } w \in \mathsf{W}(\pi)\}.
\end{aligned}$$

For $a, b, c \in \mathbb{C}^\times$, consider the Poisson structure on $\mathbb{C}^3$ given by

$$\{x_1, x_2\} = x_1x_2 + ax_3^2,$$
$$\{x_2, x_3\} = x_2x_3 + bx_1^2,$$
$$\{x_3, x_1\} = x_3x_1 + cx_2^2.$$

Its tail weights are

$$\mathrm{W}(\pi) = \{(-1,-1,2)^t, (2,-1,-1)^t, (-1,2,-1)^t\}.$$

For $m = (m_1, m_2, m_3)$, open AD conditions are

$$-m_1 - m_2 + 2m_3 > 0, \quad 2m_1 - m_2 - m_3 > 0, \quad -m_1 + 2m_2 - m_3 > 0,$$

not possible. Thus we do not always have

$$\mathcal{D}_\pi^{\mathrm{AD}}(\mathbb{Z})^o \neq \varnothing.$$

Notation and a combinatorial lemma.

For a finite set $\mathsf{W} \subset \mathbb{Z}^n \setminus \{0\}$, let

$$\mathsf{W}_{\geq p} = \text{the set of sums of at least } p \text{ elements of } \mathsf{W},$$

and let $\mathcal{D}_{\mathsf{W}}(\mathbb{Z})^o = \{m \in \mathbb{Z}^n_{\mathrm{row}} : mw > 0, \ \forall\, w \in \mathsf{W}\}$.

Gordan Lemma. For any finite set $\mathsf{W} \subset \mathbb{Z}^n \setminus \{0\}$,

$$\mathcal{D}_{\mathsf{W}}(\mathbb{Z})^o \neq \varnothing \qquad \Longleftrightarrow \qquad 0 \notin \mathsf{W}_{\geq 1},$$

and in such a case, the subset

$$\mathsf{W} \setminus \mathsf{W}_{\geq 2} := \{w \in \mathsf{W} : w \notin \mathsf{W}_{\geq 2}\}$$

is the unique minimal generating set of the semigroup $\mathsf{W}_{\geq 1}$.

Corollary: For any $\pi \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$, we have

$$\mathcal{D}_\pi^{\mathrm{AD}}(\mathbb{Z})^o \neq \varnothing \qquad \Longleftrightarrow \qquad 0 \notin \mathsf{W}(\pi)_{\geq 1},$$

and in such a case, setting $S_\pi := \mathsf{W}(\pi) \setminus \mathsf{W}(\pi)_{\geq 2}$, then

$$\mathcal{D}_\pi^{\mathrm{AD}}(\mathbb{Z})^o = \{m \in \mathbb{Z}^n_{\mathrm{row}} : \ mw > 0, \ \forall\, w \in S_\pi\}.$$

- Call elements in $S_\pi = \mathsf{W}(\pi) \setminus \mathsf{W}(\pi)_{\geq 2}$ primitive tail weights of π;
- In general S_π is a much smaller subset of $\mathsf{W}(\pi)$.

Definition. $\pi \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$ is said to be admissible in $(x_1, \ldots, x_n)$ if

$$0 \notin \mathsf{W}(\pi)_{\geq 1}.$$

We only consider admissible $\pi \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$ in these lectures.

Lemma. For any admissible $\pi = \pi_0 + \sum_{w \in \mathsf{W}(\pi)} \pi_w \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$, one has

$$[\pi_0, \pi_w] = 0, \qquad \forall \;\; w \in S_\pi = \mathsf{W}(\pi) \setminus \mathsf{W}(\pi)_{\geq 2}.$$

Proof. Fix $w \in S_\pi$ and write

$$\pi = \pi_0 + \pi_w + \pi'_w, \qquad \pi'_w = \sum_{w' \in \mathsf{W}(\pi) \setminus \{w\}} \pi_{w'}.$$

Since π is Poisson,

$$0 = [\pi, \pi] = 2[\pi_0, \pi_w] + 2[\pi_0, \pi'_w] + 2[\pi_w, \pi'_w] + [\pi_w, \pi_w] + [\pi'_w, \pi'_w].$$

A weight w-term must come from $2[\pi_0, \pi_w]$. Thus $[\pi_0, \pi_w] = 0$. $\diamond$

We are thus motivated to make another definition.

Admissible algebraic deformations of log-canonical Poisson structures:

Definition: Let π_0 be log-canonical, and $S \subset \mathrm{W}_2$ finite and $0 \notin S_{\geq 1}$.

- $\pi \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$ is an S-admissible algebraic deformation of π_0 if

$$\pi = \pi_0 + \pi_1 + \pi_{\geq 2}, \qquad \text{with}$$

$$\pi_1 \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S, \qquad [\pi_0, \pi_1] = 0, \qquad \pi_{\geq 2} \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^{S_{\geq 2}}.$$

- Also say that π is an S-admissible deformation of π_0 along π_1.

Important:

$0 \notin S_{\geq 1}$ ensures any S-admissible π still has π_0 as log-canonical term.

To construct S-admissible deformations, we impose conditions on S:

Definition. For a finite $S \subset \mathcal{W}_2$, say S has

- Property $W(-1)$ if $0 \notin S_{\geq 1}$;
- Property $W(0)$ if $S_{\geq 1} \cap (\mathbb{Z}_{\geq 0})^n = \varnothing$;
- Property $W(1)$ if $S_{\geq 1} \cap \mathcal{W}_1 = \varnothing$.

Note that

$$W(1) \Longrightarrow W(0) \Longrightarrow W(-1).$$

A combinatorial Lemma: If $\mathcal{S} \subset \mathcal{W}_2$ has Property $W(0)$, then

$$\mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^{S_m} = 0 \qquad \text{for large enough } m.$$

Admissible Deformation Lemma:

Let π_0 be log-canonical. Let $S \subset \mathrm{W}_2$ be finite, and let $\pi_1 \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S$ such that $[\pi_0, \pi_1] = 0$. Assume

$$\mathrm{H}^3_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0 \quad \text{and} \quad \mathrm{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0.$$

1. If S has $W(-1)$, i.e., if $0 \notin S_{\geq 1}$, then $\exists$

   $$\pi(t) = \pi_0 + t\pi_1 + t^2\pi_2 + \cdots, \qquad [\pi(t), \pi(t)] = 0,$$

   with $\pi_m \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^{S_m}$ for $m \geq 2$, and any two are gauge equivalent.

2. If S has $W(0)$, every such $\pi(t)$ is polynomial in t, so $\pi(1)$ is an S-admissible algebraic deformation of π_0 along π_1.

3. If S has $W(1)$, then S-admissible algebraic deformation of π_0 along π_1 exists and is unique.

Proof. We seek for

$$\pi(t) = \pi_0 + t\pi_1 + t^2\pi_2 + \cdots, \qquad [\pi(t), \pi(t)] = 0.$$

- At order $m \geq 2$ we must solve

$$2[\pi_0, \pi_m] + \sum_{k=1}^{m-1} [\pi_k, \pi_{m-k}] = 0.$$

- The obstruction

$$O_m = \sum_{k=1}^{m-1} [\pi_k, \pi_{m-k}] \in \mathfrak{X}^3_{\mathrm{Alg}}(\mathbb{C}^n)^{S_m}$$

  is d_{π_0}-closed, so $\mathrm{H}^3_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0$ gives the next π_m;
- If S has $W(0)$, the formal series terminates;
- If S has $W(1)$, no nontrivial gauge transformations, so π is unique.

Q.E.D.

We now turn to Poisson cohomology of log-canonical Poisson structures

Laurent Poisson cohomology of π_0. Let

$$\pi_0 = \sum_{j<k} \lambda_{j,k} \partial_j \wedge \partial_k, \qquad \boldsymbol{\lambda} = (\lambda_{j,k}).$$

For $w \in \mathbb{Z}^n = \mathbb{Z}^n_{\mathrm{col}}$, set $u_w = (\partial_1, \dots, \partial_n)\boldsymbol{\lambda} w$.

Key Formula. For every $V \in \mathfrak{X}^\bullet_{\mathrm{Laur}}(\mathbb{C}^n)^w$,

$$d_{\pi_0} V = [\pi_0, V] = u_w \wedge V.$$

Hence $(\mathfrak{X}^\bullet_{\mathrm{Laur}}(\mathbb{C}^n)^w, \mathrm{d}_{\pi_0} = [\pi_0, -])$ is a Koszul complex.

Corollary: For every $w \in \mathbb{Z}^n$,

$$\mathrm{H}^\bullet_{\mathrm{Laur}}(\mathbb{C}^n, \pi_0)^w \neq 0 \qquad \Longleftrightarrow \qquad \boldsymbol{\lambda} w = 0,$$

and when $\boldsymbol{\lambda} w = 0$, $\mathrm{d}_{\pi_0} = [\pi_0, -] \equiv 0$ on $\mathfrak{X}^\bullet_{\mathrm{Laur}}(\mathbb{C}^n)^w$.

Algebraic Poisson cohomology of π_0. Recall

$$J_w = \{j \in [1,n] : w_j = -1\}, \qquad \text{for } \ w = (w_1, \ldots, w_n)^t \in (\mathbb{Z}_{\geq -1})^n.$$

For $J \subset [1,n]$, set $\mathbb{C}^J = \sum_{j \in J} \mathbb{C} e_j$.

Cohomology Lemma [Matviichuk–Pym–Schedler]. For any $w \in (\mathbb{Z}_{\geq -1})^n$,

$$\boxed{\mathrm{H}^\bullet_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^w \neq 0 \qquad \Longleftrightarrow \qquad \boldsymbol{\lambda} w \in \mathbb{C}^{J_w}.} \tag{1}$$

If this condition holds, $\mathrm{d}_{\pi_0} = [\pi_0, -] \equiv 0$ on $\mathfrak{X}^\bullet_{\mathrm{Alg}}(\mathbb{C}^n)^w$.

- We call (1) the Cohomology Equation for π_0.

Proof of the Cohomology Lemma. Fix $w \in (\mathbb{Z}_{\geq -1})^n$, and let

$$E_{J_w} = \mathrm{Span}_{\mathbb{C}}\{\partial_k : k \notin J_w\}.$$

Every algebraic poly-vector field of weight w has a unique expression

$$x^w \partial_{J_w} \wedge \eta, \qquad \eta \in \bigwedge^\bullet E_{J_w}.$$

By the Key Formula,

$$\mathrm{d}_{\pi_0}(x^w \partial_{J_w} \wedge \eta) = (-1)^{|J_w|} x^w \partial_{J_w} \wedge \left(\overline{u}_w \wedge \eta\right),$$

where $\overline{u}_w$ is the projection of $u_w = (\partial_1, \ldots, \partial_n)\boldsymbol{\lambda} w$ to E_{J_w}, so the weight-w sub-complex is, up to the factor $x^w \partial_{J_w}$, the Koszul complex

$$\left(\bigwedge^\bullet E_{J_w},\ (-1)^{|J_w|}\overline{u}_w \wedge -\right).$$

It is acyclic if $\overline{u}_w \neq 0$, and its differential is zero if $\overline{u}_w = 0$. Finally,

$$\overline{u}_w = 0 \quad \Longleftrightarrow \quad \boldsymbol{\lambda} w \in \mathbb{C}^{J_w}.$$

◇

Notation:

- Denote by
$$\mathcal{W}^{\mathrm{coh}}(\pi_0) \subset (\mathbb{Z}_{\geq -1})^n$$
the set of all non-zero solutions to the cohomological equation.
- Call $w \in \mathcal{W}^{\mathrm{coh}}(\pi_0)$ a non-zero cohomology weight of π_0.
- For $J \subset [1, n]$, set
$$\mathcal{W}_J^{\mathrm{coh}}(\pi_0) = \{w \in \mathcal{W}^{\mathrm{coh}}(\pi_0) : J_w = J\}.$$

Remark:

- For $w \in (\mathbb{Z}_{\geq -1})^n$,

$$\boldsymbol{\lambda} w \in \mathbb{C}^{J_w} \implies \boldsymbol{\lambda} w \in (\mathbb{C}^{J_w})_0 := \left\{ \sum_{j \in J_w} a_j e_j : \sum_j a_j = 0 \right\}.$$

  In particular, if $|J_w| = 0$ or $|J_w| = 1$, then $\boldsymbol{\lambda} w \in \mathbb{C}^{J_w}$ iff $\boldsymbol{\lambda} w = 0$.

- If $\boldsymbol{\lambda}$ is non-degenerate, then, for $J \subset [1, n]$,

$$\mathcal{W}_J^{\mathrm{coh}}(\pi_0) = \boldsymbol{\lambda}^{-1}(\mathbb{C}^J) \cap \{w \in (\mathbb{Z}_{\geq -1})^n : J_w = J\}.$$

An example: Consider the log-symplectic log-canonical π_0 on $\mathbb{C}^4$ with

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\lambda} w = (w_4, -w_3, w_2 - w_4, -w_1 + w_3)^t.$$

Solving the Cohomology Equation gives

$$\begin{aligned} \mathcal{W}^{\mathrm{coh}}(\pi_0) = \{&(-1,-1,-1,m)^t : m \in \mathbb{Z}_{\geq 0}\} \\ &\sqcup \{(-1,-1,m,-1)^t : m \in \mathbb{Z}_{\geq 0}\} \sqcup \{(-1,-1,-1,-1)^t\}. \end{aligned}$$

An example continued: Consequently, for $0 \leq p \leq 2$,

$$\mathsf{H}^p_{\mathrm{Alg}}(\mathbb{C}^4, \pi_0) = \mathsf{H}^p_{\mathrm{Alg}}(\mathbb{C}^4, \pi_0)^0 = \mathsf{Span}_{\mathbb{C}}\{\partial_I : |I| = p\}.$$

In degree three,

$$\begin{aligned}\mathsf{H}^3_{\mathrm{Alg}}(\mathbb{C}^4, \pi_0) = \mathsf{H}^3_{\mathrm{Alg}}(\mathbb{C}^4, \pi_0)^0 & \\ & \oplus \bigoplus_{m \geq 0} \mathbb{C} x_1^{-1} x_2^{-1} x_3^{-1} x_4^m \partial_1 \wedge \partial_2 \wedge \partial_3 \\ & \oplus \bigoplus_{m \geq 0} \mathbb{C} x_1^{-1} x_2^{-1} x_3^m x_4^{-1} \partial_1 \wedge \partial_2 \wedge \partial_4,\end{aligned}$$

where $\mathsf{H}^3_{\mathrm{Alg}}(\mathbb{C}^4, \pi_0)^0 = \mathsf{Span}_{\mathbb{C}}\{\partial_I : |I| = 3\}$.

An example continued: In degree 4,

$$\begin{aligned}\mathsf{H}^4_{\mathrm{Alg}}(\mathbb{C}^4,\pi_0) = \mathsf{H}^4_{\mathrm{Alg}}(\mathbb{C}^4,\pi_0)^0 &\oplus \bigoplus_{m\geq 0} \mathbb{C} x_1^{-1}x_2^{-1}x_3^{-1}x_4^{m}\partial_{[1,4]} \\ &\oplus \bigoplus_{m\geq 0} \mathbb{C} x_1^{-1}x_2^{-1}x_3^{m}x_4^{-1}\partial_{[1,4]} \\ &\oplus \mathbb{C} x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}\partial_{[1,4]},\end{aligned}$$

with

$$\mathsf{H}^4_{\mathrm{Alg}}(\mathbb{C}^4,\pi_0)^0 = \mathbb{C}\partial_{[1,4]}.$$

$\diamond$

Summary of Lecture 1.

1. A Poisson structure is a bi-vector π satisfying $[\pi, \pi] = 0$; the differential $d_\pi = [\pi, -]$ defines Poisson cohomology.
2. We study $\pi \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$ by its $(\mathbb{C}^\times)^n$ tail weight decomposition
$$\pi = \pi_0 + \pi_{\mathrm{tail}} = \pi_0 + \sum_{w \in \mathrm{W}(\pi)} \pi_w;$$
3. When $0 \notin \mathrm{W}(\pi)_{\geq 1}$, we have the set of primitive tail weights
$$w \in S_\pi = \mathrm{W}(\pi) \backslash \mathrm{W}(\pi)_{\geq 2}$$
and non-empty open AD cone of π defined by inequalities using S_π;
4. S-admissible deformations of a log-canonical π_0 can be constructed recursively under condition
$$\mathrm{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0 \quad \text{and} \quad \mathrm{H}^3_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0.$$
Properties $W(0)$ and $W(1)$ of S give algebraicity and uniqueness.
5. Cohomology Equation for a log-canonical π_0 on $w \in (\mathbb{Z}_{\geq -1})^n$:
$$\mathrm{H}^\bullet_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^w \neq 0 \iff \boldsymbol{\lambda} w \in \mathbb{C}^{J_w}.$$