Exercises for *Hamiltonian group actions and fixed points*
Poisson 2026

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Question 1. Moment polytopes for projective spaces.

Consider the standard $\mathbb{T}^3$-action on $\mathbb{C}P^3$:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0, z_1, z_2, z_3] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3].$$

This has moment map

$$\Phi([z_0, z_1, z_2, z_3]) = \left(\frac{|z_1|^2}{\sum |z_j|^2} \ , \ \frac{|z_2|^2}{\sum |z_j|^2} \ , \ \frac{|z_3|^2}{\sum |z_j|^2} \right)$$

(a) What is the moment polytope for this action?

(b) Exhibit explicitly the subsets of $\mathbb{C}P^3$ for which the stabilizer under this action is $\{1\}, S^1, \mathbb{T}^2$, and $\mathbb{T}^3$.

Show that the images of these subsets under the moment map are the interior, the facets, the edges, and the vertices, respectively.

(c) What is the moment polytope for the $\mathbb{T}^2$-action of $\mathbb{C}P^1 \times \mathbb{C}P^1$ as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{i\theta} z_1], [w_0, e^{i\eta} w_1])?$$

Question 2. Moment polytopes for Grassmannians and flag varieties.

(a) Consider the standard $\mathbb{T}^4$-action on $\mathbb{C}^4$ by coordinate-wise multiplication:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) \cdot (z_1, z_2, z_3, z_4) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3, e^{i\theta_4} z_4).$$

This induces an action on the Grassmannian of 2-planes in $\mathbb{C}^4$

$$\mathcal{G}r_2(\mathbb{C}^4) = \left\{ V \subset \mathbb{C}^4 \ \middle| \ \dim_{\mathbb{C}}(V) = 2 \right\}.$$

What subgroup acts trivially? For the residual action, what are the fixed points? What are the submanifolds stabilized by codimension one subgroups? What are the Euler characteristic and Betti numbers of $\mathcal{G}r_2(\mathbb{C}^4)$?

(b) Consider the standard $\mathbb{T}^3$-action on $\mathbb{C}^3$ by coordinate-wise multiplication:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot (z_1, z_2, z_3) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3).$$

This induces an action on the full flag variety

$$\mathcal{F}\ell ags(\mathbb{C}^3) = \left\{ \{0\} \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3 \;\middle|\; \dim_{\mathbb{C}}(V_i) = i \right\}.$$

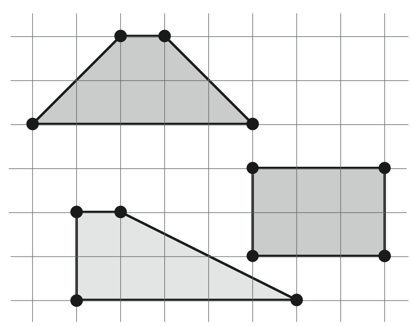
What subgroup acts trivially? For the residual action, what are the fixed points? What are the submanifolds stabilized by codimension one subgroups? What are the Euler characteristic and Betti numbers?

Question 3. Classifying some moment polytopes.

(a) Classify all 2-dimensional Delzant polytopes with 3 vertices up to translation, scaling, and the action of $GL(2, \mathbb{Z})$. Classify all 4-dimensional symplectic toric manifolds with 3 fixed points, up to weak equivalence.

(b) Similarly, classify all n-dimensional Delzant polytopes with $n + 1$ vertices up to translation, scaling, and the action of $GL(n, \mathbb{Z})$ and all $2n$-dimensional symplectic toric manifolds with $n + 1$ fixed points, up to weak equivalence.

(c) Which of the following Delzant trapezoids are equivalent? What are the corresponding toric symplectic manifolds? What are the underlying smooth manifolds?



(d) Classify all 2-dimensional Delzant polytopes with 4 vertices up to translation, scaling, and the action of $GL(2, \mathbb{Z})$.

(*Hint:* By a linear transformation in $GL(2, \mathbb{Z})$, we can make one of the angles in the polytope into a right angle. Check that this forces another angle to also become a right angle.)

(e) What are all the 4-dimensional toric symplectic manifolds with 4 fixed points?

Question 4. Combinatorial symplectic blowup.

Consider a Delzant polytope in $\mathbb{R}^n$ with a vertex P and with primitive (inward-pointing) edge vectors $u_1, \ldots, u_n$ at P. Chop off the corner to obtain a new polytope with the same vertices except P, and with P replaced by n new vertices:

$$P + \varepsilon u_j, \qquad j = 1, \ldots, n,$$

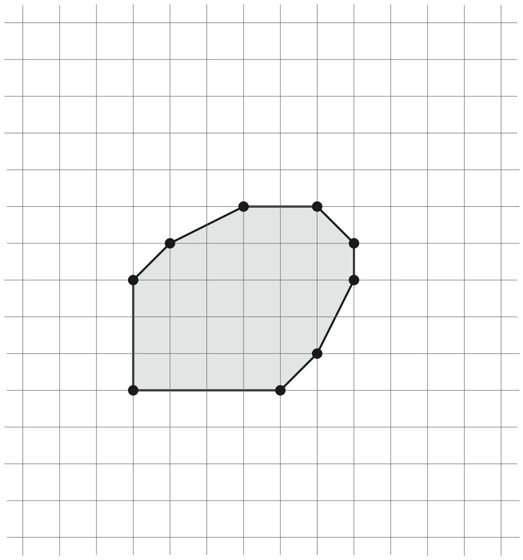
and ε is a small positive real number (small enough that you've just chopped off a corner, nothing more). Show that this new polytope is also Delzant.

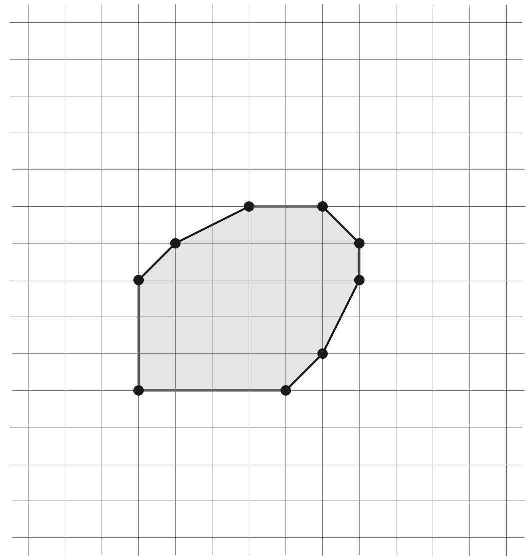
(The new corresponding toric manifold is called the ε-symplectic blowup of the original one. This is also a first example of symplectic cutting.)

Question 5. Toric four-folds.

(a) Compute the Euler characteristic, Betti numbers, and self-intersections of the T-invariant spheres in the toric four-fold with Delzant polygon shown below.

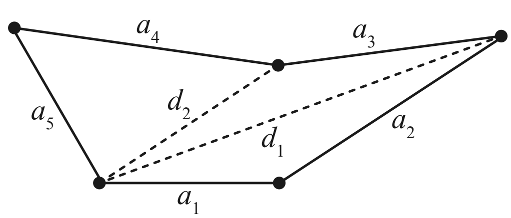
(b) Identify the toric four-fold with Delzant polygon shown below as a blowup of $\mathbb{C}P^2$. How many corner chops do you need to perform?





Question 6. Pentagon spaces.

Consider a polygon space $\mathcal{P}o\ell(a_1, a_2, a_3, a_4, a_5)$ with bending flow about the two diagonals d_1 and d_2 indicated in the figure below.



(a) When $a_1 \neq a_2$ and $a_4 \neq a_5$, bending flow is a toric action on $\mathcal{P}o\ell(a_1, a_2, a_3, a_4, a_5)$, with moment map $\Phi = (d_1, d_2)$. What inequalities must the image satisfy?

(*Hint:* Think about triangle inequalities.)

(b) What are the possible Euler characteristics of $\mathcal{P}o\ell(a_1, a_2, a_3, a_4, a_5)$? What is the diffeotype in each case?

(c) In the example of equilateral pentagon space $\mathcal{P}o\ell(1, 1, 1, 1, 1)$, the bending flow fails to define a global action. Where does the action fail to be defined? Identify these subset(s).

Question 7. Grassmannian calculations in cohomology.

(a) Use the ABBV formula to compute the self-intersection of $\mathbb{C}P^2_{e_1} \subset \mathcal{G}r_2(\mathbb{C}^4)$.

(b) Use the ABBV formula to compute the self-intersection of $\mathbb{C}P^1 \times \mathbb{C}P^1 \subset \mathcal{G}r_2(\mathbb{C}^4)$.