Exercises

Introduction to polynomial Poisson brackets

Three exercises

Throughout, $x_1, \dots, x_n$ are the standard coordinates on $\mathbb{C}^n$,

$$\partial_{x_j} = \frac{\partial}{\partial x_j} \qquad \text{and} \qquad \partial_j = x_j \partial_{x_j}.$$

For a weight $w = (w_1, \dots, w_n)^t \in (\mathbb{Z}_{\geq -1})^n$, set

$$J_w = \{j : w_j = -1\}, \qquad \mathbb{C}^{J_w} = \{a = (a_1, \dots, a_n)^t \in \mathbb{C}^n : a_k = 0 \text{ for } k \notin J_w\}.$$

Exercise 1. Tail weights, primitive weights, and the AD cone

Suppose a polynomial Poisson bivector π on $\mathbb{C}^4$ has log-canonical part π_0 and tail

$$\pi_{\text{tail}} = a x_2 \, \partial_{x_1} \wedge \partial_{x_3} + b x_3^2 \, \partial_{x_2} \wedge \partial_{x_4} + d x_3 \, \partial_{x_1} \wedge \partial_{x_4}, \qquad abd \neq 0.$$

(a) Compute the $(\mathbb{C}^\times)^4$-weight of each displayed tail monomial.

(b) Determine the primitive tail weights.

(c) Compute the AD cone of π.

(d) Find an integral point in the open AD cone and state the test degeneration it produces.

Exercise 2. Computing the full algebraic Poisson cohomology

Consider the log-canonical structure on $\mathbb{C}^4$ whose coefficient matrix is

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\lambda}^{-1} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Recall that the cohomology equation is

$$\boldsymbol{\lambda} w \in \mathbb{C}^{J_w}, \qquad w \in (\mathbb{Z}_{\geq -1})^4.$$

Compute

$$\mathsf{H}^p_{\text{Alg}}(\mathbb{C}^4, \pi_0), \qquad p = 0, 1, 2, 3, 4.$$

Exercise 3. Invariant weights, admissible deformation, and the AD cone

Let $p \in \mathbb{Z}_{>0}$ and $\mathbb{T} = \mathbb{C}^\times$. Identify $X^*(\mathbb{T}) \cong \mathbb{Z}$ and $\mathfrak{t} \cong \mathbb{C}$, and consider the action datum

$$\boldsymbol{\beta} = (p, 1, -1, -p), \qquad \mathbf{h} = (p, 1, -1, -p).$$

Thus $\mathbb{T}$ acts on $\mathbb{C}^4$ by

$$t \cdot (x_1, x_2, x_3, x_4) = (t^p x_1, t x_2, t^{-1} x_3, t^{-p} x_4).$$

Let π_0 be the action-type log-canonical structure

$$\begin{aligned}\pi_0 &= -\sum_{1 \le j < k \le 4} \beta_j(h_k) x_j x_k \partial_{x_j} \wedge \partial_{x_k} \\ &= -p x_1 x_2 \partial_{x_1} \wedge \partial_{x_2} + p x_1 x_3 \partial_{x_1} \wedge \partial_{x_3} \\ &\quad + p^2 x_1 x_4 \partial_{x_1} \wedge \partial_{x_4} + x_2 x_3 \partial_{x_2} \wedge \partial_{x_3} \\ &\quad + p x_2 x_4 \partial_{x_2} \wedge \partial_{x_4} - p x_3 x_4 \partial_{x_3} \wedge \partial_{x_4}.\end{aligned}$$

If $\boldsymbol{\lambda}$ is the coefficient matrix of π_0, recall that $\mathcal{S}(\pi_0)$ is the set of all $\mathbb{T}$-invariant degree-two cohomological weights of π_0; equivalently, it is the set of all $w \in \mathcal{W}_2$ satisfying

$$\boldsymbol{\lambda} w \in \mathbb{C}^{J_w} \qquad \text{and} \qquad \boldsymbol{\beta} w = 0.$$

(a) Determine $\mathcal{S}(\pi_0)$ by solving these two equations.

(b) Determine $\mathcal{S}(\pi_0)_{\ge 1} \cap \mathcal{W}_2$.

(c) Find the unique admissible deformation of π_0 along

$$\pi_1 = \sum_{\theta \in \mathcal{S}(\pi_0)} V_\theta.$$

(d) Determine the AD cone of the resulting Poisson structure π.