POINT VORTEX DYNAMICS ON KÄHLER TWISTOR SPACES

SONJA HOHLOCH AND GUNER MUAREM

Abstract. In this paper, we provide tools to study the dynamics of point vortex dynamics on $\mathbb{CP}^n$ and the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$. These are the only Kähler twistor spaces arising from 4-manifolds. We give an explicit expression for Green's function on $\mathbb{CP}^n$ which enables us to determine the Hamiltonian H and the equations of motions for the point vortex problem on $\mathbb{CP}^n$. Moreover, we determine the momentum map $\mu : \mathbb{F}_{1,2}(\mathbb{C}^3) \to \mathfrak{su}^*(3)$ on the flag manifold.

Contents

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1. Introduction

1.1. **Point vortex dynamics.** The problem of dynamics of interacting point vortices goes back to the work of Helmoltz [Hel67] in the 19th century and can be formulated intuitively in its simplest form as follows. Consider N points $z_1, \dots, z_N$ (which we shall refer to as 'vortices') in the plane $\mathbb{C} \simeq \mathbb{R}^2$ with coordinates $z_k = x_k + iy_k$. Let $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}^{\neq 0}$ be real, non-zero numbers simulating the 'vortex strength' of each point. The equations determining this dynamical system are given by the N differential equations

$$\dot{\bar{z}}_j = \frac{1}{2\pi i} \sum_{k=1}^{N} \frac{\Gamma_k}{z_j - z_k} \qquad \text{for } 1 \leq j \leq N.$$

The signs of the vortex strengths determine the sense of rotation of each vertex. This system is in fact an example of a Hamiltonian system. Endow $\mathbb{C}^N$ with the symplectic form $\Omega := \sum_{k=1}^{N} \Gamma_k \tau_k^* \omega_{st}$ where ω_{st} is the standard symplectic form on $\mathbb{C}$ and $\tau_k : \mathbb{C}^N \to \mathbb{C}$ the projection on the kth component. Denote by $r(z_j, z_k) := |z_j - z_k|$ the Euclidean distance between points $z_j, z_k \in \mathbb{C} \simeq \mathbb{R}^2$ and set

$$\mathsf{Diag}_N(\mathbb{C}) := \{(z_1, \dots, z_N) \in \mathbb{C}^N \mid z_j = z_k \text{ for some } 1 \leq j, k \leq N \text{ with } j \neq k\}.$$

Abbreviate $z := (z_1, \dots, z_N)$ and consider the Hamiltonian function given by

$$H : \mathbb{C}^N \setminus \mathsf{Diag}_N(\mathbb{C}) \to \mathbb{R}, \quad H(z) := -\frac{1}{4\pi} \sum_{k \neq j} \Gamma_j \Gamma_k \log(r(z_j, z_k)).$$

Identifying $z_k = x_k + iy_k \simeq (x_k, y_k)$, the Hamiltonian equations are then given by

$$\begin{cases} \dot{x}_j = -\frac{1}{\Gamma_j} \partial_{y_j} H(x_1, \dots, x_N, y_1, \dots, y_N), \\ \dot{y}_j = \frac{1}{\Gamma_j} \partial_{x_j} H(x_1, \dots, x_N, y_1, \dots, y_N), \end{cases} \qquad \text{for } 1 \leq j \leq N.$$

Then one obtains the following three constants of motion (see for instance Galajinsky [Gal22]) which reflect the invariance under translation and rotation of the system:

$$p_x(z) := \sum_{j=1}^{N} \Gamma_j \Re(z_j), \qquad p_y(z) := \sum_{j=1}^{N} \Gamma_j \Im(z_j), \qquad m(z) := \frac{1}{2} \sum_{j=1}^{N} \Gamma_j |z_j|^2$$

where $\Re$ denotes the real part and $\Im$ the imaginary part of a complex number. More generally, Lin [Lin41] showed the following to be a fitting model for point vortex dynamics: Let (M, ω) be a symplectic manifold and consider the space $\mathcal{M} := \Pi_{k=1}^{N} M \setminus \mathsf{Diag}_N(M)$ as phase space of N moving vortices $z_1, \dots, z_N$ with vortex strengths $\Gamma_k \in \mathbb{R}^{\neq 0}$ for $1 \leq k \leq N$. Let G be the fundamental solution (also called *Green's function*) of the Laplace-Beltrami operator and set

$$R : M \to \mathbb{R}, \qquad R(z) := \lim_{s \to z} \left(G(s, z) - \frac{1}{2\pi} \log r(s, z) \right)$$

which is often referred to as *Robin function*. The Hamiltonian of the system is then given by

$$H : \mathcal{M} \to \mathbb{R}, \quad H(z_1, \dots, z_N) := \sum_{1 \le j < k \le N} \Gamma_i \Gamma_j G(z_j, z_k) + \sum_{k=1}^{N} \Gamma_k^2 R_g(z_k).$$

Green's function describes the interaction between pairs of distinct vortices and R describes self-interactions of the vortices. On homogeneous manifolds, R is often neglected due to symmetry reasons.

The equations of the point vortex problem are exactly the Euler equation arising in the discretization of fluid equations in mathematical modeling problems, see Aref [Are07], and Angrand [ADG85] for the corresponding numerics.

During the past years, quite some work has been done on generalizing this approach to other symplectic manifolds, for example:

(1) On the 2-spheres $\mathbb{S}^2$ with $\mathsf{SO}(3)$-invariant symplectic form and Hamiltonian vector field, see Crowdy [Cro06], Laurent-Polz & Montaldi & Roberts [LPMR11], Lim & Montaldi & Roberts [LMR01].
(2) There has also been some research done on the cylinder concerning periodic motion, see Montaldi & Souliere & Tokieda [MST03], Dritschel & Boatto [DB15].
(3) Point vortices on the cylinder, see Montaldi & Souliere & Tokieda [MST03].
(4) Point vortices on the hyperbolic plane, see Montaldi & Nava-Gaxiola [MNG14].
(5) Point vortices on $\mathbb{CP}^2$ with underlying symmetry group $\mathsf{SU}(3)$, see Montaldi & Shaddad [MS19a, MS19b].

A natural question is if the examples from above can be generalised to higher dimension. This naturally yields larger and more complicated symmetry groups. For example, in the case of the 2-sphere, the symmetry group is $G = \mathsf{SO}(3)$. But since the spheres $\mathbb{S}^{2n}$ for $n > 1$ do not admit a symplectic structure, generalizing straightforward to higher dimensions with $\mathsf{SO}(m)$-symmetry does not necessarily make sense. Nevertheless, since $\mathbb{CP}^1 \cong \mathbb{S}^2$ on can intuitively think of $\mathbb{CP}^n$ as the 'best symplectic approximation' of the $(n+1)$-sphere, albeit with underlying higher dimensional symmetry group $\mathsf{SU}(n)$.

1.2. **Relation to twistor spaces.** In the late seventies, Atiyah [AHS78] introduced the twistor theory for a 4-dimensional Riemannian manifold, relating it to 3-dimensional complex analysis. A few years later, in a paper by Hitchin [Hit81], the question arose which complex manifold could be obtained by using Atiyah's twistor construction on compact 4-manifolds. More precisely, the question was: which 4-manifolds have a twistor space which is Kähler?

Surprisingly, there are not many, namely only the 4-sphere $\mathbb{S}^4$ and the projective plane $\mathbb{CP}^2$ have Kähler twistor spaces. More specifically, the twistor space $\mathcal{T}(\mathbb{S}^4)$ is the complex projective space $\mathbb{CP}^3$ and $\mathcal{T}(\mathbb{CP}^2)$ is the 6-dimensional flag manifold (or Wallach space) $\mathbb{F}_{1,2}(\mathbb{C}^3) = \mathbb{W}^6 = \mathsf{SU}(3)/\mathbb{T}^2$. In this paper, we will be in particular interested in the spaces $\mathbb{CP}^n$ and the 6-dimensional flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$ in the context of point vortex dynamics.

1.3. **Main results.** One of the goals of this paper is to obtain an explicit expression for the Hamiltonian of the point vortex problem on certain coadjoint orbits in order to write down explicitly the equations of motion, look for conserved quantities, and analyse the underlying algebraic structure.

We are interested in symplectic manifolds with canonical $\mathsf{SU}(n)$-symmetry obtained by the coadjoint action of $\mathsf{SU(n)}$ on its dual Lie algebra $\mathfrak{su}(n)^*$. Specifically, we will focus on the case $n = 3$. There exist exactly two coadjoint orbits, namely the six dimensional 'generic' orbit

$$\mathcal{O}^{\mathsf{SU}(3)} = \frac{\mathsf{SU}(3)}{\mathsf{U}(1) \times \mathsf{U}(1)} \cong \mathbb{F}_{1,2}(\mathbb{C}^3)$$

and the four dimension 'degenerate' orbit (the meaning of this will become clear later on)

$$\mathcal{O}_d^{\mathsf{SU}(3)} = \frac{\mathsf{SU}(3)}{\mathsf{SU}(2) \times \mathsf{SU}(1)} \cong \mathbb{CP}^2$$

where the index d refers to degenerate.

The first main question that we address in this paper arose from the following context: Montaldi & Shaddad [MS19a, MS19b] studied the point vortex dynamics only on the degenerate orbit $\mathcal{O}_d^{\mathsf{SU}(3)}$ but not on the generic one. Thus a natural question is to investigate the point vortex problem on the generic orbit $\mathcal{O}^{\mathsf{SU}(3)}$. Before we start with the associated momentum map we need some notation. Let B be the subgroup of upper triangular matrices of $\mathsf{SL}(3, \mathbb{C})$. Then we may identify $\mathbb{F}_{1,2}(\mathbb{C}^3) \simeq \mathsf{SL}(3, \mathbb{C})/B$ (see Lemma 3.7 for details) of which the elements are of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} =: Z \in \mathsf{SL}(3, \mathbb{C})/B$$

with $z_1, z_2, z_3 \in \mathbb{C}$. Define the functions $K_1, K_2 : \mathbb{F}_{1,2}(\mathbb{C}^3) \to \mathbb{R}$ by

$$K_1(Z) := 1 + |z_1|^2 + |z_2|^2 \quad \text{and} \quad K_2(Z) = 1 + |z_3|^2 + |z_1 z_3 - z_2|^2.$$

Theorem 1.1. *The momentum map of the left action of* $\mathsf{SU}(3)$ *on the generic coadjoint orbit* $\mathcal{O}^{\mathsf{SU}(3)} = \mathbb{F}_{1,2}(\mathbb{C}^3) \simeq \mathsf{SL}(3, \mathbb{C})/B$ *is explicitly given by*

$$\mu : \mathsf{SL}(3, \mathbb{C})/B \to \mathfrak{su}(3)^*, \qquad \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} \mapsto (\mu_{ij})_{1 \le i,j \le 3}$$

where $(\mu_{ij})_{1 \le i,j \le 3}$ *is the traceless, anti-Hermitian matrix with entries*

$$\mu_{11} = \frac{1}{3} \left(\frac{x_3^2 + y_3^2 + 2}{K_2} - \frac{x_2^2 + y_2^2 - 1}{K_1} \right),$$

$$\mu_{22} = \frac{1}{3} \left(-\frac{2x_2^2 + 2y_2^2 + 1}{K_1} - \frac{x_3^2 + y_3^2 - 1}{K_2} \right),$$

$$\mu_{33} = -(\mu_{11} + \mu_{22}),$$

$$\mu_{12} = \frac{(iy_1 - x_1)(x_3 - iy_3) - iy_2 + x_2}{K_2} - \frac{x_1 - iy_1}{K_1},$$

$$\mu_{13} = \frac{(iy_1 - x_1)(x_3 - iy_3) - iy_2 + x_2}{K_2} - \frac{x_1 - iy_1}{K_1},$$
$$\mu_{23} = \frac{iy_3 + x_3}{K_2} - \frac{(x_1 + iy_1)(x_2 - iy_2)}{K_1}.$$

The remaining entries are determined by the fact that the matrix is anti-Hermitian.

Theorem 1.1 is restated as Theorem 4.3 and proven in Section 4. We do not yet have an explicit formula for the Hamiltonian H on $\mathbb{F}_{1,2}(\mathbb{C}^3)$ since this requires an explicit expression of Green's function on $\mathbb{F}_{1,2}(\mathbb{C}^3)$ which turned out to be more involved than expected and will be treated in a future work.

The second main question that we solved in this paper was motivated by the works of Montaldi & Shaddad [MS19a, MS19b] about the dynamics of the generalised vortex problem on $\mathbb{CP}^2$ on which they worked without an explicit expression for the Hamiltonian. In fact, it is possible to compute the Hamiltonian of the point vortex problem for general $\mathbb{CP}^n$ explicitly. The answer involves first computing Green's function on $\mathbb{CP}^n$ (stated and proven as Theorem 5.5 in Section 5):

Theorem 1.2. *Consider $\mathbb{CP}^n$ with the Fubini-Study metric. Then Green's function is given by $G : (\mathbb{CP}^n \times \mathbb{CP}^n) \setminus \mathsf{Diag}_2(\mathbb{CP}^n) \to \mathbb{R}$ with*

$$G(\xi, \eta) = -\frac{1}{2n \cdot \mathsf{vol}(\mathbb{CP}^n)} \left(\log(\sin(r(\xi,\eta))) - \sum_{j=1}^{n-1} \frac{1}{2j \sin^{2j}(r(\xi,\eta))} \right)$$

where $r(\xi,\eta) = \arccos \sqrt{\frac{\langle \xi,\eta\rangle_H \langle \eta,\xi\rangle_H}{\langle \xi,\xi\rangle_H \langle \eta,\eta\rangle_H}}$ is the geodesic distance between two point in $\mathbb{CP}^n$ and $\langle\cdot,\cdot\rangle_H$ is the Hermitian inner product.

The following theorem is stated and proven as Theorem 5.6 in Section 5.

Theorem 1.3. *The Hamiltonian for the N point vortex dynamics on the projective space $\mathbb{CP}^n$ is explicitly given by*

$$H : (\mathbb{CP}^n)^N \setminus \mathsf{Diag}_N(\mathbb{CP}^n) \to \mathbb{R},$$
$$H(\zeta) = -\frac{1}{2(n-1)!\,\pi^n} \sum_{\alpha<\beta}^{N} \Gamma_\alpha \Gamma_\beta \left(\log(\sin(r(\zeta_\alpha,\zeta_\beta))) - \sum_{j=1}^{n-1} \frac{1}{2j \sin^{2j}(r(\zeta_\alpha,\zeta_\beta))} \right)$$

where $\zeta = (\zeta_1, \ldots, \zeta_N)$ and $r(\zeta_\alpha, \zeta_\beta)$ is the geodesic distance on $\mathbb{CP}^n$ between the two points given by

$$r(\zeta_\alpha,\zeta_\beta) = \arccos \sqrt{\frac{\langle \zeta_\alpha,\zeta_\beta\rangle_H \langle \zeta_\beta,\zeta_\alpha\rangle_H}{\langle \zeta_\alpha,\zeta_\alpha\rangle_H \langle \zeta_\beta,\zeta_\beta\rangle_H}},$$

where $\langle\cdot,\cdot\rangle_H$ is the Hermitian inner product.

As already explained above, we do not yet have an explicit formula for the Hamiltonian H on $\mathbb{F}_{1,2}(\mathbb{C}^3)$ since we do not yet have an explicit expression of Green's function on $\mathbb{F}_{1,2}(\mathbb{C}^3)$. The hope (see Section 5.4) is to make use of the fibration

$$\mathbb{S}^2 \longrightarrow \mathbb{F}_{1,2}(\mathbb{C}^3) \longrightarrow \mathbb{CP}^2$$

and to obtain Green's function on $\mathbb{F}_{1,2}(\mathbb{C}^3)$ from Green's functions on $\mathbb{S}^2$ and $\mathbb{CP}^2$ and thus obtain the Hamiltonian on $\mathbb{F}_{1,2}(\mathbb{C}^3)$. This is planned to be carried out in a future work.

1.4. **Organisation of the paper.** We give a quick overview of this paper:

- In Section 2, we recall necessary notions and results from Lie algebras, representation theory and differential geometry.
- In Section 3, we consider geometric and algebraic features and properties of the two coadjoint orbits $\mathcal{O}_d^{\mathsf{SU}(3)} \simeq \mathbb{CP}^2$ and $\mathcal{O}^{\mathsf{SU}(3)} \simeq \mathbb{F}_{1,2}(\mathbb{C}^3)$.
- In Section 4, we analyse the momentum map for various situations.
- In Section 5, we study Green's function in various settings and compute an explicit formula for it on $\mathbb{CP}^n$. After that, we compute the associated Hamiltonian function.

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2. Preliminaries

2.1. **Notions and conventions from group actions and Lie theory.** Let G be a compact Lie group with Lie algebra $\mathsf{Lie}(G) = \mathfrak{g}$ and dual algebra $\mathfrak{g}^*$. The (left) *action* of a Lie group G on a manifold M is denoted by the map $\Phi : G \times M \to M$ which satisfies for all $x \in M$

(1) $\Phi(e, x) = x$,
(2) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ for all $g, h \in G$.

We usually write the action briefly as $g \cdot x$ or simply gx. Recall that Lie group actions are smooth. Moreover, the *isotropy subgroup* or *stabilizer* of a point m is given by the closed subgroup $G_m := \{g \in G \mid gm = m\}$. The orbit under G of a point $m \in M$ is given by

$$\mathcal{O}_m := \{gm \in M \mid g \in G\} \subseteq M.$$

Lying in the same orbit gives rise to an equivalence relation on the manifold M via $x \sim y \Leftrightarrow gx = y$ for $x, y \in M$ and $g \in G$. The space consisting of all these equivalence classes is called the *orbit space* and denoted by M/G. Now consider the action of a Lie group G on itself by conjugation

$$c_g : G \to G, \quad h \mapsto ghg^{-1}.$$

Identifying the Lie algebra $\mathfrak{g}$ with the tangent space T_eG at the neutral element $e \in G$ and differentiating c_g in e, we obtain for all $g \in G$

$$\mathsf{Ad}_g : T_eG \simeq \mathfrak{g} \;\to\; T_eG \simeq \mathfrak{g}, \quad \mathsf{Ad}_g X = gXg^{-1}$$

with adjoint representation

$$\mathsf{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad (g, X) \mapsto \mathsf{Ad}_g(X).$$

As dual notation, we have the *coadjoint representation*

$$\mathsf{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, \alpha) \mapsto \mathsf{Ad}^*_{g^{-1}} \alpha = g \alpha g^{-1}.$$

For every $\mu \in \mathfrak{g}^*$, the set

$$\mathcal{O}_\mu := \{\mathsf{Ad}^*_g \mu \mid \text{for all } g \in G\} \subseteq \mathfrak{g}^*$$

is the *coadjoint orbit* of G through μ.

Note that, in general, the adjoint and coadjoint representations (and thus the resulting orbits) are *not* isomorphic, see for instance counterexamples given by *groups of Euclidean type* (cf. Arathoon & Montaldi [AM18]).

In this paper, we are working with Lie algebras that consist of matrices. Here the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by the so-called Killing form $\kappa(\cdot, \cdot)$ which is a multiple of the trace of the product of the two matrices. For example, for $\mathfrak{su}(n)$, the Killing form is given by

$$\kappa_{\mathfrak{su}(n)}(X, Y) = \mathsf{trace}(XY) \quad \text{for all } X, Y \in \mathfrak{su}(n). \tag{2.1.1}$$

We denote the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ by

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \to \mathfrak{g}^*, \quad X \mapsto \kappa_X \text{ where } \kappa_X(Y) := \kappa(X, Y) \text{ for all } X, Y \in \mathfrak{g}.$$

2.2. **Exponential of a matrix.** In the context of Lie groups and Lie algebras, the exponential map is defined as

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \gamma(1)$$

where $\gamma : \mathbb{R} \to G$ is the unique one-parameter subgroup of G for which the tangent vector at the identity is X. In the case of a matrix Lie group, the exponential map is given by

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k \quad \text{for all } (n \times n)\text{-matrices } A,$$

briefly called the *exponential* of the matrix A. If $A = \mathsf{Diag}(a_{11}, \dots, a_{nn})$ is a diagonal matrix we obtain $A^k = \mathrm{Diag}(a_{11}^k, \dots, a_{nn}^k)$ and therefore

$$\begin{aligned}\exp(A) &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{Diag}(a_{11}^k, \dots, a_{nn}^k) = \mathsf{Diag}\left(\sum_{k=0}^{\infty} \frac{1}{k!} a_{11}^k, \dots, \sum_{k=0}^{\infty} \frac{1}{k!} a_{nn}^k \right) \\ &= \mathsf{Diag}\left(e^{a_{11}}, \dots, e^{a_{nn}}\right).\end{aligned}$$

If A is diagonalisable with $A = PDP^{-1}$, where P is the matrix of eigenvectors and D is the diagonal matrix with the eigenvalues on the diagonal, then

$$A^k = (PDP^{-1})^k = PDP^{-1} \cdots PDP^{-1} = PD^k P^{-1}$$

and therefore

$$\exp(A) = P \exp(D) P^{-1}. \tag{2.2.1}$$

2.3. **Symplectic manifolds, Hamiltonian dynamics, and momentum maps.** A symplectic manifold (M, ω) is a smooth manifold M equipped with a *symplectic form* ω which is a closed non-degenerate differential 2-form, i.e. $d\omega = 0$ and whenever $\omega_x(u, v) = 0$ for all $u \in T_xM$ then $v = 0$. This implies in particular that symplectic manifolds are always even dimensional.

Given a symplectic manifold (M, ω), the map

$$TM \to T^*M, \quad X \to \iota_X\omega \quad \text{where} \quad \iota_X\omega(Y) := \omega(X, Y) \text{ for all } Y \in TM$$

is an isomorphism, often referred to as *contraction* of ω by a vector field. Symplectic manifolds are the natural geometric background for Hamiltonian dynamics: Given a smooth function $H : M \to \mathbb{R}$, its *Hamiltonian vector field* (also called *symplectic gradient*) is defined via $\iota_{X_H}\omega = dH$. In this situation, H is often referred to as *Hamiltonian function*. Let G be a Lie group G with Lie algebra $\mathsf{Lie}(G) = \mathfrak{g}$ and assume that the action $G \times M \to M$ is by symplectomorphisms, i.e, for all $g \in G$, the map $M \to M$, $x \mapsto g.x$ is a symplectomorphism. Denote by $\langle \cdot, \cdot \rangle$ the dual pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Every $\xi \in \mathfrak{g}$ gives rise to a vector field X_ξ via

$$X_\xi(x) = \frac{d}{dt}\bigg|_{t=0} \exp(t\xi) \cdot x$$

for all $x \in M$. The *momentum map* for this G-action on (M, ω) is a map $\mu : M \to \mathfrak{g}^*$ such that

$$d(\langle \mu, \xi \rangle) = \iota_{X_\xi}\omega$$

for all $\xi \in \mathfrak{g}$ where

$$\langle \mu, \xi \rangle : M \to \mathbb{R}, \quad x \mapsto \langle \mu(x), \xi \rangle.$$

2.4. **Weyl group and coadjoint orbits of $\mathsf{SU}(n)$.** In this paper, we are in particular interested in the coadjoint orbits of the Lie group $\mathsf{SU}(3)$ since they are can be chosen to be the setting for vortex dynamics on the projective plane $\mathbb{CP}^2$ and the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$.

Let us start with fixing notation and recalling some properties of this Lie group and its Lie algebra. The general linear group is defined as

$$\mathsf{GL}(n, \mathbb{C}) = \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid \det A \neq 0\}.$$

The maximal compact simply connected Lie subgroup of $\mathsf{GL}(n, \mathbb{C})$ is given by

$$\mathsf{SU}(n) = \{U \in \mathrm{Mat}_n(\mathbb{C}) \mid U\overline{U}^T = 1,\ \det U = 1\}.$$

We have $\dim_{\mathbb{R}} \mathsf{SU}(n) = n^2 - 1$. The Lie algebra $\mathfrak{su}(n)$ of $\mathsf{SU}(n)$ can be identified with

$$\{U \in \mathrm{Mat}_n(\mathbb{C}) \mid \overline{U}^T = -U,\ \mathsf{trace}\, U = 0\},$$

meaning all $(n \times n)$-matrices which are skew-Hermitian matrices with trace zero. Our convention for the Lie bracket is $[A, B] = AB - BA$ for all $A, B \in \mathfrak{su}(n)$. We say that $A, B \in \mathfrak{su}(n)$ *commute* if $[A, B] = 0$.

In what follows, we will often work with the following basis of $\mathfrak{su}(3)$:

Notation 2.1. The following eight traceless traceless (3×3)-matrices are known as the Gell-Mann matrices.

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We have $[\tilde{\lambda}_3, \tilde{\lambda}_8] = 0$ and no other of these matrices commute with both $\tilde{\lambda}_3$ and $\tilde{\lambda}_8$. The set

$$\left\{ \lambda_k := \frac{i}{2} \tilde{\lambda}_k \;\middle|\; k = 1, \ldots, 8 \right\}$$

forms a (rescaled) basis for the Lie algebra $\mathfrak{su}(3)$, often called *Gell-Mann basis*.

Remark 2.2. The fact that only λ_3 and λ_8 commute (see Notation 2.1) implies that $\mathsf{SU}(3)$ has rank two so that there are two Casimirs denoted by $C_1, C_2 \in \mathfrak{su}(3)$, i.e., $[C_1, \lambda_k] = [C_2, \lambda_k] = 0$ for all $k = 1, \ldots, 8$. Explicitly, we have $C_1 = \sum_{k=1}^{8} \lambda_k \lambda_k$ and $C_2 = 8 \sum_{j,k,\ell=1}^{8} d_{jk\ell} \lambda_j \lambda_k \lambda_\ell$ where I_3 is the (3×3) unit matrix and $d_{jk\ell}$ the so-called structure constants of $\mathfrak{su}(3)$.

We will now recall the so-called *Weyl group*. Let E be a finite dimensional vector space over $\mathbb{R}$ and let $\langle \cdot, \cdot \rangle_E$ be an inner product on E. A *roots system* $\Phi \subset E$ is a finite set of non-zero vectors, called *roots* such that:

(1) The set of roots Φ spans the space E.
(2) If $\alpha \in \Phi$ and $c \in \mathbb{R}$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$.
(3) $\sigma_\alpha(\beta) := \beta - 2\frac{\langle \alpha, \beta \rangle_E}{\langle \alpha, \alpha \rangle_E} \alpha \in \Phi$ for all $\alpha, \beta \in \Phi$, i.e. Φ is invariant under σ_α for all $\alpha \in \Phi$ which is the reflection about the hyperplane orthogonal to α.
(4) $(\alpha, \beta) := 2\frac{\langle \alpha, \beta \rangle_E}{\langle \alpha, \alpha \rangle_E} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$, i.e. the projection of β onto the line through α is an integer or half-integer multiple of α.

A subset $\Phi^+ \subset \Phi$ is called a *positive root system* if

(1) for all $\alpha \in \Phi$, either $\alpha \in \Phi$ or $-\alpha \in \Phi$,
(2) for all $\alpha, \beta \in \Phi^+$, we have $\alpha + \beta \in \Phi^+$.

Denote by $\mathsf{O}(E) := \{A \in \mathsf{GL}(E) \mid \langle Av, Aw \rangle_E = \langle v, w \rangle_E, \text{ for all } v, w \in E\}$ the orthogonal group consisting of all elements in E preserving the inner product. The (finite) subgroup $\mathcal{W} \leq \mathsf{O}(E)$ generated by all reflections σ_α with $\alpha \in \Phi$ is called the *Weyl group* associated to Φ. Denote the hyperplane perpendicular to $\alpha \in \Phi$ by Π_α. The closure of a connected component of $E \setminus \{\Pi_\alpha \mid \alpha \in \Phi\}$ is called a *Weyl chamber*. We define the *positive Weyl chamber* (with respect to a fixed choice of Φ^+) as the closed set

$$\mathcal{C} = \{x \in E \mid \langle x, \alpha \rangle_E \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

Given a positive root system, there is only one positive Weyl chamber. Let G be a Lie group and consider a maximal torus $H \subset G$ (i.e. a compact, connected, abelian Lie subgroup of G which is maximal with respect to these properties) and the Cartan algebra $Lie(H) := \mathfrak{h} \subset \mathfrak{g}$ with dual $\mathfrak{h}^*$. Note that a maximal torus is unique up to conjugation. In the case of $\mathsf{SU}(n)$ the situation is as follows:

Example 2.3. The maximal torus T of $\mathsf{SU}(n)$ is given by the diagonal matrices $\mathsf{Diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ such that $\prod_{j=1}^n e^{i\theta_j} = 1$. The Lie algebra $\mathsf{Lie}(T) =: \mathfrak{t}$ (i.e. the Cartan algebra) is then given by the space of traceless diagonal matrices $\mathfrak{t} = \left\{ \mathsf{Diag}(\theta_1, \ldots, \theta_n) \,\middle|\, \sum_{j=1}^n \theta_j = 0 \right\}$. Thus, the interior $\mathfrak{t}_+^0$ of the positive Weyl chamber $\mathfrak{t}_+$ is given by

$$\mathfrak{t}_+ = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, x_1 > x_2 > \cdots > x_n \text{ and } \sum x_i = 0 \right\}$$

and the closure $\bar{\mathfrak{t}}_+ = \mathfrak{t}_+$ is given by replacing $>$ by $\geq$ in the above set. Here the Weyl group is the symmetric group $\mathsf{Sym}(3)$ generated by the positive roots α, β, $\alpha + \beta$ sketched in Figure 1. It permutes in fact all roots.

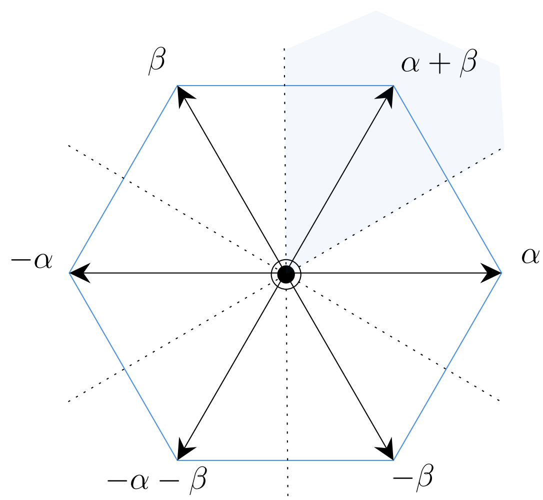

FIGURE 1. Root diagram of $\mathfrak{su}(3)$ where the positive roots are given by $\alpha = (1, 0)$, $\beta = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\alpha + \beta = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The simple roots are given by α and β. The blue shaded area is the positive Weyl chamber.

The orbits of the Weyl group are described by Bott's theorem:

Theorem 2.4 (Bott). *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$ and dual $\mathfrak{h}^*$. Let $\mathcal{O}_\mu$ be a coadjoint orbit of G. Then $\mathcal{O}_\mu \cap \mathfrak{h}^*$ is an orbit of the Weyl group.*

This implies that each coadjoint orbit $\mathcal{O}$ of G is uniquely defined by a starting point $\mu_0 \in \mathfrak{h}^*$ in the (closed) positive Weyl chamber (for more details, see for example Bernatska & Holod et al. [BH08]).

Definition 2.5. Denote the interior of the positive Weyl chamber W by W° and its boundary by ∂W. The coadjoint orbit of a point $\mu_0 \in W^\circ$ is said to be a *generic* orbit. The coadjoint orbit of a point $\mu_0 \in \partial W$ is called a *degenerate* orbit.

Example 2.6. The group $\mathsf{SU}(3)$ has exactly two coadjoint orbits, a generic one of dimension six and a degenerate one of dimension four. The generic orbit is denoted by $\mathcal{O}^{\mathsf{SU}(3)}$ and can be identified with

$$\mathcal{O}^{\mathsf{SU}(3)} = \frac{\mathsf{SU}(3)}{\mathsf{U}(1)\times\mathsf{U}(1)}.$$

The degenerate orbit is denoted by $\mathcal{O}_d^{\mathsf{SU}(3)}$ and can be identified with

$$\mathcal{O}_d^{\mathsf{SU}(3)} = \frac{\mathsf{SU}(3)}{\mathsf{SU}(2)\times\mathsf{SU}(1)} \cong \mathbb{CP}^2.$$

3. Geometric structures of coadjoint orbits of $\mathsf{SU}(3)$

In this section, we characterize coadjoint orbits of $\mathsf{SU}(3)$ by their algebraic and geometric properties and, eventually, describe their Kähler structure.

3.1. **Coadjoint orbits characterized by eigenvalues.** Let $(X, \{\cdot,\cdot\})$ be a Poisson manifold. A maximal connected submanifold $Y \subset X$ for which the Poisson structure descends to a symplectic structure is called a *symplectic leaf*. Moreover, the Poisson manifold is foliated by its symplectic leaves. Let $\mathfrak{g}$ be a Lie algebra with dual $\mathfrak{g}^*$. Then there is a canonical Poisson structure on $\mathfrak{g}^*$ called the Lie-Poisson structure. In this case, the symplectic leaves are the coadjoint orbits. A smooth function $C : \mathfrak{g} \to \mathbb{R}$ is called a *Casimir function* if C is constant on each coadjoint orbit, or equivalently, if C is invariant under the coadjoint action of G on $\mathfrak{g}^*$. Recall that level sets of Casimir functions $C : \mathfrak{g}^* \to \mathbb{R}$ are symplectic manifolds (see for instance Arnaudon & De Castro & Holm [ADCH18]) and that coadjoint orbits lie on level sets of Casimir functions (since they are invariant under the coadjoint action).

In the case of $\mathfrak{su}(n)$, the Casimir functions are given by

$$C_j : \mathfrak{su}^*(n) \to \mathbb{R}, \quad A \mapsto \mathsf{trace}(A^j)$$

for $j = 1, \ldots, n-1$ where $A^j := A \circ \ldots \circ A$ is the j-fold composition. For $n = 3$, we may therefore study the dynamics on the intersection of the Casimir level sets $C_1 = c_1$ and $C_2 = c_2$ for some constants c_1 and c_2. The set $\{C_1 = c_1\} \cap \{C_2 = c_2\}$ can be identified with the space

$$\mathcal{M} := \{A \in \mathrm{Mat}_3(\mathbb{C}) \mid A = \overline{A}^T,\ \mathsf{trace}(A) = 0,\ \mathsf{trace}(A^2) = c_1,\ \mathsf{trace}(A^3) = c_2\}$$

which can be seen as phase space of a $\mathsf{SU}(3)$-invariant dynamical system. In the generic case, we have $\dim(\mathcal{M}) = 6$ and, in the degenerate case, $\dim(\mathcal{M}) = 4$ which corresponds to the dimensions of two coadjoint orbits of $\mathsf{SU}(3)$, see Example 2.6.

Lemma 3.1. *The phase space $\mathcal{M}$ is isomorphic as vector space to the space*

$$\{A \in \mathrm{Mat}_3(\mathbb{C}) \mid A = \overline{A}^T, A \text{ has eigenvalues } \lambda_1 \geq \lambda_2 \geq \lambda_3 \text{ with } \lambda_1 + \lambda_2 + \lambda_3 = 0\}.$$

Proof. The characteristic polynomial χ_A for a traceless Hermitian matrix A can be written non-factorized and factorized:

$$\chi_A = \det(A - \lambda I_3) = -\lambda^3 - \frac{1}{2}\mathsf{trace}(A^2)\lambda + \frac{1}{3}\mathsf{trace}(A^3) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda).$$

In the non-factorized version, two coefficients are expressed by the trace and, in the factorized version, the eigenvalues of the matrix appear. Comparing the coefficients in the expression above with the definition of $\mathcal{M}$ and the claim in the statement yields the bijection. □

Let $\Lambda := \mathsf{Diag}(\lambda_1, \lambda_2, \lambda_3)$ with $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ and set $\mathcal{O}_\Lambda = \{A\Lambda A^{-1} \mid A \in \mathsf{SU}(3)\}$. The spectral theorem for Hermitian matrices states that the eigenvalues of a Hermitian matrix are real and that the eigenvectors corresponding to these eigenvalues are orthogonal. This allows to deduce the following bijective correspondence:

$$\left\{\begin{matrix}\text{coadjoint orbits}\\ \text{of } \mathsf{SU}(3)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\mathcal{O}_\Lambda \text{ with } \lambda_1+\lambda_2+\lambda_3=0\\ \text{and } \lambda_1 \geq \lambda_2 \geq \lambda_3\end{matrix}\right\}$$

This leads to three types orbits (of which one is trivial):

(i) *All three eigenvalues are distinct.* Then the stabilizer is given by $\mathsf{Diag}(\alpha, \beta, \overline{\alpha\beta})$ with $\alpha, \beta \in \mathbb{C}$ and the coadjoint orbit is
$$\mathcal{O}^{\mathsf{SU}(3)} = \frac{\mathsf{SU}(3)}{\mathsf{U}(1)\times\mathsf{U}(1)} = \frac{\mathsf{U}(3)}{\mathsf{U}(1)\times\mathsf{U}(1)\times\mathsf{U}(1)}.$$
We will see in Section 3.2 that this orbit can be identified with a six dimensional flag manifold.

(ii) *Two eigenvalues are equal.* The stabilizer is given by the block diagonal matrix $\mathsf{Diag}(A, \overline{\det A})$ where $A \in \mathsf{U}(2)$. In this case the coadjoint orbit can be identified with
$$\mathcal{O}_d^{\mathsf{SU}(3)} = \mathsf{SU}(3)/\mathsf{U}(2) \cong \mathbb{CP}^2.$$

(iii) *All eigenvalues are equal.* In this case, the stabilizer is $\mathsf{SU}(3)$ so that the orbit is trivial (since $\lambda_1 + \lambda_2 + \lambda_3 = 3\lambda = 0$ implies $\lambda = 0$).

Remark 3.2. More generally, setting $\Lambda := \mathsf{Diag}(\lambda_1, \dots, \lambda_n)$ with $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $\lambda_1 \geq \dots \geq \lambda_n$ and $\sum_{i=1}^n \lambda_i = 0$, the coadjoint orbits of $\mathsf{SU}(n)$ are of the form $\mathcal{O}_\Lambda = \{A\Lambda A^{-1} \mid A \in \mathsf{SU}(n)\}$.

3.2. **Coadjoint orbits seen as flag manifolds.** The four dimensional degenerate orbit space has a nice geometrical interpretation as $\mathbb{CP}^2$. We will now see that there is also a nice geometric characterisation of the generic orbit as a so-called flag manifold (of which the degenerate orbit $\mathbb{CP}^2$ is a special case).

Definition 3.3. Consider $\mathbb{C}^n$ and let $r \in \{1, \dots, n\}$. A *flag* $f_{k_1,\dots,k_r;n}$ in $\mathbb{C}^n$ is a nested sequence of vector subspaces $V_{k_1} \subsetneq \dots \subsetneq V_{k_r}$ in $\mathbb{C}^n$ such that $\dim_{\mathbb{C}} V_{k_j} = k_j$ for all $1 \leq j \leq r$. The space of all such flags is denoted by $\mathbb{F}_{k_1,\dots,k_r}(\mathbb{C}^n)$.

Remark 3.4. $\mathbb{F}_{k_1,\dots,k_r}(\mathbb{C}^n)$ is a compact, complex and smooth manifold and is usually referred to as the *flag manifold*. Note that all flag manifolds are in fact generalisations of projective spaces. The flag manifold $\mathbb{F}_1(\mathbb{C}^n)$ is precisely $\mathbb{CP}^{n-1}$. Moreover, the flag manifold $\mathbb{F}_k(\mathbb{C}^n)$ is the space of k-dimensional vector subspaces of $\mathbb{C}^n$, i.e., the Grassmannian.

For $n = 3$, $k_1 = 1$ and $k_2 = 2$, we obtain the generic coadjoint orbit of $\mathsf{SU}(3)$
$$\mathcal{O}^{\mathsf{SU}(3)} = \mathbb{F}_{1,2}(\mathbb{C}^3) = \{(L, P) \mid L \subset P \subset \mathbb{C}^3 \text{ with } \dim_{\mathbb{C}}(L) = 1, \dim_{\mathbb{C}}(P) = 2\}.$$

This space also appears in the context of so-called Wallach manifolds introduced by Wallach [Wal72] which we will describe now. Consider the linear map $J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ defined as

$$J(z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}) = (z_{2n}, \dots, z_{n+1}, -z_n, \dots, -z_1).$$

The group $\mathsf{Sp}(n)$ is defined as

$$\mathsf{Sp}(n) = \{A \in \mathsf{SU}(2n) \mid AJ = J\bar{A}\}.$$

Let $\mathsf{F}(4)$ be the 52-dimensional exceptional simple Lie group. Moreover, recall that the universal cover of the orthogonal group $\mathsf{SO}(8)$ is called the spin group and is denoted by $\mathsf{Spin}(8)$. The Wallach manifolds W^6 of dimension six, W^{12} of dimension twelve, and W^{24} of dimension twenty-four are given by

$$W^6 := \frac{\mathsf{SU}(3)}{\mathsf{U}(1) \times \mathsf{U}(1)}, \quad W^{12} := \frac{\mathsf{Sp}(3)}{\mathsf{Sp}(1) \times \mathsf{Sp}(1) \times \mathsf{Sp}(1)} \quad \text{and} \quad W^{24} := \frac{\mathsf{F}(4)}{\mathsf{Spin}(8)}.$$

These are all compact Riemannian manifolds of positive curvature. Moreover, these manifolds can be thought of as the total space of the following homogeneous fibrations:

$$\mathbb{S}^2 \longrightarrow W^6 \longrightarrow \mathbb{CP}^2,$$
$$\mathbb{S}^4 \longrightarrow W^{12} \longrightarrow \mathbb{HP}^2,$$
$$\mathbb{S}^8 \longrightarrow W^{24} \longrightarrow \mathbb{OP}^2.$$

For more details on these fibrations, we refer the reader to Dearricott & Galaz-Garciá et al. [Dea14] and the references therein.

Remark 3.5. On each flag manifold there is a natural action of the isometry group of $\mathbb{CP}^2$, resp. $\mathbb{HP}^2$, resp. $\mathbb{OP}^2$. When looking at these fibrations, one might ask if it is possible to generalise the point vortex dynamics to $\mathbb{HP}^2$ and $\mathbb{OP}^2$. However, neither $\mathbb{HP}^2$ nor $\mathbb{OP}^2$ admit a symplectic structure since $\mathbb{HP}^2$ is a quaternionic Kähler manifold and these are not symplectic. Moreover, any symplectic manifold admits compatible almost-complex structures, but $\mathbb{OP}^2$ does not admit one.

3.3. **Examples of coadjoint orbits of $\mathsf{SU}(4)$.** In this article, we mainly focus on the two coadjoint orbits of $\mathsf{SU}(3)$ given by $\mathbb{CP}^2$ and $\mathbb{F}_{1,2}(\mathbb{C}^3)$. Now we want to have a brief glance at the situation for $\mathsf{SU}(4)$ (see Bernatska & Holod [BH08]).

coadjoint orbit	type	dimension	name
$\mathsf{SU}(4)/\mathsf{U}(1)^3$	generic	12	$\mathbb{F}_{1,2,3}(\mathbb{C}^4)$ ('full flag')
$\mathsf{SU}(4)/\mathsf{SU}(2) \times \mathsf{U}(1) \times \mathsf{U}(1)$	degenerate	10	$\mathbb{F}_{1,2}(\mathbb{C}^4)$ ('partial flag')
$\mathsf{SU}(4)/S(\mathsf{U}(2) \times \mathsf{U}(2))$	degenerate	8	$\mathsf{Gr}_2(\mathbb{C}^4)$ (Grassmannian)
$\mathsf{SU}(4)/\mathsf{SU}(3) \times \mathsf{U}(1)$	degenerate	6	$\mathbb{CP}^3$ (projective space)

The 'partial flag manifold' $\mathbb{F}_{1,2}(\mathbb{C}^4)$ consists of pairs (L, P) where L is a one-dimensional and P is a two-dimensional subspace of $\mathbb{C}^4$ such that $L \subset P$. There are natural projections $\varphi_1(L, P) = L$ and $\varphi_2(L, P) = P$.

A fibre bundle over A with fibre B with map τ is denoted by $\mathcal{E}(A, B, \tau)$. We have in particular that the generic orbit is a fibre bundle over a degenerate orbit, more precisely (see Bernatska & Holod [BH08])

$$\frac{\mathsf{SU}(4)}{\mathsf{U}(1)^3} = \mathcal{E}(\mathbb{CP}^3, \mathbb{F}_{1,2}(\mathbb{C}^3), \tau_2) = \mathcal{E}(\mathsf{Gr}_2(\mathbb{C}^4), \mathbb{CP}^1, \tau_2),$$

and the three degenerate spaces are in connection with each other by the following double fibration:

$$\begin{array}{ccc} \mathbb{F}_{1,2}(\mathbb{C}^4) & \xrightarrow{\tau_1} & \mathbb{CP}^3 \\ \tau_2 \big\downarrow & & \\ \mathsf{Gr}_2(\mathbb{C}^4) & & \end{array}$$

As a general remark, the partial flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^4)$ can also be described in terms of algebraic geometry by using the *Plücker embedding*. In the case of the Grassmannian this embeddings has an interesting interpretation:

$$\mathsf{Gr}_2(\mathbb{C}^4) \hookrightarrow \mathbb{P}\left(\bigwedge^2 \mathbb{C}^4\right) \cong \mathbb{CP}^5.$$

Thus, the image of the Grassmannian is a quadric in $\mathbb{CP}^5$, often called the *Klein quadric*.

3.4. **Bruhat decomposition and induced coordinates.** So far, we described the coadjoint orbits in terms of matrices and gave a geometrical interpretation in terms of flag manifolds. Now we will focus on the analytical structure which will allow us to determine the Laplace operator to the aim of finding the corresponding Green's function. For that, we need a bit of notation:

A closed subgroup P of a Lie group G is *parabolic* if the quotient variety G/P satisfies the following property: for any variety Y, the projection map $(G/P) \times Y \to Y$ maps closed set to closed sets. Furthermore, a closed, connected and solvable subgroup of G is called a *Borel subgroup*. Note that all Borel subgroups are mutually conjugate.

Given a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, its complexification is defined by $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, recall that a real subalgebra $\mathfrak{f}$ of a complex Lie algebra $\mathfrak{h}$ is called a *real form* of $\mathfrak{h}$ if every $h \in \mathfrak{h}$ can be uniquely written as $h = h_1 + ih_2$ with $h_1, h_2 \in \mathfrak{f}$. The complexification of $\mathfrak{f}$ yields again $\mathfrak{h}$, i.e., $\mathfrak{f}^{\mathbb{C}} \cong \mathfrak{h}$. Note that not every complex Lie algebra has a real form. Moreover, there are in general several non-isomorphic real forms for a given complex Lie algebra:

Example 3.6. The Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ has the following (non-isomorphic) real forms:

$$\mathfrak{sl}(3, \mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -(a+e) \end{pmatrix} \middle| \, a, b, c, d, e, f, g, h \in \mathbb{R} \right\},$$

$$\mathfrak{su}(1,2) = \left\{ \begin{pmatrix} a+bi & c+di & ei \\ f+gi & -2bi & -c+di \\ hi & -f+gi & -a+bi \end{pmatrix} \middle| \, a,b,c,d,e,f,g,h \in \mathbb{R} \right\},$$

$$\mathfrak{su}(3) = \left\{ \begin{pmatrix} ai & c+di & g+hi \\ -(c-di) & ib & e+fi \\ -(g-hi) & -(e-fi) & -i(a+b) \end{pmatrix} \middle| \, a,b,c,d,e,f,g,h \in \mathbb{R} \right\}.$$

The first one is called the *split real form*, the second one *quasi-split form*, and the last one is referred to as *compact form*.

Let G be a compact and connected Lie group. The complexification of G is defined as the complex Lie group $G^{\mathbb{C}}$ that contains G as a closed subgroup and that has the following (universal) property: every homomorphism $f : G \to L$, for every complex Lie group L, lifts to a homomorphism $G^{\mathbb{C}} \to L$. Moreover, on the level of Lie algebras, $\mathsf{Lie}(G^{\mathbb{C}}) = \mathfrak{g}^{\mathbb{C}}$ is the complexification of $\mathsf{Lie}(G) = \mathfrak{g}$.

The following result was proven in more generality by Picken [Pic90], but we sketch the proof here for the reader's convenience.

Lemma 3.7. *Denote by B the subgroup of upper triangular matrices of $\mathsf{SL}(3,\mathbb{C})$. Then there is an isomorphism $\mathsf{SU}(3)/\mathbb{T}^2 \cong \mathsf{SL}(3,\mathbb{C})/B$.*

Proof. Take a matrix $g \in \mathsf{SL}(3,\mathbb{C})$ and denote its columns by $\underline{g}_k$ for $k = 1,2,3$ so that the matrix can be written as $g = (\underline{g}_1, \underline{g}_2, \underline{g}_3)$. Let $\langle \cdot, \cdot \rangle_H$ be the Hermitian inner product on $\mathbb{C}^3$ given by $\langle x, y \rangle_H = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$ for $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{C}^3$. A priori, the vectors $\underline{g}_k$ are not orthonormal. Nevertheless, using the Gramm-Schmidt procedure they can be made orthonormal:

$$\begin{aligned} \underline{g}'_1 &:= \underline{g}_1 \\ \underline{g}'_2 &:= \underline{g}_2 - \frac{\langle \underline{g}_2, \underline{g}'_1 \rangle}{\langle \underline{g}'_1, \underline{g}'_1 \rangle} \underline{g}'_1 \\ \underline{g}'_3 &:= \underline{g}_3 - \frac{\langle \underline{g}_3, \underline{g}'_2 \rangle}{\langle \underline{g}'_2, \underline{g}'_2 \rangle} \underline{g}'_2 - \frac{\langle \underline{g}_3, \underline{g}'_1 \rangle}{\langle \underline{g}'_1, \underline{g}'_1 \rangle} \underline{g}'_1 \end{aligned}$$

Normalising each of them via $\underline{v}_k := \frac{\underline{g}'_k}{||\underline{g}'_k||}$, the matrix given by $U = (\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is an element of $\mathsf{SU}(3)$. Moreover, it satisfies $U = gb'$ for some upper triangular matrix $b' \in B$ which performs the Gramm-Schmidt procedure on g. Thus $g = U(b')^{-1}$ lies in $\mathsf{SL}(3,\mathbb{C})/B$. This means that we can write $\mathsf{SL}(3,\mathbb{C}) = \mathsf{SU}(3)B$ and that $\mathsf{SU}(3) \cap B = \mathbb{T}^2$ (this intersection exactly result in the 2-torus). This induces the wanted isomorphism in the following way: take an equivalence class $[g]_B \in \mathsf{SL}(3,\mathbb{C})/B$ which can also be written as $[Ub]_B$ (see Gramm-Schmidt procedure). It is then mapped to $[U]_{\mathbb{T}^2} \in \mathsf{SU}(3)/\mathbb{T}^2$. □

The power of this isomorphism lies in the fact that one can make the transition from the geometrical picture of the coadjoint orbit (as being the flag manifold realised as the homogeneous space $\mathsf{SU}(3)/\mathbb{T}^2$) to the complex manifold $\mathsf{SL}(3,\mathbb{C})/B$. This is convenient as there exists a well-developed theory of so-called Bruhat coordinates on the complex manifold, which will be useful for our approach.

Definition 3.8. Let $\mathfrak{g}$ be a semi-simple Lie algebra with Cartan algebra $\mathfrak{h}$ and root system Φ. The weight space decomposition of a Lie algebra $\mathfrak{g}$ is given by the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H) \text{ for } H \in \mathfrak{h}\}$. All $\alpha \in \mathfrak{h}^*$ that are non zero are called *roots*.

Recall that a Lie algebra is *simple* if $\mathfrak{g}$ is not abelian and $\mathfrak{g}$ has no non-trivial ideals. For a simple Lie algebra $\mathfrak{g}$, we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}_\pm := \bigoplus_{\alpha\in\Phi^\pm} \mathfrak{g}_\alpha$ are the so-called upper and lower nilpotent subalgebras consisting of the positive (resp. negative) roots of $\mathfrak{g}$. Moreover, set

$$\mathfrak{b}_\pm := \mathfrak{h} \oplus \mathfrak{n}_\pm$$

and call them the *upper* and *lower Borel subalgebras*. On the Lie group level, $B^\pm$ and $N^\pm$ are called the *Borel subgroups* and *unipotent subgroups* of the Lie group G. In particular, we say that N^- is the *opposite* unipotent subgroup.

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Consider a Borel subgroup $B \leq G$ and the Weyl group $\mathcal{W}$ associated with G. Then the *Bruhat decomposition* of G is given by $G = \bigcup_{w\in\mathcal{W}} BwB$. This decomposition gives rise to the *cell decomposition* of the homogeneous space $G/B = \bigcup_{w\in\mathcal{W}} BwB/B$. Each of the BwB/B corresponds to an affine space of dimension $\ell(w)$ where $\ell(w)$ is the length of the Weyl group element w given by the minimal k such that w can be written as a product of k generators of the Weyl group. Note that there is always an element in the Weyl group $\mathcal{W}$ which has maximal length, in this case the length of this element equals the number of positive roots Φ^+. This element in the Weyl group is denoted by $w_o \in \mathcal{W}$. It has the property that $w_0(\Phi^+) = \Phi^-$, i.e. it interchanges the positive and negative roots. In the Bruhat decomposition

$$G/B = \bigcup_{w\in\mathcal{W}} BwB/B,$$

the identity element e gives rise to an open subset BeB which is called the *big cell* and is denoted by X_e. The flag manifold is identified with the space $\mathsf{SL}(3, \mathbb{C})/B$ where B is the subgroup of upper triangular matrices. In this context, we have

$$N^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} \middle| \, z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

Proposition 3.9 ([BMHM94])**.** *N^- acts freely and transitively on the big cell X_e. Thus we may identify X_e with N^-.*

Note that the translates $g \cdot N_-$ of the big cell (under the G-action) cover the whole flag manifold.

The flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$ can also be seen as

$$\mathbb{F}_{1,2}(\mathbb{C}^3) = \{(V_1, V_2) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \mid V_2(V_1) = 0\}. \tag{3.4.1}$$

This means intuitively that the manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$ consists of all pairs (V_1, V_2) where V_2 is a projective line in $\mathbb{CP}^2$ and V_1 is a point on the line.

Theorem 3.10 ([BMHM94]). *Consider the group* $\mathsf{SL}(3,\mathbb{C})$ *and the Borel subgroup* B *of upper triangular matrices. Let* $R_1 := [0:0:1] \in \mathbb{CP}^2$ *and* $R_2 := [0:x:y] \in (\mathbb{CP}^2)^*$. *Then the (Bruhat) cell-decomposition of* $\mathsf{SL}(3,\mathbb{C})/B$ *consists of the following six cells (where the indices in the Bruhat cell correspond to the group elements of the symmetric group in three elements, see Table 1):*

(1) The big cell, *which has codimension zero, is given by*

$$X_e := \{(V_1,V_2) \in \mathbb{F}_{1,2}(\mathbb{C}^3) \mid R_2(V_1) \neq 0,\ V_2(R_1) \neq 0\}.$$

(2) There are two Bruhat cells of codimension one given by

$$X_{(1,2)} := \{(V_1,V_2) \in \mathbb{F}_{1,2}(\mathbb{C}^3) \mid R_2(V_1) = 0,\ R_1 \neq V_1,\ R_2 \neq V_2\},$$
$$X_{(2,3)} := \{(V_1,V_2) \in \mathbb{F}_{1,2}(\mathbb{C}^3) \mid V_2(R_1) = 0,\ R_1 \neq V_1,\ R_2 \neq V_2\}.$$

(3) There are two Bruhat cells of codimension two given by

$$X_{(1,2,3)} := \{(V_1,V_2) \in \mathbb{F}_{1,2}(\mathbb{C}^3) \mid V_1 = R_1,\ R_2 \neq V_2\},$$
$$X_{(1,3,2)} := \{(V_1,V_2) \in \mathbb{F}_{1,2}(\mathbb{C}^3) \mid V_2 = R_2,\ R_1 \neq V_1\}.$$

(4) The 0*-cell, which has codimension three, is given by*

$$X_{(1,3)} := \{(V_1,V_2) \in \mathbb{F}_{1,2}(\mathbb{C}^3) \mid V_1 = R_1,\ R_2 = V_2\}.$$

Moreover, the opposite unipotent subgroup N^- *acts transitively on the big cell* X.

Recall that the Weyl group of $\mathsf{SL}(3,\mathbb{C})$ is isomorphic to $\mathsf{Sym}(3)$, the symmetric group of order three. We now give an overview how the elements of $\mathsf{Sym}(3)$ correspond to the elements of the Weyl group and their associated Bruhat decomposition and Weyl length.

To work on the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$, we need explicit coordinates.

Corollary 3.11. *The big cell of the flag manifold* $\mathbb{F}_{1,2}(\mathbb{C}^3) = \mathsf{SL}(3,\mathbb{C})/B$ *can be identified with* N^-. *This leads to the following coordinate chart for the big cell of* $\mathbb{F}_{1,2}(\mathbb{C}^3)$*:*

$$N^- \to \mathbb{C}^3 \simeq \mathbb{R}^6, \qquad \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} \mapsto (z_1,z_2,z_3) \simeq (x_1,x_2,x_3,y_1,y_2,y_3). \tag{3.4.2}$$

3.5. **The Kähler structure of coadjoint orbits.** Let M be a complex manifold of complex dimension n with local coordinates $(z_1,\dots,z_n) \in \mathbb{C}^n$. Then a *Hermitian metric* h is of the form

$$h = \sum_{i,j=1}^{n} h_{ij} dz_i \otimes d\overline{z}_j$$

where $(h_{ij})_{1\le i,j\le n}$ is a positive-definite Hermitian matrix. A complex manifold M equipped with a Hermitian metric h is called a *Hermitian manifold*. A Hermitian manifold (M,h) carries a natural symplectic form, more precisely, the $(1,1)$-form

Group element	Bruhat expression	Length	Matrix representation
e	empty word	0	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
$(1,2)$	s_1	1	$\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$
$(2,3)$	s_2	1	$\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$
$(1,2,3)$	s_1s_2	2	$\begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix}$
$(1,3,2)$	s_2s_1	2	$\begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix}$
$(1,3)$	$s_1s_2s_1$	3	$\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}$

Table 1. Bruhat expressions.

given by imaginary part of the Hermitian metric h is symplectic and has the explicit expression

$$\omega := -\Im(h) = -\frac{1}{2i}(h-\overline{h}) = \frac{i}{2}\sum_{1\le i,j\le n} h_{ij}dz_i\otimes d\overline{z}_j - h_{ji}d\overline{z}_i\otimes dz_j$$
$$= \frac{i}{2}\sum_{1\le i,j\le n} h_{ij}(dz_i\otimes d\overline{z}_j - d\overline{z}_j\otimes dz_i) = \frac{i}{2}\sum_{1\le i,j\le n} h_{ij}dz_i\wedge d\overline{z}_j.$$

This ω is often referred to as the *fundamental form* on (M,h).

An almost complex structure J on a smooth manifold M is an isomorphism $J: TM\to TM$ with $J^2 = -\operatorname{Id}$. Such a J is integrable if the so-called *Nijenhuis tensor*

$$N_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

vanishes for all vector fields X,Y on the manifold M.

A symplectic manifold (M,ω) is *Kähler* if there exists an integrable almost complex structure J such that the bilinear form $g(u,v) := \omega(u,Jv)$ is symmetric and positive definite for all $u,v\in TM$, i.e., g is a Riemannian metric.

A Hermitian manifold (M,h) resp. h is *Kähler* if the fundamental form $-\Im(h)$ is closed, i.e. $-d\Im(h) = 0$. Moreover, in this situation, $-\Im(h)$ is in fact a (real) symplectic form on (M,h).

Lemma 3.12. *Let (M,h) be a Kähler manifold. Then for all $p \in M$ there exists a open neighbourhood U of p and a function $K_U : U \to \mathbb{R}$ such that*

$$h_{ij} = \partial_{z_i}\partial_{\bar{z}_j} K_U(z_i, \bar{z}_j) \quad \text{for all } 1 \leq i,j \leq n$$

for local complex coordinates z on U. This locally defined function is usually called the Kähler potential and denoted by K_M.

Note that $(h_{ij})_{1\leq i,j\leq n}$ is defined globally, whereas the potential is only defined locally. Using the Dolbeault operators

$$\partial := \sum_{k=1}^{n} \partial_{z_k} dz_k \qquad \text{and} \qquad \overline{\partial} := \sum_{k=1}^{n} \partial_{\overline{z}_k} d\overline{z}_k,$$

the fundamental $(1,1)$-form can be expressed as $\omega = i\partial\overline{\partial}K_M$.

Example 3.13. On $\mathbb{R}^{2n} \cong \mathbb{C}^n$, consider the Euclidean metric g_E, the standard symplectic form ω_{st}, and standard compatible complex structure J_{st} given in matrix notation by

$$g_E = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}, \qquad \omega_{st} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad J_{st} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $(n \times n)$-unit matrix. Then $K_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{R}$ given by $K_{\mathbb{C}^n}(z) := \frac{|z|}{2}$ is a Kähler potential since

$$i\partial\overline{\partial}\left(\frac{|z|}{2}\right) = \frac{i}{2}\partial\overline{\partial}\sum_{k=1}^{n} z_k\overline{z}_k = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k.$$

An important class of Kähler manifolds is given by coadjoint orbits:

Theorem 3.14 (Bott)**.** *Let G be a semi-simple compact Lie group. Each (co)adjoint orbit has a G-equivariant Kähler structure.*

Later on, we will study the point vortex dynamics modelled on the degenerate orbit $\mathbb{CP}^2$ and the generic orbit given by flag manifold $F_{1,2}(\mathbb{C}^3)$. Therefore it is useful to know their Kähler potentials.

Lemma 3.15 (Picken [Pic90])**.** *The Kähler potentials on $\mathbb{CP}^n$ and on the flag manifold are given by the following logarithmic functions depending on local coordinates $(1 : z_1 : z_2 : \cdots : z_n)$ on $\mathbb{CP}^n$ and the local coordinates from Corollary 3.11 for the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$.*

$$K_{\mathbb{CP}^n} = \log\left(1 + \sum_{k=1}^{n-1}|z_k|^2\right), \tag{3.5.1}$$

$$K_{\mathbb{F}_{1,2}(\mathbb{C}^3)} = \log\big((1 + |z_1|^2 + |z_2|^2)(1 + |z_3|^2 + |z_1 z_3 - z_2|^2)\big) =: \log(K_1K_2). \tag{3.5.2}$$

Moreover,

Lemma 3.16 (Muñoz & González-Prieto & Rojo [MGPR20])**.** *In homogeneous coordinates $(1 : z_1 : z_2 : \cdots : z_n) \in \mathbb{CP}^n$, the Hermitian metric $h = (h_{ij})_{1\leq i,j\leq n}$ on $\mathbb{CP}^n$*

takes the following form:

$$h_{ij} = \frac{(1+|z_1|^2+|z_2|^2)\delta_{ij} - \overline{z}_i z_j}{(1+|z_1|^2+|z_2|^2)^2} = \frac{(1+|z|^2)\delta_{ij} - \overline{z}_i z_j}{(1+|z|^2)^2}$$

where δ_{ij} *is the Kronecker symbol. Written as matrix, we have*

$$(h_{ij})_{ij} = \frac{1}{(1+|z|^2)^2}\begin{pmatrix} 1+|z|^2-|z_1|^2 & -\bar{z}_1 z_2 & \cdots & -\bar{z}_1 z_n \\ -\bar{z}_2 z_1 & 1+|z|^2-|z_2|^2 & \cdots & -\bar{z}_2 z_n \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{z}_n z_1 & -\bar{z}_n z_2 & \cdots & 1+|z|^2-|z_n|^2 \end{pmatrix}.$$

Its determinant is given by $\det(h_{ij}) = \frac{1}{(1+|z|^2)^{n+1}}$.

Lemma 3.17. *Recall from Lemma 3.15 the real valued functions*

$$K_1 = 1+|z_1|^2+|z_2|^2 \quad \textit{and} \quad K_2 = 1+|z_3|^2+|z_1 z_3 - z_2|^2.$$

Then the Hermitian metric $(h_{ij})_{1\le i\le j\le 3}$ *on* $\mathbb{F}_{1,2}(\mathbb{C}^3)$ *has the following matrix representation:*

$$(h_{ij})_{1\le i,j\le n} = \begin{pmatrix} \frac{1+|z_2|^2}{K_1^2} + \frac{|z_3|^2(1+|z_3|^2)}{K_2^2} & -\frac{\overline{z}_1 z_2}{K_1^2} - \frac{z_3(1+|z_3|^2)}{K_2^2} & \frac{z_3(\overline{z}_1+\overline{z}_2 z_3)}{K_2^2} \\ -\frac{z_1\overline{z}_2}{K_1^2} - \frac{\overline{z}_3(1+|z_3|^2)}{K_2^2} & \frac{1+|z_1|^2}{K_1^2} + \frac{1+|z_3|^2}{K_2^2} & -\frac{\overline{z}_1+\overline{z}_2 z_3}{K_2^2} \\ \frac{\overline{z}_3(z_1+z_2\overline{z}_3)}{K_2^2} & -\frac{z_1+z_2\overline{z}_3}{K_2^2} & \frac{K_1}{K_2^2} \end{pmatrix}.$$

Its determinant is given by $\det((h_{ij})_{1\le i,j\le n}) = \frac{2}{K_1^2 K_2^2}$.

Proof. Recall from Lemma 3.15 the expression for the Kähler potential

$$\log(1+|z_1|^2+|z_2|^2)(1+|z_3|^2+|z_1 z_3 - z_2|^2) = \log K_1 K_2 = \log K_1 + \log K_2.$$

The entries of the matrix $(h_{ij})_{1\le i\le j\le 3}$ are computed via $h_{ij} = \partial_{z_i}\partial_{\bar{z}_j} K_M(z_i, \bar{z}_j)$. Exemplarily we now compute the entry

$$h_{11} = \partial_{z_1}\partial_{\overline{z}_1}\log((1+|z_1|^2+|z_2|^2)) + \partial_{z_1}\partial_{\overline{z}_1}\log(1+|z_3|^2+|z_1 z_3 - z_2|^2).$$

The first term in this expression becomes

$$\partial_{z_1}\frac{z_1}{1+|z_1|^2+|z_2|^2} = \frac{(1+|z_1|^2+|z_2|^2) - z_1\overline{z}_1}{(1+|z_1|^2+|z_2|^2)^2} = \frac{1+|z_2|^2}{K_1^2}$$

and the second one

$$\begin{aligned} \partial_{z_1}\frac{\overline{z}_3(z_1 z_3 - z_2)}{1+|z_3|^2+|z_1 z_3 - z_2|^2} &= \frac{|z_3|^2(1+|z_3|^2+|z_1 z_3 - z_2|^2) - |z_3|^2|z_1 z_3 - z_2|^2}{(1+|z_3|^2+|z_1 z_3 - z_2|^2)^2} \\ &= \frac{|z_3|^2(1+|z_3|^2)}{K_2^2}. \end{aligned}$$

Altogether, we obtain

$$h_{11} = \frac{1+|z_2|^2}{K_1^2} + \frac{|z_3|^2(1+|z_3|^2)}{K_2^2}.$$

The other entries are computed analogously. □

A straightforward computation yields:

Corollary 3.18. *The inverse matrix* $((h_{ij})_{1\leq i\leq j\leq 3})^{-1} =: (h^{ij})_{1\leq i\leq j\leq 3}$ *is given by:*

$$\begin{pmatrix} K_1\left(1+|z_1|^2+\frac{K_1}{K_2}\right) & K_1\left(\overline{z}_1 z_2+\frac{K_1}{K_2}z_3\right) & (\overline{z}_1+z_3\overline{z}_2)(\overline{z}_1 z_2 - z_3 - z_3|z_1|^2) \\ K_1\left(z_1\overline{z}_2+\frac{K_1}{K_2}\overline{z}_3\right) & K_1\left((1+|z_2|^2)+\frac{K_1}{K_2}|z_3|^2\right) & (\overline{z}_1+\overline{z}_2 z_3)\left((1+|z_2|^2)-z_1\overline{z}_2 z_3\right) \\ (z_1+\overline{z}_3 z_2)(z_1\overline{z}_2-\overline{z}_3-z_3|z_1|^2) & (z_1+z_2\overline{z}_3)\left((1+|z_2|^2)-\overline{z}_1 z_2\overline{z}_3\right) & K_1(1+|z_3|^2)+\frac{K_2^2}{K_1} \end{pmatrix}$$

where $K_1 = 1+|z_1|^2+|z_2|^2$ *and* $K_2 = 1+|z_3|^2+|z_1 z_3 - z_2|^2$.

3.6. **Different symplectic structures.** Important for us is the following result due to Kirillov, Kostant and Souriau:

Theorem 3.19. *Let* G *be a Lie group and* $\mathfrak{g}$ *its Lie algebra with dual* $\mathfrak{g}^*$ *and* $\mu \in \mathfrak{g}^*$. *Then the coadjoint orbit* $\mathcal{O}_\mu$ *carries the canonical symplectic form*

$$\omega_\nu^{KKS}(\mathrm{ad}_\xi^*\,\nu, \mathrm{ad}_\eta^*\,\nu) := \langle \nu, [\xi,\eta]\rangle,$$

where $\xi, \eta \in \mathfrak{g}$ *and* $\nu \in \mathcal{O}_\mu$. *This symplectic form is usually called the* Kirillov-Kostant-Souriau (KKS) symplectic form.

This implies that, considered as coadjoint orbit, the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathbb{T}^2$ can be endowed with ω^{KKS} as symplectic form. Note that there is an additional way to consider $\mathbb{F}_{1,2}(\mathbb{C}^3)$ as symplectic manifold: We consider the complexification of $\mathsf{SU}(3)/\mathbb{T}^2$ given by $(\mathsf{SU}(3)/\mathbb{T}^2)^{\mathbb{C}} = \mathsf{SL}(3,\mathbb{C})/B$. Using the coordinates on the big cell given in (3.4.2) and the Hermitian metric from Lemma 3.17, Picken & Duistermaat [Pic90] give the following formula for the symplectic form on $(W^6)^{\mathbb{C}}$:

$$\omega^{(W^6)^{\mathbb{C}}} := \frac{i}{2}\left(\partial\overline{\partial}\log\left(1+\sum_{k=1}^{2}|z_k|^2\right)+\partial\overline{\partial}\log\left(1+|z_3|^2+|z_1z_3-z_2|^2\right)\right).$$

Note that, if $\omega^{\mathbb{CP}^2}$ denotes the Fubini-Study form on $\mathbb{CP}^2$ then the symplectic form $\omega^{(W^6)^{\mathbb{C}}}$ consists of the Fubini-Study form $\omega^{\mathbb{CP}^2} = \frac{i}{2}\partial\overline{\partial}\log\left(1+\sum_{k=1}^2|z_k|^2\right)$ on $\mathbb{CP}^2$ plus the correction term $\tilde{\omega} = \frac{i}{2}\partial\overline{\partial}\log\left(1+|z_3|^2+|z_1z_3-z_2|^2\right)$, i.e., $\omega^{(W^6)^{\mathbb{C}}} = \omega^{\mathbb{CP}^2}+\tilde{\omega}$.

Remark 3.20 (Picken & Duistermaat [Pic90], Bernatska & Holod et al. [BH08])**.** $\left((W^6)^{\mathbb{C}},\ \omega^{(W^6)^{\mathbb{C}}}\right)$ and $\left(\mathbb{F}_{1,2}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathbb{T}^2,\ \omega^{KKS}\right)$ are symplectomorphic.

Summarized, we have the following types of coadjoint orbits of $\mathsf{SU}(3)$, each endowed with its natural symplectic structure:

coadjoint orbit	symplectic form	(real) dimension
$\frac{\mathsf{SU}(3)}{\mathsf{U}(1)\times\mathsf{U}(1)}$	Kirillov-Kostant-Souriau	6
$\mathbb{CP}^2$	Fubini-Study	4
point	trivial	0

TABLE 2. Coadjoint orbits of $\mathsf{SU}(3)$.

4. The point vortex momentum map on $\mathbb{CP}^2$ and $\mathbb{F}_{1,2}(\mathbb{C}^3)$

In this section, we will study the Hamiltonian action of $\mathsf{SU}(3)$ on (products of) coadjoint orbits. In the case of the degenerate orbit, the dynamics have been studied before: for example, the Hamiltonian action of $\mathsf{SU}(3)$ on $\mathbb{CP}^2 \times \mathbb{CP}^2$ has been studied by Beddulli & Gori [BG07]. Moreover, Montaldi & Shaddad [MS19a] considered a similar problem but added a copy of the projective plane. To be more precise, they considered the (diagonal) action of $\mathsf{SU}(3)$ on $\mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2$ and the associated properties of the (weighted) momentum map. We will focus on the generic orbit, which is the six-dimensional flag manifold and construct a momentum map $\mu : \mathcal{O}^{\mathsf{SU}(3)} \to \mathfrak{su}(3)^*$ explicitly.

4.1. **The momentum map for vortex dynamics.** Let $N \in \mathbb{N}$ and, for $1 \leq k \leq N$, let $\Gamma_k \in \mathbb{R}^{\neq 0}$ ('weight') and let (M_k, ω_k) be a symplectic manifold. Let G be a Lie group that acts on each (M_k, ω_k) with momentum map $\mu_k : M_k \to \mathfrak{g}^*$. Now set $M := \Pi_{k=1}^N M_k$ and equip it with the weighted symplectic form $\omega_M := \sum_{k=1}^N \Gamma_k \tau_k^* \omega_k$ where $\tau_k : M \to M_k$ is the projection on the kth factor. The diagonal action of G on M is given by $g.m := (g.m_1, \dots, g.m_N)$ for $g \in G$ and $m = (m_1, \dots, m_N) \in M$ and its momentum map is given by

$$\mu_M : M \to \mathfrak{g}^*, \qquad \mu_M(m_1, \dots, m_N) = \sum_{k=1}^N \Gamma_k \mu_k(m_k).$$

We are interested in the special situation when the symplectic manifolds M_k are coadjoint orbits i.e. $(M_k, \omega_k) = (\mathcal{O}, \omega_{\mathcal{O}})$. In the next subsections, we study momentum maps of vortex dynamics for the following two situations:

$$\mu_{\mathcal{O}_d^{\mathsf{SU}(3)}} : \mathcal{O}_d^{\mathsf{SU}(3)} \simeq \mathbb{CP}^2 \to \mathfrak{su}(3)^* \quad \text{and} \quad \mu_{\mathcal{O}^{\mathsf{SU}(3)}} : \mathcal{O}^{\mathsf{SU}(3)} \simeq \mathbb{F}_{1,2}(\mathbb{C}^3) \to \mathfrak{su}(3)^*.$$

Recall that we identify $\mathfrak{su}(3)$ with $\mathfrak{su}(3)^*$ using the Killing form. Thus, $\mathfrak{su}(3) \cong \mathfrak{su}(3)^*$ is identified with the space of complex skew-Hermitian matrices with trace zero.

4.2. **The momentum map of the degenerate orbit $\mathcal{O}_d^{\mathsf{SU}(3)} \simeq \mathbb{CP}^2$.** In this subsection, we recall some facts from Montaldi & Shaddad [MS19b] concerning the momentum map of the degenerate coadjoint orbit $\mathcal{O}_d^{\mathsf{SU}(3)} \simeq \mathbb{CP}^2$ of $\mathsf{SU}(3)$.

Theorem 4.1 (Montaldi & Shaddad [MS19b]). *The momentum map for the Fubini-Study form on $\mathbb{CP}^2$ is given by*

$$\mu : \mathbb{CP}^2 \to \mathfrak{su}(3)^*, \quad [x : y : z] \mapsto \begin{pmatrix} |x|^2 - \frac{1}{3} & x\overline{y} & x\overline{z} \\ \overline{x}y & |y|^2 - \frac{1}{3} & y\overline{z} \\ \overline{x}z & \overline{y}z & |z|^2 - \frac{1}{3} \end{pmatrix}.$$

Furthermore, the map satisfies the following properties:

(i) *μ is $\mathsf{SU}(3)$-equivariant for the left action, i.e. $\mu(gZ) = g\mu(Z)$ for all $g \in \mathsf{SU}(3)$ and all $Z \in \mathbb{CP}^2$.*

(ii) *The image of μ consists of (3×3) Hermitian matrices with eigenvalues $-\frac{1}{3}, -\frac{1}{3}$ and $\frac{2}{3}$.*

Proof. We briefly sketch a part of the proof: the characteristic polynomial $\chi(u)$ of the matrix $\mu(x, y, z)$ is given by

$$\chi(u) = (|x|^2+|y|^2+|z|^2)\left(u^2 + \frac{2}{3}u + \frac{1}{9}\right) - u^3 - u^2 - \frac{u}{3} - \frac{1}{27}.$$

Solving the equation $\chi(u) = 0$ and using the fact that $|x|^2+|y|^2+|z|^2= 1$ gives the three eigenvalues $u_1 = -\frac{1}{3}$, $u_2 = -\frac{1}{3}$ and $u_3 = \frac{2}{3}$. □

4.3. **The momentum map of the generic orbit $\mathcal{O}^{\mathsf{SU}(3)} \simeq \mathbb{F}_{1,2}(\mathbb{C}^3)$.** In order to obtain the momentum map on the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3) \simeq \mathcal{O}^{\mathsf{SU}(3)}$ we need to have the infinitesimal generators of the Lie algebra $\mathfrak{su}(3)$ at our disposal. They are provided by the following statement:

Lemma 4.2. *Let $\lambda_1, \dots, \lambda_8$ be the rescaled basis from Notation 2.1. Then the infinitesimal vector fields of the Lie algebra $\mathfrak{su}(3)$ on the flag manifold are given by*

$$\begin{aligned}
X_{\lambda_1} &= \frac{i}{2}\left((1-z_1^2)\partial_{z_1} - z_1 z_2 \partial_{z_2} + (z_1 z_3 - z_2)\partial_{z_3}\right),\\
X_{\lambda_2} &= \frac{1}{2}\left((-z_1^2-1)\partial_{z_1} - z_1 z_2 \partial_{z_1} + (z_1 z_3 - z_2)\partial_{z_3}\right),\\
X_{\lambda_3} &= \frac{i}{2}\left(-2z_1\partial_{z_1} - z_2\partial_{z_2} + z_3\partial_{z_3}\right),\\
X_{\lambda_4} &= \frac{i}{2}\left(-z_1 z_2\partial_{z_1} + (1-z_2^2)\partial_{z_2} - z_3(z_2 - z_1 z_3)\partial_{z_3}\right),\\
X_{\lambda_5} &= \frac{1}{2}\left(-z_1 z_2\partial_{z_1} - (1+z_2^2)\partial_{z_2} - z_3(z_2 - z_1 z_3)\partial_{z_3}\right),\\
X_{\lambda_6} &= \frac{i}{2}\left(z_2\partial_{z_1} + z_1\partial_{z_2} + (1-z_3^2)\partial_{z_3}\right),\\
X_{\lambda_7} &= \frac{1}{2}\left(z_2\partial_{z_1} - z_1\partial_{z_2} - (1-z_3^2)\partial_{z_3}\right),\\
X_{\lambda_8} &= -\frac{i\sqrt{3}}{2}\left(z_2\partial_{z_2} + z_3\partial_{z_3}\right).
\end{aligned}$$

Proof. In order to obtain the fundamental vector fields associated to $\mathfrak{su}(3)$ it is sufficient to determine the vector fields associated to the basis $\lambda_1, \dots, \lambda_8$ from Notation 2.1. The vector fields are determined by the equation

$$X_{\lambda_k} = \left.\frac{d}{dt}\right|_{t=0} \exp(t\lambda_k)\cdot\mathcal{Z}$$

where $\mathcal{Z} \in \mathbb{F}_{1,2}(\mathbb{C}^3) \simeq \mathcal{O}^{\mathsf{SU}(3)}$ (see local coordinates expression from Corollary 3.11) and $\exp(t\lambda_k)\cdot\mathcal{Z}$ is defined as multiplication of the matrices $\exp(t\lambda_k)$ and $\mathcal{Z}$ and corresponds to the left action of $\mathsf{SU}(3)$ on the flag manifold.

Without loss of generality, $\mathcal{Z}$ may lie in the big cell and thus is of the form $\mathcal{Z} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{smallmatrix}\right)$. When $\exp(t\lambda_k)$ acts on $\mathcal{Z}$, the result lies not necessarily again in the big cell. But, due to the fact that the flag manifold is identified with the (complexified) homogeneous space $\mathsf{SL}(3,\mathbb{C})/B$ where B is the Borel subgroup of upper triangular matrices, we can always multiply (from the right) with elements from B to get again an element in the big cell. Using formula (2.2.1), we compute

$$\exp(t\lambda_1) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & i\sin\left(\frac{t}{2}\right) & 0 \\ i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(t\lambda_2) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) & 0 \\ -\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp(t\lambda_3) = \begin{pmatrix} e^{\frac{it}{2}} & 0 & 0 \\ 0 & e^{-\frac{it}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(t\lambda_4) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & 0 & i\sin\left(\frac{t}{2}\right) \\ 0 & 1 & 0 \\ i\sin\left(\frac{t}{2}\right) & 0 & \cos\left(\frac{t}{2}\right) \end{pmatrix},$$

$$\exp(t\lambda_5) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & 0 & \sin\left(\frac{t}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(\frac{t}{2}\right) & 0 & \cos\left(\frac{t}{2}\right) \end{pmatrix}, \quad \exp(t\lambda_6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{t}{2}\right) & i\sin\left(\frac{t}{2}\right) \\ 0 & i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix},$$

$$\exp(t\lambda_7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) \\ 0 & -\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}, \quad \exp(t\lambda_8) = \begin{pmatrix} e^{\frac{it}{2\sqrt{3}}} & 0 & 0 \\ 0 & e^{\frac{it}{2\sqrt{3}}} & 0 \\ 0 & 0 & e^{-\frac{it}{\sqrt{3}}} \end{pmatrix}.$$

Now we need to compute $\exp(t\lambda_k) \cdot \mathcal{Z}$. Note that, as mentioned above, the result may not lie in the big cell. Thus we need to multiply in addition from the right with an element $\mathbf{b} = \left(\begin{smallmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{smallmatrix}\right) \in B$, i.e.,

$$\exp(\lambda_k t) \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix}$$

in order to obtain as element of the big cell

$$A_k := \begin{pmatrix} 1 & 0 & 0 \\ f_{1,k}(z_1,t) & 1 & 0 \\ f_{2,k}(z_2,t) & f_{3,k}(z_3,t) & 1 \end{pmatrix}$$

for some functions $f_{j,k}(z_j,t)$ with $j \in \{1,2,3\}$ and $k \in \{1,\dots,8\}$ depending on the complex variables z_j and the real variable t. Now, for $k \in \{1,\dots,8\}$, we solve $\exp(\lambda_k t)\mathcal{Z}\mathbf{b} = A_k$ for $\mathbf{b} \in B$. The solutions are denoted by $\beta_k \in B$ and are given as follows:

$$\beta_1 = \begin{pmatrix} \frac{1}{\cos\left(\frac{t}{2}\right)+iz_1\sin\left(\frac{t}{2}\right)} & -i\sin\left(\frac{t}{2}\right) & 0 \\ 0 & \cos\left(\frac{t}{2}\right)+iz_1\sin\left(\frac{t}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} \frac{1}{z_1\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)} & -\sin\left(\frac{t}{2}\right) & 0 \\ 0 & z_1\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} e^{\frac{it}{2}} & 0 & 0 \\ 0 & e^{-\frac{it}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta_4 = \begin{pmatrix} \frac{1}{\cos\left(\frac{t}{2}\right)+iz_2\sin\left(\frac{t}{2}\right)} & \frac{z_3\sin\left(\frac{t}{2}\right)}{-z_2\sin\left(\frac{t}{2}\right)+z_1z_3\sin\left(\frac{t}{2}\right)+i\cos\left(\frac{t}{2}\right)} & -i\sin\left(\frac{t}{2}\right) \\ 0 & 1-\frac{z_1\left(z_3\sin\left(\frac{t}{2}\right)\right)}{-z_2\sin\left(\frac{t}{2}\right)+z_1z_3\sin\left(\frac{t}{2}\right)+i\cos\left(\frac{t}{2}\right)} & iz_1\sin\left(\frac{t}{2}\right) \\ 0 & 0 & \cos\left(\frac{t}{2}\right)+i\left(z_2-z_1z_3\right)\sin\left(\frac{t}{2}\right) \end{pmatrix},$$

$$\beta_5 = \begin{pmatrix} \frac{1}{z_2\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)} & -\frac{z_3\sin\left(\frac{t}{2}\right)}{z_2\sin\left(\frac{t}{2}\right)-z_1z_3\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)} & -\sin\left(\frac{t}{2}\right) \\ 0 & \frac{z_2\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)}{z_2\sin\left(\frac{t}{2}\right)-z_1z_3\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)} & z_1\sin\left(\frac{t}{2}\right) \\ 0 & 0 & z_2\sin\left(\frac{t}{2}\right)-z_1z_3\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right) \end{pmatrix},$$

$$\beta_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\cos\left(\frac{t}{2}\right)+iz_3\sin\left(\frac{t}{2}\right)} & -\frac{i}{\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\cot\left(\frac{t}{2}\right)} \\ 0 & 0 & \cos\left(\frac{t}{2}\right)+iz_3\sin\left(\frac{t}{2}\right) \end{pmatrix},$$

$$\beta_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{z_3\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)} & -\frac{1}{\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\cot\left(\frac{t}{2}\right)} \\ 0 & 0 & z_3\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right) \end{pmatrix},$$

$$\beta_8 = \begin{pmatrix} e^{-\frac{it}{2\sqrt{3}}} & 0 & 0 \\ 0 & e^{-\frac{it}{2\sqrt{3}}} & 0 \\ 0 & 0 & e^{\frac{it}{\sqrt{3}}} \end{pmatrix}.$$

In order to obtain the vector fields, we must compute the derivatives of A_k with respect to t and evaluate in $t = 0$. This reduces to determining the derivatives of the coordinate functions

$$\frac{d}{dt} f_{j,k}(z_j, t)$$

in $t = 0$ for all $j \in \{1, 2, 3\}$ and $k \in \{1, \ldots, 8\}$. The corresponding vector fields are then, for $k \in \{1, \ldots, 8\}$, given by

$$X_{\lambda_k} = \left(\left.\frac{d}{dt}\right|_{t=0} f_{1,k}(z_1,t)\right)\partial_{z_1} + \left(\left.\frac{d}{dt}\right|_{t=0} f_{2,k}(z_2,t)\right)\partial_{z_2} + \left(\left.\frac{d}{dt}\right|_{t=0} f_{3,k}(z_3,t)\right)\partial_{z_3}$$

which yields the claim. □

Now we compute the explicit formula for the momentum map on $\mathcal{O}^{\mathsf{SU}(3)} \simeq \mathbb{F}_{1,2}(\mathbb{C}^3) \simeq \mathsf{SL}(3,\mathbb{C})/B$ associated with vortex dynamics.

Theorem 4.3. *Let $K_1 = 1 + |z_1|^2 + |z_2|^2$ and $K_2 = 1 + |z_3|^2 + |z_1 z_3 - z_2|^2$. The momentum map for the left action of $\mathsf{SU}(3)$ on the generic coadjoint orbit $\mathsf{SL}(3,\mathbb{C})/B$ is given by*

$$\mu : \mathsf{SL}(3,\mathbb{C})/B \to \mathfrak{su}(3)^*, \qquad \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} \mapsto (\mu_{ij})_{1\le i,j\le 3}$$

where $(\mu_{ij})_{1\le i,j\le 3}$ is the traceless, anti-Hermitian matrix with entries

$$\mu_{11} = \frac{1}{3}\left(\frac{x_3^2+y_3^2+2}{K_2} - \frac{x_2^2+y_2^2-1}{K_1}\right),$$

$$\mu_{22} = \frac{1}{3}\left(-\frac{2x_2^2+2y_2^2+1}{K_1} - \frac{x_3^2+y_3^2-1}{K_2}\right),$$
$$\mu_{33} = -(\mu_{11}+\mu_{22}),$$
$$\mu_{12} = \frac{(iy_1-x_1)(x_3-iy_3)-iy_2+x_2}{K_2} - \frac{x_1-iy_1}{K_1},$$
$$\mu_{13} = \frac{(iy_1-x_1)(x_3-iy_3)-iy_2+x_2}{K_2} - \frac{x_1-iy_1}{K_1},$$
$$\mu_{23} = \frac{iy_3+x_3}{K_2} - \frac{(x_1+iy_1)(x_2-iy_2)}{K_1}.$$

The remaining entries are determined by the fact that the matrix is anti-Hermitian.

Let us prepare for the proof of Theorem 4.3 by computing the momentum map of an easy example:

Example 4.4. Let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Consider the symplectic manifold $(\mathbb{R}^2, \omega_0 := dx \wedge dy)$ and the standard action of $\mathsf{SO}(2) \cong \mathsf{U}(1) \cong \mathbb{T}$ on $\mathbb{R}^2$ by rotations, i.e., $\exp(\theta J) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ acts on $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

The vector field associated with the element $\exp(\theta J)$ is given by

$$J_M = \frac{d}{d\theta}\bigg|_{\theta=0} \exp(\theta J)\cdot\begin{pmatrix} x \\ y \end{pmatrix} = y\partial_x - x\partial_y.$$

The contraction of the symplectic form ω_0 via the vector field J_M is

$$\begin{pmatrix} y & -x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = xdx + ydy.$$

The map

$$\mu : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \mu(x,y) = \frac{1}{2}(x^2+y^2)$$

is a moment map since it satisfies $\iota_{J_M}\omega = d\langle \mu, J\rangle$.

Recall that a 1-form η is called *exact* if there exists a function F such that $dF = \eta$. The following example illustrate a technique how to find such function F.

Example 4.5. Let $dF = y^2 e^{xy^2} dx + 2xye^{xy^2} dy$ be a 1-form on $\mathbb{R}^2$. Thus the (still unknown) function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following system of partial differential equations:

$$\partial_x F(x,y) = y^2 e^{xy^2},$$
$$\partial_y F(x,y) = 2xye^{xy^2}.$$

Integrating the first equation gives

$$F(x,y) = \int y^2 e^{xy^2} dx = e^{xy^2} + C(y),$$

where $C(y)$ is a function depending on y. Using the second equation leads to

$$\partial_y F(x,y) = 2xye^{xy^2} + C'(y) = 2xye^{xy^2} \iff C'(y) = 0.$$

This means that $C \equiv c$ is constant and therefore the wanted function F is given by $F(x,y) = e^{xy^2} + c$.

Now we are ready for

Proof of Theorem 4.3: By definition of the moment map, we must have

$$d\langle \mu, \lambda_k \rangle = \iota_{X_{\lambda_k}} \omega,$$

for all $\lambda_k \in \mathfrak{g}$ and induced vector fields X_{λ_k} from Lemma 4.2. Moreover, recall from (2.1.1) that the dual pairing is given by the trace. Therefore we have

$$\langle \mu, \lambda_k \rangle : \mathsf{SL}(3,\mathbb{C})/B \to \mathbb{R}, \quad x \mapsto \langle \mu(x), \lambda_k \rangle = \mathsf{trace}(\mu(x)\lambda_k).$$

The dual pairing explicitly becomes

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 0 & \frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \frac{i\mu_{12}}{2} + \frac{i\mu_{21}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \frac{\mu_{21}}{2} - \frac{\mu_{12}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \frac{i\mu_{11}}{2} - \frac{i\mu_{22}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}\right) = \frac{i\mu_{13}}{2} + \frac{i\mu_{31}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}\right) = \frac{\mu_{31}}{2} - \frac{\mu_{13}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} \\ 0 & \frac{i}{2} & 0 \end{pmatrix}\right) = \frac{i\mu_{23}}{2} + \frac{i\mu_{32}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}\right) = \frac{\mu_{32}}{2} - \frac{\mu_{23}}{2},$$

$$\mathsf{trace}\left(\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{pmatrix} \frac{i}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{i}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{i}{\sqrt{3}} \end{pmatrix}\right) = \frac{i\left(\mu_{11} + \mu_{22} - 2\mu_{33}\right)}{2\sqrt{3}}.$$

In Lemma 3.17, we obtained the Hermitian metric $h = (h_{kl})_{1 \le k,l \le n}$ on the flag manifold. Moreover, the (real) symplectic form is given by $\omega = \frac{i}{2}\sum_{k,l=1}^{n} h_{kl} dz_k \wedge d\overline{z}_l$.

In terms of real coordinates $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6$ the matrix representing the symplectic form is given by

$$\omega := \begin{pmatrix} \Im(h) & -\Re(h) \\ \Re(h) & \Im(h) \end{pmatrix}$$

with

$$\Im(h) = \begin{pmatrix} 0 & \frac{x_2y_1-x_1y_2}{K_1^2} - \frac{y_3\left(x_3^2+y_3^2+1\right)}{K_2^2} & \frac{-(x_3(y_1-2x_2y_3))-x_3^2y_2+y_3(x_1+y_2y_3)}{K_2^2} \\ \frac{x_1y_2-x_2y_1}{K_1^2} + \frac{y_3\left(x_3^2+y_3^2+1\right)}{K_2^2} & 0 & \frac{x_3y_2-x_2y_3+y_1}{K_2^2} \\ \frac{x_3^2y_2+x_3(y_1-2x_2y_3)-y_3(x_1+y_2y_3)}{K_2^2} & -\frac{x_3y_2-x_2y_3+y_1}{K_2^2} & 0 \end{pmatrix}$$

and

$$\Re(h) = \begin{pmatrix} \frac{x_2^2+y_2^2+1}{K_1^2} + \frac{x_3^2\left(2y_3^2+1\right)+x_3^4+y_3^4+y_3^2}{K_2^2} & -\frac{x_1x_2+y_1y_2}{K_1^2} - \frac{x_3\left(x_3^2+y_3^2+1\right)}{K_2^2} & \frac{y_3(2x_3y_2+y_1)+x_2\left(x_3^2-y_3^2\right)+x_1x_3}{K_2^2} \\ -\frac{x_1x_2+y_1y_2}{K_1^2} - \frac{x_3\left(x_3^2+y_3^2+1\right)}{K_2^2} & \frac{x_1^2+y_1^2+1}{K_1^2} + \frac{x_3^2+y_3^2+1}{K_2^2} & -\frac{x_1+x_2x_3+y_2y_3}{K_2^2} \\ \frac{y_3(2x_3y_2+y_1)+x_2\left(x_3^2-y_3^2\right)+x_1x_3}{K_2^2} & -\frac{x_1+x_2x_3+y_2y_3}{K_2^2} & \frac{K_1}{K_2^2} \end{pmatrix}.$$

Evaluating $d\langle\mu, \lambda_k\rangle = \iota_{X_{\lambda_k}}\omega$ using the matrix representing ω, we obtain the equations

$$\begin{cases} d\left(\frac{i\mu_{12}}{2} + \frac{i\mu_{21}}{2}\right) & = \iota_{X_{\lambda_1}}\omega, \\ d\left(\frac{\mu_{21}}{2} - \frac{\mu_{12}}{2}\right) & = \iota_{X_{\lambda_2}}\omega, \\ d\left(\frac{i\mu_{13}}{2} + \frac{i\mu_{31}}{2}\right) & = \iota_{X_{\lambda_4}}\omega, \\ d\left(\frac{\mu_{31}}{2} - \frac{\mu_{13}}{2}\right) & = \iota_{X_{\lambda_5}}\omega, \\ d\left(\frac{i\mu_{23}}{2} + \frac{i\mu_{32}}{2}\right) & = \iota_{X_{\lambda_6}}\omega, \\ d\left(\frac{\mu_{32}}{2} - \frac{\mu_{23}}{2}\right) & = \iota_{X_{\lambda_7}}\omega, \\ d\left(\frac{i\mu_{11}}{2} - \frac{i\mu_{22}}{2}\right) & = \iota_{X_{\lambda_3}}\omega, \\ d\left(\frac{i(\mu_{11}+\mu_{22}-2\mu_{33})}{2\sqrt{3}}\right) & = \iota_{X_{\lambda_8}}\omega \end{cases} \tag{4.3.1}$$

where we need to solve for the components μ_{ij} of the momentum map. The 1-forms on the left hand side are exact and therefore of the general form $dF = \sum_{k=1}^3 \frac{\partial F}{\partial x_k}dx_k + \sum_{k=1}^3 \frac{\partial F}{\partial y_k}dy_k$. The 1-forms on the right hand side are contractions and of the general form $\sum_{k=1}^3 G_k dx_k + \sum_{k=1}^3 G_{3+k}dy_k$. Thus we must solve $\frac{\partial F}{\partial x_k} = G_k$ and $\frac{\partial F}{\partial y_k} = G_{k+3}$ for all $k \in \{1,2,3\}$. We now proceed as in Example 4.5. Considering the coordinate x_1, we find

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = \int G_1(x_1, x_2, x_3, y_1, y_2, y_3)dx_1 + C(x_2, x_3, y_1, y_2, y_3)$$

and obtain therefore an expression for F. Using $\frac{\partial F}{\partial x_k} = G_2$ we obtain

$$\begin{aligned} G_2(x_1, x_2, x_3, y_1, y_2, y_3) &= \frac{\partial F}{\partial x_2}(x_1, x_2, x_3, y_1, y_2, y_3) \\ &= \frac{\partial}{\partial x_2}\int G_1(x_1, x_2, x_3, y_1, y_2, y_3)dx_1 + \frac{\partial}{\partial x_2}C(x_2, x_3, y_1, y_2, y_3) \end{aligned}$$

and therefore

$$\frac{\partial}{\partial x_2}C = G_2 - \frac{\partial}{\partial x_2}\int G_1$$

which yields

$$C = \int \left(G_2 - \frac{\partial}{\partial x_2}\int G_1 \right) dx_2 + C'(x_3, y_1, y_2, y_3).$$

Iterating this procedure, we determine F in terms of $G_1, \ldots, G_6$. Therefore, if $G_1, \ldots, G_6$ are explicitly given, we can find an explicit formula for F.

Now we will apply this procedure to the systems of equations given in (4.3.1). We start with the coupled system

$$\begin{cases} d\left(\frac{i\mu_{12}}{2} + \frac{i\mu_{21}}{2}\right) & = \iota_{X_{\lambda_1}}\Omega, \\ d\left(\frac{\mu_{21}}{2} - \frac{\mu_{12}}{2}\right) & = \iota_{X_{\lambda_2}}\Omega. \end{cases}$$

By integration we obtain the following expressions. Note that, in our situation, the function C from above is constant and may be chosen to be zero.

$$\frac{i}{2}(\mu_{12} + \mu_{21}) = \frac{1}{2}\left(-\frac{x_1(x_3^2 + y_3^2) - x_2x_3 - y_2y_3}{K_2} - \frac{x_1}{K_1} \right) =: \alpha,$$
$$\frac{1}{2}(\mu_{21} - \mu_{12}) = \frac{1}{2}\left(\frac{x_3y_2 - x_3^2y_1 - y_3(x_2 + y_1y_3)}{K_2} - \frac{y_1}{K_1} \right) =: \beta.$$

We now can solve for μ_{21} via

$$\begin{aligned} i\mu_{21} = \alpha + i\beta &= \frac{1}{2}\left(\frac{-(x_1 + iy_1)(x_3 + iy_3) + x_2 + iy_2}{K_2} - \frac{x_1 + iy_1}{K_1} \right) \\ &= \frac{1}{2}\left(-\frac{z_1}{K_1} + \frac{-z_1z_3 + z_2}{K_2} \right) \end{aligned}$$

and find

$$\mu_{21} = -\frac{i}{2}\left(-\frac{z_1}{K_1} + \frac{-z_1z_3 + z_2}{K_2} \right). \tag{4.3.2}$$

The second pair of coupled equations is

$$\begin{cases} d\left(\frac{i\mu_{13}}{2} + \frac{i\mu_{31}}{2}\right) & = \iota_{X_{\lambda_4}}\Omega, \\ d\left(\frac{\mu_{31}}{2} - \frac{\mu_{13}}{2}\right) & = \iota_{X_{\lambda_5}}\Omega \end{cases}$$

which can be integrated as

$$\frac{i\mu_{13}}{2} + \frac{i\mu_{31}}{2} = \frac{1}{2}\left(\frac{-x_2 + x_1x_3 - y_1y_3}{K_2} - \frac{x_2}{K_1} \right) =: \gamma,$$
$$\frac{\mu_{31}}{2} - \frac{\mu_{13}}{2} = \frac{1}{2}\left(\frac{x_3y_1 + x_1y_3 - y_2}{K_2} - \frac{y_2}{K_1} \right) =: \delta$$

We optain

$$i\mu_{31} = \gamma + i\delta = -\frac{x_2 + iy_2}{2K_1} - \frac{-x_3y_1 - x_1(x_3 + y_3) + x_2 + y_2 + y_1y_3}{2K_2}$$

and finally

$$\mu_{31} = \frac{-i}{2}\left(-\frac{x_2+iy_2}{K_1} - \frac{-(x_1+iy_1)(x_3+iy_3)+x_2+iy_2}{K_2}\right).$$

The next pair of equations is

$$\begin{cases} d\left(\frac{i\mu_{23}}{2}+\frac{i\mu_{32}}{2}\right) & = \iota_{X_{\lambda_6}}\Omega, \\ d\left(\frac{\mu_{32}}{2}-\frac{\mu_{23}}{2}\right) & = \iota_{X_{\lambda_7}}\Omega. \end{cases}$$

We compute

$$\begin{aligned} \frac{i\mu_{23}}{2}+\frac{i\mu_{32}}{2} &= \frac{1}{2}\left(\frac{x_3}{K_2}-\frac{x_1x_2+y_1y_2}{K_1}\right) =: \zeta, \\ \frac{\mu_{32}}{2}-\frac{\mu_{23}}{2} &= \frac{1}{2}\left(\frac{x_2y_1-x_1y_2}{K_1}-\frac{y_3}{K_2}\right) =: \eta \end{aligned}$$

and obtain $\zeta + i\eta = i\mu_{32}$ and therefore

$$\begin{aligned} i\mu_{32} &= \frac{1}{2}\left(\frac{x_3}{K_2}-\frac{x_1x_2+y_1y_2}{K_1}\right) + i\frac{1}{2}\left(\frac{x_2y_1-x_1y_2}{K_1}-\frac{y_3}{K_2}\right) \\ &= \frac{1}{2}\left(\frac{-x_1x_2-y_1y_2+i(x_2y_1-x_1y_2)}{K_1}+\frac{x_3-iy_3}{K_2}\right). \end{aligned}$$

In order to determine μ_{33}, we integrate the last equation in (4.3.1) and obtain

$$\frac{i(\mu_{11}+\mu_{22}-2\mu_{33})}{2\sqrt{3}} = \frac{\sqrt{3}}{4}\left(\frac{x_2^2+y_2^2}{K_1}-\frac{1}{K_2}\right).$$

Using the fact that the matrix M is traceless, i.e. $\mu_{11}+\mu_{22}+\mu_{33}=0$, the left-hand side reduces to

$$\frac{-3i\mu_{33}}{2\sqrt{3}} = \frac{\sqrt{3}}{4}\left(\frac{x_2^2+y_2^2}{K_1}-\frac{1}{K_2}\right)$$

and therefore

$$\mu_{33} = \frac{i}{2}\left(\frac{x_2^2+y_2^2}{K_1}-\frac{1}{K_2}\right).$$

On the other hand, using $2(\mu_{11}+\mu_{22}) = -2\mu_{33}$, we have

$$\frac{i}{2\sqrt{3}}(\mu_{11}+\mu_{22}-2\mu_{33}) = \frac{3i}{2\sqrt{3}}(\mu_{11}+\mu_{22}) = \frac{\sqrt{3}}{4}\left(\frac{x_2^2+y_2^2}{K_1}-\frac{1}{K_2}\right).$$

Moreover, integrating the remaining Cartan equation $d\left(\frac{i\mu_{11}}{2}-\frac{i\mu_{22}}{2}\right) = \iota_{X_{\lambda_3}}\omega$ gives

$$\frac{i}{2}(\mu_{11}-\mu_{22}) = \kappa.$$

This yields the system of equations

$$\begin{cases} \frac{3i}{2\sqrt{3}}(\mu_{11}+\mu_{22}) & = \lambda, \\ \frac{i}{2}(\mu_{11}-\mu_{22}) & = \kappa \end{cases} \iff \begin{cases} \frac{3i}{2\sqrt{3}}(\mu_{11}+\mu_{22}) & = \lambda, \\ \frac{3i}{2\sqrt{3}}(\mu_{11}-\mu_{22}) & = \frac{3}{\sqrt{3}}\kappa. \end{cases}$$

Adding (resp. subtracting) the equations gives

$$\mu_{11} = -\frac{\sqrt{3}}{3}i\left(\alpha + \frac{3}{\sqrt{3}}\beta\right) = -\frac{i}{\sqrt{3}}\left(\frac{x_2^2+y_2^2-1}{2\sqrt{3}K_1} - \frac{x_3^2+y_3^2+2}{2\sqrt{3}K_2}\right),$$
$$\mu_{22} = -\frac{\sqrt{3}}{3}i\left(\alpha - \frac{3}{\sqrt{3}}\beta\right) = -\frac{i}{\sqrt{3}}\left(\frac{2x_2^2+2y_2^2+1}{2\sqrt{3}K_1} + \frac{x_3^2+y_3^2-1}{2\sqrt{3}K_2}\right),$$
$$\mu_{33} = -(\mu_{11}+\mu_{22}).$$

Now as we are working over $\mathfrak{su}^*(3)$ we have obtained all the entries of the matrix. □

5. Green's function and the vortex Hamiltonian for $\mathbb{CP}^n$

In Section 4, we recalled the point vortex momentum map on $\mathbb{CP}^2$ and computed the one on $\mathbb{F}_{1,2}(\mathbb{C}^3)$, i.e., on the degenerate and generic orbits of the action of $\mathsf{SU}(3)$. The aim of this section is to determine the Hamiltonian for the point vortex problem on $\mathbb{CP}$. An important ingredient here is the fundamental solution of the Laplace-Beltrami operator.

5.1. **The Laplace-Beltrami operator.** Let (M, g) be a Riemannian manifold of dimension n with local coordinates $x = (x_1, \ldots, x_n)$ in which we express the metric by the symmetric $(n \times n)$-matrix $(g_{ij}) := (g_{ij})_{1\le i,j\le n}$. The inverse of this matrix is denoted by $(g_{ij})^{-1} =: (g^{ij})$. The Riemannian volume form on (M, g) is given in local coordinates by

$$\sqrt{|\det(g)|}dx_1 \wedge \cdots \wedge dx_n =: d\mu.$$

Let $f : M \to \mathbb{R}$ a smooth function. Then the *Laplace-Beltrami operator* Δ on M is given in local coordinates $(x_1, \ldots, x_n)$ by

$$\Delta f = -\frac{1}{\sqrt{|\det(g_{ij})|}}\sum_{i,j=1}^{n} \partial_{x_j} g^{ij}\sqrt{|\det(g_{ij})|}\,\partial_{x_i} f.$$

which, on $\mathbb{R}^n$ equipped with the Euclidean metric, yields $\Delta = \sum_{j=1}^n \partial_{x_j}^2$. The *big diagonal* $\mathsf{Diag}_N(M)$ of the N-fold product $M \times \ldots \times M$ is defined as

$$\mathsf{Diag}_N(M) := \{(y_1, \ldots, y_N) \in M^N \mid y_j = y_k \text{ for some } j \neq k \text{ with } 1 \le j,k \le N\},$$

i.e., it consists of all N-tuple point on M for which at least two points coincide. If (M, g) is compact, then, for the Laplace-Beltrami operator, there exists a function $G : (M\times M)\setminus\mathsf{Diag}_2(M) \to \mathbb{R}$, referred to as *Green's function*, satisfying the following properties (see for instance Section 2.3 in Aubin [Aub98]):

(1) For all functions $\phi \in C^2(M, \mathbb{R})$, we have

$$\phi(p) = \frac{1}{\mathsf{vol}(M)}\int_M \phi(q)\ d\mu(q) + \int_M G(p,q)\Delta\phi(q)\ d\mu(q). \tag{5.1.1}$$

(2) G is smooth on $(M \times M) \setminus \mathsf{Diag}_2(M)$.
(3) G is symmetric, i.e. $G(p, q) = G(q, p)$ for all $p, q \in M$.
(4) We have $\int_M G(p, q)\ d\mu(q) = \text{constant}$ for all $p \in M$.

Green's function is also called a *fundamental solution* of the Laplace-Beltrami operator.

Examples 5.1.

(1) Green's function on $\mathbb{R}^2$ equipped with the Euclidean metric is given by

$$G(x,y) = -\frac{1}{2\pi}\ln|x-y|$$

where $|x-y| := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ is the Euclidean distance. Note that $|x-y|$ coincides with the geodesic distance $r(x,y)$ of the Euclidean metric.

(2) Consider the unit sphere $\mathbb{S}^2$ with $x = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \in \mathbb{S}^2$ and $y = (\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta') \in \mathbb{S}^2$. Then Green's function $G(x,y)$ for the spherical Laplace operator is given by (see Dritschel [Dri88])

$$G(x,y) = \frac{1}{2\pi}\log(1-\cos\Theta),$$

where $\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

On the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$, we have

Proposition 5.2. *The Laplace operator on the flag manifold is given by*

$$\Delta_{\mathbb{F}_{1,2}(\mathbb{C}^3)} = \Delta_{\mathbb{CP}^2} + \Delta_R$$

where

$$\Delta_{\mathbb{CP}^2} = \sum_{j,k=1}^{2}(1+\delta_{jk}z_k\overline{z}_j)\partial_{z_j}\partial_{\overline{z}_k}$$

is the Laplace operator on $\mathbb{CP}^2$ *and* Δ_R *a correction term given by*

$$\begin{aligned}\Delta_R = {} & \frac{K_1^2}{K_2}\left(\partial_{z_1}\partial_{\overline{z}_1} + z_3\partial_{z_1}\partial_{\overline{z}_2} + \overline{z}_3\partial_{z_2}\partial_{\overline{z}_1} + |z_3|^2\partial_{z_2}\partial_{\overline{z}_2}\right) + \left(K_1(1+|z_3|^2) + \frac{K_2^2}{K_1}\right)\partial_{z_3}\partial_{\overline{z}_3} \\ & + (\overline{z}_1 + z_3\overline{z}_2)(\overline{z}_1 z_2 - z_3 - z_3|z_1|^2)\partial_{z_1}\partial_{\overline{z}_3} + (\overline{z}_1 + \overline{z}_2 z_3)\left((1+|z_2|^2) - z_1\overline{z}_2 z_3\right)\partial_{z_2}\partial_{\overline{z}_3} \\ & + (z_1 + \overline{z}_3 z_2)(z_1\overline{z}_2 - \overline{z}_3 - z_3|z_1|^2)\partial_{z_3}\partial_{\overline{z}_1} + (z_1 + z_2\overline{z}_3)\left((1+|z_2|^2) - \overline{z}_1 z_2\overline{z}_3\right)\partial_{z_3}\partial_{\overline{z}_2}.\end{aligned}$$

where K_1 *and* K_2 *are the functions from Lemma 3.15.*

Proof. On an n-dimensional Kähler manifold (M,h) with Kähler potential K_M, the Laplace operator Δ is given by

$$\Delta = 2\sum_{i,j=1}^{n} h^{ij}\partial_i\overline{\partial}_j K_M.$$

Using the expression for the Kähler potential on the flag manifold from Lemma 3.15, one obtains the expression in the proposition by direct calculations. □

5.2. **The Hamiltonian for point vortex dynamics.** Let (M,ω) be a symplectic manifold and $N \in \mathbb{N}$. For $1 \leq k \leq N$, let $\tau_k : \Pi_{k=1}^N M_k \to M$ be the projection on the kth factor and let $\Gamma_1,\dots,\Gamma_N \in \mathbb{R}^{\neq 0}$. Consider the space $\mathcal{M} := \Pi_{k=1}^N M \setminus \mathsf{Diag}_N(M)$ and endow it with the symplectic form $\Omega := \Omega(\Gamma) := \sum_{k=1}^N \Gamma_k\tau_k^*\omega$. Recall Green's function $G : (M \times M) \setminus \mathsf{Diag}_N(M) \to \mathbb{R}$ defined by the expression (5.1.1) and define the so-called *Robin function* (see also Dritschel & Boatto [DB15])

$$R : M \to \mathbb{R}, \quad R(t) := \lim_{\tilde{t} \to t} \left(G(\tilde{t}, t) - \frac{1}{2\pi} \log d(\tilde{t}, t) \right).$$

The Hamiltonian $H : \mathcal{M} \to \mathbb{R}$,

$$H(s_1, \ldots, s_N) := \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j G(s_i, s_j) + \sum_{k=1}^{N} \Gamma_k^2 R_g(s_k) \tag{5.2.1}$$

describes the dynamics of N vortices with vortex strength $\Gamma_k \in \mathbb{R}^{\neq 0}$ for $k = 1, \ldots, N$ on the phase space $\mathcal{M}$. Green's function G describes the interaction between pairs of distinct vortices and the Robin function takes self-interactions into account (for more details, see Lin [Lin41]).

To study the vortex dynamics on an explicitly given symplectic manifold, we need explicit formulas for Green's function. Unfortunately, Green's functions are explicitly known only for certain classes of manifolds as for instance planes, hyperbolic planes, and 2-spheres, see Galajinsky [Gal22], Lim & Montaldi & Roberts [LMR01], Montaldi & Nava-Gaxiola [MNG14].

Motivated by the study of the generic and degenerate coadjoint orbits of $\mathsf{SU}(3)$, we naturally are interested in the Green's functions on these coadjoint orbits. The degenerate coadjoint orbit is $\mathbb{CP}^2$ and, in the case of $\mathsf{SU}(n+1)$, one of the degenerate coadjoint orbits is $\mathbb{CP}^n$.

5.3. **Green's function and the Hamiltonian on the coadjoint orbit $\mathbb{CP}^n$.** The aim of this subsection is to obtain an explicit formula for Green's function on $\mathbb{CP}^n$. This will allow us then to write down the Hamiltonian for the point vortex dynamics on $\mathbb{CP}^n$ explicitly.

Given an arbitrary compact Riemannian manifold, the explicit computation of Green's function is not obvious. However, there are certain homogeneous spaces for which methods are available to obtain an explicit formula for Green's function. Among these spaces, there are compact rank one symmetric spaces, briefly CROSS spaces. All CROSS spaces are given by the following list: the n-sphere $\mathbb{S}^n$, the projective spaces $\mathbb{KP}^n$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and the octonionic plane $\mathbb{OP}^2$.

CROSS spaces are special examples of so-called *locally harmonic Blaschke manifolds* (see Besse [Bes78]) for which the following result was established.

Proposition 5.3 (Beltrán & Corral & Criado del Rey [BCdR19]). *Let M be a locally harmonic Blaschke manifold and denote the geodesic distance between two points $x, y \in M$ by $r(x,y) =: r$. Then, Green's function on M is given by $G(x,y) = \varphi(r)$ where φ is determined by the differential equation*

$$\varphi'(r) = -\frac{1}{r^{n-1} V_M(r) \, \mathsf{vol}(M)} \int_r^{\mathrm{inj}(M)} t^{n-1} V_M(t) dt \tag{5.3.1}$$

where V_M is the so-called volume density and $\mathrm{inj}(M)$ *is the injectivity radius of the manifold.*

Recall that the *diameter* of a Riemannian manifold (M, g) is defined by

$$\mathsf{diam}(M) := \sup_{x,y\in M} r(x, y)$$

where $r(x, y)$ is the geodesic distance between $x, y \in (M, g)$. We now focus on the CROSS space $M = \mathbb{CP}^n$ where, in fact, the injectivity radius is equal to the diameter. The density function $V_{\mathbb{CP}^n}$ was explicitly determined by Kreyssig [Kre10] as

$$V_{\mathbb{CP}^n}(r) = \frac{2^{2n-1}}{r^{n-1}} \sin^{2n-1}(r) \cos(r). \tag{5.3.2}$$

In order to solve the differential equation (5.3.1) for $M = \mathbb{CP}^n$ we need the following technical result.

Lemma 5.4. *Let $n \in \mathbb{N}^{>0}$. Then*

$$\int \frac{1 - \sin^{2n}(x)}{\sin^{2n-1}(x)\cos(x)}\, dx = \log(\sin(x)) - \sum_{j=1}^{n-1} \frac{1}{2j \sin^{2j}(x)}.$$

Proof. We start with

$$\begin{aligned}\int \frac{1 - \sin^{2n}(x)}{\sin^{2n-1}(x)\cos(x)}\, dx &= \int \frac{1}{\sin^{2n-1}(x)\cos(x)}\, dx - \int \tan(x)\, dx \\ &= \int \frac{1}{\sin^{2n-1}(x)\cos(x)}\, dx + \log(\cos(x)).\end{aligned}$$

Using $1 + \cot^2(x) = \csc^2(x)$, we obtain

$$\begin{aligned}\int \frac{1}{\sin^{2n-1}(x)\cos(x)}\, dx = \int \sec(x)\csc^{2n-1}(x)\, dx &= \int \sec(x)\csc^{2n-2}(x)\csc(x)\, dx \\ &= \int \sec(x)\csc(x)(1 + \cot^2(x))^{n-1}\, dx.\end{aligned}$$

Using the binomial formula

$$(1 + \cot^2(x))^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \cot^{2k}(x)$$

we obtain

$$\int \sec(x)\csc(x) \sum_{k=0}^{n-1} \binom{n-1}{k} \cot^{2k}(x)\, dx = \sum_{k=0}^{n-1} \binom{n-1}{k} \int \sec(x)\csc(x)\cot^{2k}(x)\, dx.$$

Since

$$\sec(x)\csc(x)\cot^{\ell}(x) = \frac{1}{\sin(x)\cos(x)} \frac{\cos^{\ell}(x)}{\sin^{\ell}(x)} = \frac{\cos^{\ell-1}(x)}{\sin^{\ell+1}(x)} = \frac{\cot^{\ell-1}(x)}{\sin^2(x)}$$

the integral becomes

$$\int \cot^{\ell-1} \csc^2(x)\, dx.$$

By using the substitution $v = \cot(x)$ and $dv = -\csc^2(x)dx$ we have

$$-\int v^{\ell-1}\, dv = -\frac{v^\ell}{\ell} + C = -\frac{\cot^\ell(x)}{\ell} + C$$

where C is the integration constant. Now, for $\ell = 2k$, we can rewrite

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\int \sec(x)\csc(x)\cot^{2k}(x)\,dx = \int \sec(x)\csc(x)\,dx - \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{\cot^{2k}(x)}{2k}.$$

We compute

$$\int \sec(x)\csc(x)dx = \log(\sin(x)) - \log(\cos(x)).$$

This means for our original integral

$$\begin{aligned}&\int \frac{1-\sin^{2n}(x)}{\sin^{2n-1}(x)\cos(x)}dx\\ &= \log(\sin(x)) - \log(\cos(x)) - \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{\cot^{2k}(x)}{2k} + \log(\cos(x))\\ &= \log(\sin(x)) - \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{\cot^{2k}(x)}{2k}.\end{aligned}$$

By using repeatedly

$$\cot^2(x) = \frac{1}{\sin^2(x)},$$

we finally obtain

$$\int \frac{1-\sin^{2n}(x)}{\sin^{2n-1}(x)\cos(x)}dx = \log(\sin(x)) - \sum_{j=1}^{n-1}\frac{1}{2j\sin^{2j}(x)}.$$

□

We now obtain

Theorem 5.5. *Green's function on* $\mathbb{CP}^n$ *with the Fubini-Study metric is given by* $G : \mathbb{CP}^n \times \mathbb{CP}^n \setminus \mathsf{Diag}_2(\mathbb{CP}^n) \to \mathbb{R}$ *with*

$$G(\xi,\eta) = -\frac{1}{2n\cdot \mathsf{vol}(\mathbb{CP}^n)}\left(\log(\sin(r(\xi,\eta))) - \sum_{j=1}^{n-1}\frac{1}{2j\sin^{2j}(r(\xi,\eta))}\right)$$

where $r(\xi,\eta) = \arccos\sqrt{\frac{\langle\xi,\eta\rangle\langle\eta,\xi\rangle_H}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle_H}}$ *is the geodesic distance between the two point in* $\mathbb{CP}^n$ *and* $\langle\cdot,\cdot\rangle_H$ *is the Hermitian inner product.*

Proof. Recall from Equation (5.3.2) that the volume density of $\mathbb{CP}^n$ is

$$V_{\mathbb{CP}^n}(r) = \frac{2^{2n-1}\sin^{2n-1}(r)\cos(r)}{r^{n-1}}$$

and that

$$\mathrm{inj}(\mathbb{CP}^n) = \mathsf{diam}(\mathbb{CP}^n) = \frac{\pi}{2} \quad \text{and} \quad \mathsf{vol}(\mathbb{CP}^n) = \frac{\pi^n}{n!}.$$

Therefore the ODE for $\varphi(r)$ from (5.3.1) can be written as

$$\varphi'(r) = -\frac{1}{r^{n-1}V_{\mathbb{CP}^n}(r)\,\mathsf{vol}(\mathbb{CP}^n)} \int_r^{\mathsf{diam}(\mathbb{CP}^n)} t^{n-1}V_{\mathbb{CP}^n}(t)\ dt$$

and gives rise to the following equation:

$$\begin{aligned}\varphi'(r) &= -\frac{1}{\mathsf{vol}(\mathbb{CP}^n)\sin^{2n-1}(r)\cos(r)} \int_r^{\frac{\pi}{2}} \sin^{2n-1}(t)\cos(t)dt \\ &= -\frac{1}{\mathsf{vol}(\mathbb{CP}^n)\sin^{2n-1}(r)\cos(r)}\ \frac{1}{2n}(1-\sin^{2n}(r)).\end{aligned}$$

Solving this ODE gives the following formula for the fundamental solution

$$\varphi(r) = -\frac{1}{2n\cdot\mathsf{vol}(\mathbb{CP}^n)} \int \frac{1-\sin^{2n}(r)}{\sin^{2n-1}(r)\cos(r)}\ dr.$$

Using Lemma 5.4 now gives the result. □

Theorem 5.6. *The Hamiltonian for the N point vortex dynamics on the projective space $\mathbb{CP}^n$ is explicitly given by*

$$H : (\mathbb{CP}^n)^N \setminus \mathsf{Diag}_N(\mathbb{CP}^n) \to \mathbb{R},$$

$$H(\zeta) = -\frac{1}{2(n-1)!\,\pi^n}\sum_{\alpha<\beta}^{N}\Gamma_\alpha\Gamma_\beta\left(\log(\sin(r(\zeta_\alpha,\zeta_\beta))) - \sum_{j=1}^{n-1}\frac{1}{2j\sin^{2j}(r(\zeta_\alpha,\zeta_\beta))}\right)$$

where $\zeta = (\zeta_1,\dots,\zeta_N)$ and $r(\zeta_\alpha,\zeta_\beta)$ is the geodesic distance on $\mathbb{CP}^n$ between the two points given by

$$r(\zeta_\alpha,\zeta_\beta) = \arccos\sqrt{\frac{\langle\zeta_\alpha,\zeta_\beta\rangle_H\langle\zeta_\beta,\zeta_\alpha\rangle_H}{\langle\zeta_\alpha,\zeta_\alpha\rangle_H\langle\zeta_\beta,\zeta_\beta\rangle_H}},$$

where $\langle\cdot,\cdot\rangle_H$ is the Hermitian inner product.

Proof. Equation (5.2.1) gives the formal expression for the Hamiltonian of the N point vortex problem on a manifold. Using the explicit expression for the Green function on $\mathbb{CP}^n$ from Theorem 5.5 yields the result. □

The Hamiltonian vector field $\mathcal{X}_H$ (and then also the equations of motion) for the Hamiltonian from Theorem 5.6 can be computed either by using the implicit formula involving the symplectic form or by making use of an compatible almost complex structure J for the Fubini-Study metric, i.e., via $X^H = J\,\mathrm{grad}(H)$.

5.4. **The Hamiltonian on the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$.** As we saw in previous subsections, the explicit knowledge of Green's function is quite rare which makes it complicated to obtain an explicit expression for the Hamiltonian of the point vortex problem in many situations.

We do not yet have an explicit formula for Green's function and the Laplacian on the flag manifold $\mathbb{F}_{1,2}(\mathbb{C}^3)$ so that we also do not yet have an explicit expression for the Hamiltonian of the point vortex problem. One idea to approach this open question may be the fibration

$$\mathbb{S}^2 \longrightarrow W^6 \simeq \mathbb{F}_{1,2}(\mathbb{C}^3) \longrightarrow \mathbb{CP}^2$$

together with the hope to deduce Green's function on $W^6 \simeq \mathbb{F}_{1,2}(\mathbb{C}^3)$ from those on $\mathbb{S}^2$ and $\mathbb{CP}^2$ and thus obtain the Hamiltonian on $W^6 \simeq \mathbb{F}_{1,2}(\mathbb{C}^3)$.

In fact, this poses the more general question of the behaviour of Green's function with respect to fibrations in general. But this is beyond the scope of the present paper.

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Sonja Hohloch
Department of Mathematics
University of Antwerp
Middelheimlaan 1
2020 Antwerp, Belgium
Email address: sonja.hohloch@uantwerpen.be

Guner Muarem
Department of Mathematics
University of Antwerp
Middelheimlaan 1
2020 Antwerp, Belgium
Email address: guner.muarem@uantwerpen.be