Introduction to polynomial Poisson brackets, II

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Lecture 2:

- Review of Lecture 1;
- Introduce $\mathbb{T}$-log-symplectic log-canonical Poisson structures;
- Deform and get interesting examples!

Review of Lecture 1.

- We consider $\mathbb{C}^n$ with standard coordinates $(x_1, \ldots, x_n)$.
- The torus $(\mathbb{C}^\times)^n$ acts on $\mathbb{C}^n$ by

$$(\mu_1, \ldots, \mu_n) \cdot (x_1, \ldots, x_n) = (\mu_1 x_1, \ldots, \mu_n x_n).$$

- Thus $(\mathbb{C}^\times)^n$ acts on the space of algebraic poly-vector fields;
- For each $j \in [1, n]$, $\partial_j := x_j \frac{\partial}{\partial x_j}$ has $(\mathbb{C}^\times)^n$-weight 0;.

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{'th}$$

- A monomial algebraic bi-vector field

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} = x^a \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} = x^{a - e_j - e_k} \partial_j \wedge \partial_k$$

has $(\mathbb{C}^\times)^n$-weight $a - e_j - e_k \in (\mathbb{Z}_{\geq -1})^n$, which has at most two entries of -1.

- For Example, on $\mathbb{C}^4$,

$$x_1^2x_3x_4^5\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+x_1x_3x_4^6\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_4}=x_1x_2^{-1}x_3x_4^5(\partial_1\wedge\partial_2+\partial_2\wedge\partial_4)$$

has $(\mathbb{C}^\times)^n$-weight $(1,-1,1,5)^4$ t $\in(\mathbb{Z}_{\geq -1})^4$.

- Let $J_w=\{j\in[1,n]:w_j=-1\}$ for $w=(w_1,\ldots,w_n)^t\in(\mathbb{Z}_{\geq -1})^n$;
- Set $\mathcal{W}_2=\{w\in(\mathbb{Z}_{\geq -1})^n:|J_w|\leq 2\}$.
- Then we have $(\mathbb{C}^\times)^n$-weight decomposition

$$\mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)=\bigoplus_{w\in\mathcal{W}_2}\mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^w.$$

- A log-canonical Poisson structure

$$\pi_0 = \sum_{1 \le j < k \le n} \lambda_{j,k} x_j x_k \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} = \sum_{j<k} \lambda_{j,k} \partial_j \wedge \partial_k,$$

where $\boldsymbol{\lambda} = (\lambda_{j,k})$ is skew-symmetric, has $(\mathbb{C}^\times)^n$-weight 0;

- We want to deform π_0 to Poisson $\pi = \pi_0 + \pi_{\mathrm{tail}}$, where

$$\pi_{\mathrm{tail}} = \bigoplus_{w \in \mathrm{W}(\pi)} \pi_w, \qquad \pi_w \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^w,$$

by controlling the set $\mathrm{W}(\pi) \subset \mathcal{W}_2 \backslash \{0\}$ of tail weights.

For a finite set $S \subset \mathcal{W}_2$ and $m \geq 1$, write

$$S_m = \{w^{(1)} + \cdots w^{(m)} : w^{(i)} \in S\}, \qquad S_{\geq m} = \bigcup_{m' \geq m} S_{m'}.$$

Definition: Let π_0 be log-canonical, and $S \subset \mathcal{W}_2$ finite and $0 \notin S_{\geq 1}$.

- $\pi \in \mathrm{Poi}_{\mathrm{Alg}}(\mathbb{C}^n)$ is an S-admissible algebraic deformation of π_0 if

$$\pi = \pi_0 + \pi_1 + \pi_{\geq 2}, \qquad \text{with}$$

$$\pi_1 \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S, \qquad [\pi_0, \pi_1] = 0, \qquad \pi_{\geq 2} \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^{S_{\geq 2}}.$$

- Also say that π is an S-admissible deformation of π_0 along π_1.

Important observation:

$0 \notin S_{\geq 1}$ ensures any S-admissible π still has π_0 as log-canonical term.

Admissible Deformation Lemma:

Let π_0 be log-canonical. Let $S \subset \mathcal{W}_2$ be finite with $0 \notin S_{\geq 1}$. Let

$$\pi_1 \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S \quad \text{such that} \quad [\pi_0, \pi_1] = 0.$$

Assume

$$\boxed{\mathrm{H}^3_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0 \quad \text{and} \quad \mathrm{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0.}$$

1. Always $\exists\ \pi_m \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^{S_m}$ for $m \geq 2$, such that

   $$\pi(t) = \pi_0 + t\pi_1 + t^2\pi_2 + \cdots$$

   satisfies $[\pi(t), \pi(t)] = 0$, and any two are gauge equivalent.
2. If $\mathfrak{X}^0_{\mathrm{Alg}}(\mathbb{C}^n)^{S_{\geq 1}} = 0$, every such $\pi(t)$ is polynomial in t;
3. If $\mathfrak{X}^1_{\mathrm{Alg}}(\mathbb{C}^n)^{S_{\geq 1}} = 0$, then $\pi(t)$ is polynomial in t and is unique.

Recall the Cohomological Equation: For $w \in (\mathbb{Z}_{\geq -1})^n$,

$$\mathrm{H}^{\bullet}_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^w \neq 0 \quad \overset{(1)}{\Longleftrightarrow} \quad \lambda w \in \mathbb{C}^{J_w} \quad \overset{(2)}{\Longleftrightarrow} \quad \lambda w \in (\mathbb{C}^{J_w})_0,$$

where

$$\mathbb{C}^{J_w} = \sum_{j \in J_w} \mathbb{C} e_j, \qquad (\mathbb{C}^{J_w})_0 = \{\sum_{j \in J_w} a_j e_j : \sum a_j = 0\}.$$

Proof: of (2): Write $\lambda w = \sum_{j=1}^{n} a_j e_j$. Then

$(1) \Rightarrow w^t \lambda w = (w^t \lambda w)^t = -w^t \lambda w$

$\Rightarrow 0 = w^t \lambda w = (w_1, \cdots, w_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{j \in J_w} a_j w_j$

$= -\sum_{j \in J_w} a_j$. //

Observation:

The conditions

$$\mathsf{H}^3_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0 \quad \text{and} \quad \mathsf{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0 \tag{1}$$

require certain non-degeneracy on π_0!

The answer: Yes if π_0 is $\mathbb{T}$-log-symplectic log-canonical.

Want $\mathbb{T} \cong (\mathbb{C}^\times)^r$ to act on $\mathbb{C}^n$

$\mathbb{T}$-log-symplectic log-canonical Poisson structures:

$\beta_j : \mathbb{T} \to \mathbb{C}^\times : t \mapsto t^{\beta_j}$.

Notation: Let $\mathbb{T} \cong (\mathbb{C}^\times)^r$ act on $\mathbb{C}^n$ via

$$t \cdot (x_1, \ldots, x_n) = (t^{\beta_1} x_1, \ldots, t^{\beta_n} x_n), \qquad \beta_j \in X^*(\mathbb{T}), \qquad j \in [1, n].$$

Write $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, and define

$$\mathbb{C}^n \longrightarrow \mathfrak{t}^* : \quad w \longmapsto \boldsymbol{\beta} w = \sum_{j=1}^{n} w_j \beta_j.$$

$\mathfrak{t} = \mathrm{Lie}(\mathbb{T})$

Definition: A log-canonical

$$\pi_0 = \sum_{j<k} \lambda_{j,k} \partial_j \wedge \partial_k, \qquad \boldsymbol{\lambda} = (\lambda_{j,k}),$$

$\left\{ \begin{array}{l} \lambda w = 0 \in \mathbb{C}^n \\ \beta w = 0 \in \mathfrak{t}^* \end{array} \right.$

is said to be $\mathbb{T}$-log-symplectic if $\ker \boldsymbol{\lambda} \cap \ker \boldsymbol{\beta} = \{0\}$., i.e., if

$\Rightarrow w = 0$.

$$\boxed{\text{Poisson tensor and the torus action are jointly nondegenerate.}}$$

$\mathbb{T}$-invariance

Notation: For any log-canonical Poisson structure π_0 on $\mathbb{C}^n$, let

$$\boxed{\mathcal{S}(\pi_0) = \{\theta \in \mathcal{W}_2 \setminus \{0\} : \mathrm{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^\theta \neq 0, \quad \beta\theta = 0\},}$$

the set of nonzero $\mathbb{T}$-invariant degree-two cohomological weights.

Theorem [Lu–Matviichuk]: Let π_0 be $\mathbb{T}$-log-symplectic. Then

- $\mathcal{S}(\pi_0)$ is a finite set;
- For any $S \subset \mathcal{S}(\pi_0)$ and $0 \notin S_{\geq 1}$, one has

$$\boxed{\mathrm{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0, \qquad \mathrm{H}^3_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^{S_{\geq 2}} = 0.}$$

(log-symplectic case proved by Matviichuk - Pym - Schedler)

Lemma. If π_0 is $\mathbb{T}$-log-symplectic, then every $\theta \in \mathcal{S}(\pi_0)$ has $|J_\theta| = 2$, and

$$\boxed{\mathrm{H}^2_{\mathrm{Alg}}(\mathbb{C}^n, \pi_0)^\theta = \mathbb{C}V_\theta,}$$

where for $J_\theta = \{j < k\}$, ie. $\theta_j = \theta_k = -1$, $\theta_i \in \mathbb{Z}_{\geq 0}$ for $i \neq j, k$.

$$V_\theta = x^\theta \partial_j \wedge \partial_k = \left(\prod_{a \neq j,k} x_a^{\theta_a} \right) \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

Proof. $\theta \in S(\pi_0) \Rightarrow \lambda\theta \in (\mathbb{C}^{J_\theta})_0$

If $|J_\theta| = 0$, or 1, then $(\mathbb{C}^{J_\theta})_0 = 0 \Rightarrow \lambda\theta = 0$.

Thus $\begin{cases} \lambda\theta = 0 \\ \beta\theta = 0 \end{cases} \Longrightarrow \theta = 0$ by log-symplecticity.

Contradiction to assumption that $\theta \neq 0$. //

We can move to the systematic examples of

- log-canonical Poisson structures of $\mathbb{T}$-action type!

$\mathfrak{t} = \mathrm{Lie}(\mathbb{T})$

Definition. An n-dimensional $\mathbb{T}$-action datum is a pair

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in X^*(\mathbb{T})^n \subset \mathfrak{t}^*, \quad \mathbf{h} = (h_1, \ldots, h_n) \in \mathfrak{t}^n$$

such that $\lambda_j := \beta_j(h_j) \neq 0$ for every $j \in [1, n]$.

Construction. Given $\mathbb{T}$-action datum $(\boldsymbol{\beta}, \mathbf{h})$, define log-canonical

$$\pi_0 = - \sum_{1 \le j < k \le n} \beta_j(h_k) x_j x_k \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k},$$

Note that $j < k$ here.

and say π_0 is of $\mathbb{T}$-action type in the ordered coordinates $(x_1, \ldots, x_n)$.

Next, we show that π_0 is $\mathbb{T}$-log-symplectic for the $\mathbb{T}$-action

$$t \cdot x_j = t^{\beta_j} x_j, \qquad j \in [1, n].$$

Need to show $\begin{cases} \lambda w = 0 \in \mathbb{C}^n \\ \beta w = 0 \in \mathfrak{t}^* \end{cases}$ for $w \in \mathbb{C}^n \Rightarrow w = 0$

Notation. Poisson coefficient matrix $\boldsymbol{\lambda}$ of π_0 is

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & -\beta_1(h_2) & -\beta_1(h_3) & \cdots & -\beta_1(h_n) \\ \beta_1(h_2) & 0 & -\beta_2(h_3) & \cdots & -\beta_2(h_n) \\ \beta_1(h_3) & \beta_2(h_3) & 0 & \cdots & -\beta_3(h_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1(h_n) & \beta_2(h_n) & \beta_3(h_n) & \cdots & 0 \end{pmatrix}.$$

Evaluating each β_j against each h_k gives $\lambda_j = \beta_j(h_j) \neq 0, \; j = 1, \ldots, n$

$$\boldsymbol{\sigma} = (\beta_k(h_j))_{j,k=1}^{n} = \begin{pmatrix} \lambda_1 & \beta_2(h_1) & \beta_3(h_1) & \cdots & \beta_n(h_1) \\ \beta_1(h_2) & \lambda_2 & \beta_3(h_2) & \cdots & \beta_n(h_2) \\ \beta_1(h_3) & \beta_2(h_3) & \lambda_3 & \cdots & \beta_n(h_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1(h_n) & \beta_2(h_n) & \beta_3(h_n) & \cdots & \lambda_n \end{pmatrix}.$$

Set $\boldsymbol{\nu} = \boldsymbol{\sigma} - \boldsymbol{\lambda}$.

Explicitly,

$$\boldsymbol{\nu} = \boldsymbol{\sigma} - \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & \beta_1(h_2) + \beta_2(h_1) & \cdots & \beta_1(h_n) + \beta_n(h_1) \\ 0 & \lambda_2 & \cdots & \beta_2(h_n) + \beta_n(h_2) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Here $\lambda_j = \beta_j(h_j) \neq 0$. Thus $\boldsymbol{\nu}$ is invertible.

Lemma. Every log-canonical π_0 of $\mathbb{T}$-action type is $\mathbb{T}$-log-symplectic.

Proof: Let $w \in \mathbb{C}^n$ be such that $\boldsymbol{\lambda} w = 0$ and $\boldsymbol{\beta} w = 0$.

- $\boldsymbol{\beta} w = 0$ implies $\boldsymbol{\sigma} w = 0$;
- $\boldsymbol{\lambda} w = 0$ and $\boldsymbol{\sigma} w = 0$ given $\boldsymbol{\nu} w = 0$, so $w = 0$.

$\square$

Consequences: For any log-canonical π_0 of $\mathbb{T}$-action type,

- The H^2- and H^3-vanishing statements hold;
- Thus the Admissible Deformation Lemma applies.

Need to know how to compute $\mathcal{S}(\pi_0)$!

Proposition. If π_0 is of $\mathbb{T}$-action type, then $\mathcal{S}(\pi_0)$ consists of all

$$\theta^{(j,k)} = \lambda_k \boldsymbol{\nu}^{-1}(e_j - e_k) \in \mathcal{W}_2 : \ 1 \leq j < k \leq n, \ \boldsymbol{\beta}\boldsymbol{\nu}^{-1} e_j = \boldsymbol{\beta}\boldsymbol{\nu}^{-1} e_k,$$

and we have

$$\theta^{(j,k)} = (\theta_1, \ldots, \theta_{j-1}, -1, \theta_{j+1}, \ldots, \theta_{k-1}, -1, 0, \ldots, 0)^t,$$

with every $\theta_a \geq 0$ for $a \neq j, k$.

$\theta \in S(\pi_0)$ $\quad J_\theta = \{ j < k \}$

$$\begin{cases} \lambda\theta = a(e_j - e_k) & (1) \\ \beta\theta = 0 & (2) \end{cases}$$

Proof. Linear algebra!

$(2) \Rightarrow \begin{cases} \sigma\theta = 0 \\ \lambda\theta = a(e_j - e_k) \end{cases}$

$+ (1)$

$\Rightarrow \nu\theta = (\sigma - \lambda)\theta = -a(e_j - e_k)$

Definitions.

- A weight $w \in (\mathbb{Z}_{\geq -1})^n$ is negatively bordered on the right if

$$w = (w_1, \ldots, w_{k-1}, -1, 0, \ldots, 0)^t$$

for some $k \in [1, n]$. We then write $\mathrm{rb}(w) = k$.

- $S \subset (\mathbb{Z}_{\geq -1})^n$ is negatively bordered on the right if every $w \in S$ is.

Lemma. If $S \subset \mathcal{W}_2$ negatively bordered on the right, then

$$S_{\geq 1} \cap (\mathbb{Z}_{\geq 0})^n = \varnothing, \quad \text{equivalently} \quad \mathfrak{X}^0_{\mathrm{Alg}}(\mathbb{C}^n)^{S_{\geq 1}} = 0.$$

Proof. Almost obvious. $\square$

dim $= |S|$

Corollary. For any log-canonical π_0 of $\mathbb{T}$-action type, any

$$S \subset \mathcal{S}(\pi_0) \quad \text{and any} \quad \pi_1 \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S,$$

$\Rightarrow [\pi_0, \pi_1] = 0$

there exists an S-admissible algebraic deformation of π_0 along π_1.

In practice, given $S \subset \mathcal{S}(\pi_0)$ and

$$\pi_1 = \sum_{\theta \in S} c_\theta V_\theta \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S, \qquad \text{with } (c_\theta)_\theta \in \mathbb{C}^S \text{ arbitrary,}$$

how to find all S-admissible algebraic deformations π of π_0 along π_1?

$S_{\geq 1} \cap (\mathbb{Z}_{\geq 0})^n = \emptyset$ implies $S_m \cap \mathcal{W}_2 = \varnothing$ for $m > N$. Write

$$\pi = \pi_0 + \pi_1 + \cdots + \pi_N, \qquad \pi_m \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^{S_m}.$$

Method 1.: Construct π_m recursively from

$$2[\pi_0, \pi_m] + \sum_{\substack{a+b=m \\ a,b\geq 1}} [\pi_a, \pi_b] = 0, \qquad m \geq 2.$$

Method 2 (Method of undetermined coefficients).

finite set

1. For each $m = 2, \ldots, N$, list the weights $w \in S_m \cap \mathcal{W}_2$.
2. Use the monomial basis $x^w \partial_j \wedge \partial_k$, where $J_w \subset \{j, k\}$, and write π_m as their linear combination with unknown coefficients.
3. Substitute into $[\pi, \pi] = 0$ and solve for coefficients.

Example. On $\mathbb{C}^4$, take the action-type log-canonical structure

$\mathbb{T} = (\mathbb{C}^\times)^2$

$$\{x_1, x_2\}_0 = x_1x_2, \qquad \{x_1, x_3\}_0 = -x_1x_3,$$
$$\{x_2, x_4\}_0 = x_2x_4, \qquad \{x_3, x_4\}_0 = -x_3x_4,$$

with the other brackets zero, and

$S(\mathbb{T}_0) = \{\theta^{(2,3)}, \theta^{(1,4)}\}$

$$\theta^{(2,3)} = (1, -1, -1, 0)^t, \qquad \theta^{(1,4)} = (-1, 1, 2, -1)^t.$$

For the fixed first-order term $\pi_1 = c_1x_1\partial_2 \wedge \partial_3 + c_2x_2x_3^2\partial_1 \wedge \partial_4$, the solutions contain an arbitrary $A \in \mathbb{C}$:

$$\begin{aligned}
\{x_1, x_4\} &= c_2x_2x_3^2 + Ax_1x_3,\\
\{x_2, x_3\} &= c_1x_1,\\
\{x_2, x_4\} &= x_2x_4 - Ac_1x_1 + (A - 2c_1c_2)x_2x_3,\\
\{x_3, x_4\} &= -x_3x_4 + (c_1c_2 - A)x_3^2.
\end{aligned}$$

Thus an S-admissible deformation along π_1 need not be unique.

We have better examples!.

Definition. A log-canonical π_0 is said to be of symmetric $\mathbb{T}$-action type if it is of $\mathbb{T}$-action type in both ordered sets of coordinates

$$(x_1, \ldots, x_n) \quad \text{and} \quad (x_n, \ldots, x_1).$$

Proposition. For π_0 of symmetric $\mathbb{T}$-action type every $\theta \in \mathcal{S}(\pi_0)$ has form

$$\theta = (0, \ldots, 0, -1, \theta_{j+1}, \ldots, \theta_{k-1}, -1, 0, \ldots, 0)^t, \qquad \theta_a \in \mathbb{Z}_{\geq 0}.$$

Obvious :

$$\mathcal{X}^1_{Alg}(\mathbb{C}^n)^{S \geq 1} = 0 \qquad \forall\ S \subset S(\pi_0)$$

Corollary. If π_0 is of symmetric $\mathbb{T}$-action type, any $S \subset \mathcal{S}(\pi_0)$ satisfies

$$\mathfrak{X}^1_{\mathrm{Alg}}(\mathbb{C}^n)^{S_{\geq 1}} = 0.$$

Consequently, for every

$$\pi_1 = \sum_{\theta \in S} c_\theta V_\theta \in \mathfrak{X}^2_{\mathrm{Alg}}(\mathbb{C}^n)^S, \qquad \text{with } (c_\theta)_\theta \in \mathbb{C}^S \text{ arbitrary,}$$

there is a unique S-admissible algebraic deformation of π_0 along π_1.

$\mathbb{T} \simeq (\mathbb{C}^\times)^r, \quad \mathfrak{t} = \mathrm{Lie}(\mathbb{T})$

Example.

Let $\langle\, , \,\rangle$ be a symmetric bilinear form on $\mathfrak{t}^*$, and let

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in X^*(\mathbb{T})^n, \qquad \langle \beta_j, \beta_j \rangle \neq 0, \quad \forall\, j \in [1, n].$$

Then

$$\pi_0 := \sum_{1 \leq j < k \leq n} \langle \beta_j, \beta_k \rangle x_j x_k \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

of symmetric $\mathbb{T}$-action type.

Log-canonical π_0 of symmetric $\mathbb{T}$-action type are abundant!

Primary examples from Lie theory: Let

$$A = (a_{i,i'})_{i,i'=1}^{r}$$

be a symmetrizable generalized Cartan matrix, with symmetrizer (d_i):

$$d_i a_{i,i'} = d_{i'} a_{i',i}.$$

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots, and define

$$\langle \alpha_i, \alpha_{i'} \rangle_A = d_i a_{i,i'}. \qquad \underline{\langle\ \ \rangle_A \text{ on}}\ \ t_A^*$$

For a sequence $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$, the roots

$$\boldsymbol{\beta}(\mathbf{i}) = \beta_1(\mathbf{i}), \ldots, \beta_n(\mathbf{i}), \qquad \beta_j(\mathbf{i}) = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j} \in t_A^*$$

real roots

satisfy $\langle \beta_j(\mathbf{i}), \beta_j(\mathbf{i}) \rangle_A \neq 0$ for every j. Let $\mathbb{T}_A$ be the torus with the root lattice as its character lattice.

$\langle \alpha_{i_j}, \alpha_{i_j} \rangle = 2 d_{i_j}$

The corresponding log-canonical Poisson structure is

$$\pi_0 = \pi_0^{(A,\mathbf{i})} = -\sum_{1\le j<k\le n} \langle \beta_j(\mathbf{i}), \beta_k(\mathbf{i})\rangle_A x_j x_k \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

For $j \in \{1, \dots, n\}$, define the successor

$$j^+ = \min\{k > j : i_k = i_j\},$$

with $j^+ = +\infty$ if there is no later occurrence, and set

$$\mathcal{J} = \{j : j^+ \le n\}.$$

Proposition. The nonzero $\mathbb{T}_A$-invariant H^2-weights of $\pi_0^{(A,\mathbf{i})}$ are

$$\mathcal{S}\big(\pi_0^{(A,\mathbf{i})}\big) = \{\theta^{(j,j^+)} : j \in \mathcal{J}\}, \qquad \text{where}$$

$$\theta^{(j,j^+)} = (0, \dots, 0, -1, -a_{i_{j+1},i_j}, \dots, -a_{i_{j^+-1},i_j}, -1, 0, \dots, 0)^t.$$

The family $\pi^{(A,\mathbf{i})}(c)$. For $c = (c_j)_{j\in\mathcal{J}} \in \mathbb{C}^{\mathcal{J}}$, set

$$\pi_1^{(A,\mathbf{i})}(c) = \sum_{j\in\mathcal{J}} c_j \left(\prod_{j<k<j^+} x_k^{-a_{i_k,i_j}} \right) \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_{j^+}}.$$

Theorem. There is a unique family of algebraic Poisson structures

$$\pi^{(A,\mathbf{i})}(c) = \pi_0^{(A,\mathbf{i})} + \pi_1^{(A,\mathbf{i})}(c) + \cdots + \pi_N^{(A,\mathbf{i})}(c), \qquad c = (c_j)_{j\in\mathcal{J}} \in \mathbb{C}^{\mathcal{J}},$$

s.t. $\pi_m^{(A,\mathbf{i})}(c)$ has weights in $\mathcal{S}(\pi_0^{(A,\mathbf{i})})_m$ and is degree m homogeneous in c.

The maximal normalized admissible deformation of $\pi_0^{(A,\mathbf{i})}$ is defined as

$$\pi^{(A,\mathbf{i})} := \pi^{(A,\mathbf{i})}\left(c_j = -\langle \alpha_{i_j}, \alpha_{i_j}\rangle_A = -2d_{i_j}\right)_{j\in\mathcal{J}}.$$

When A is of finite type, let G be the connected semisimple complex group with Cartan matrix A. The word $\mathbf{i}$ defines a Bott-Samelson cell

$$\mathcal{O}(s_{i_1},\ldots,s_{i_n}) \cong \mathbb{C}^n.$$

Theorem. In its standard affine Bott-Samelson coordinates,

the standard Poisson structure on the Bott–Samelson cell is $\pi^{(A,\mathbf{i})}$.

An example: Take A of affine $G_2^{(1)}$:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix},$$

$$(d_1, d_2, d_3) = (3, 3, 1), \qquad \mathbf{i} = (1, 2, 3, 1, 2, 3).$$

The successor pairs are

$$1^+ = 4, \qquad 2^+ = 5, \qquad 3^+ = 6, \qquad \mathcal{J} = \{1, 2, 3\}.$$

The primitive tail weights are

$$\begin{aligned} \theta_1 &= (-1, 1, 0, -1, 0, 0)^t, \\ \theta_2 &= (0, -1, 3, 1, -1, 0)^t, \\ \theta_3 &= (0, 0, -1, 0, 1, -1)^t. \end{aligned}$$

Thus

$$\begin{aligned} \pi_1(c) = c_1 x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + c_2 x_3^3 x_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_5} \\ + c_3 x_5 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_6}. \end{aligned}$$

Affine $G_2^{(1)}$ continued: the complete list of brackets for $j < k$ is

$$\begin{aligned}
\{x_1, x_2\} &= -3x_1x_2, \\
\{x_1, x_3\} &= -3x_1x_3, \\
\{x_1, x_4\} &= 3x_1x_4 + c_1x_2, \\
\{x_1, x_5\} &= -3x_1x_5 - \frac{c_1c_2}{6}x_3^3, \\
\{x_1, x_6\} &= \frac{c_1c_2c_3}{12}x_3^2, \\
\{x_2, x_3\} &= -3x_2x_3, \\
\{x_2, x_4\} &= -3x_2x_4, \\
\{x_2, x_5\} &= -6x_2x_5 + c_2x_3^3x_4,
\end{aligned}$$

$$\begin{aligned}
\{x_2, x_6\} &= -3x_2x_6 - \frac{c_2c_3}{2}x_3^2x_4, \\
\{x_3, x_4\} &= 0, \\
\{x_3, x_5\} &= -3x_3x_5, \\
\{x_3, x_6\} &= -x_3x_6 + c_3x_5, \\
\{x_4, x_5\} &= -3x_4x_5, \\
\{x_4, x_6\} &= -3x_4x_6, \\
\{x_5, x_6\} &= -3x_5x_6.
\end{aligned}$$

The higher sums of primitive tail weights are

$$\theta_1 + \theta_2, \qquad \theta_2 + \theta_3, \qquad \theta_1 + \theta_2 + \theta_3.$$

Affine $G_2^{(1)}$ continued: For $m = (m_1, \ldots, m_6) \in \mathbb{Z}^6_{\mathrm{row}}$,

$$\mathcal{D}^{\mathrm{AD}}(\mathbb{Z})^o = \left\{ m : \begin{array}{rcl} -m_1 + m_2 - m_4 & > & 0, \\ -m_2 + 3m_3 + m_4 - m_5 & > & 0, \\ -m_3 + m_5 - m_6 & > & 0 \end{array} \right\}.$$

For example, $m = (0, 1, 2, 0, 3, 0) \in \mathcal{D}^{\mathrm{AD}}(\mathbb{Z})^o$, so

$$\lim_{a \to 0} (\varphi_{a,m}^{-1})_* \pi^{(A,\mathbf{i})} = \pi_0^{(A,\mathbf{i})}.$$

Lecture 1.

- Use tail weights to study a polynomial Poisson structure π on $\mathbb{C}^n$;
- Admissible deformations of a log-canonical π_0 are controlled by their primitive tail weights;
- The Admissible Deformation Lemma under H^2- and H^3-vanishing conditions and $W(0)$ or $W(1)$.

Lecture 2.

- $\mathbb{T}$-log-symplecticity implies H^2 and H^3 vanishing conditions;
- π_0 of $\mathbb{T}$-action type implies $\mathbb{T}$-log-symplecticity and primitive weights can be computed explicitly.
- Symmetric action type gives the strongest deformation results, including examples from Lie theory.