Examples of
(a) subRiemanniann [sR] geometries which
(b) fiber 'metrically' over the plane and
(c) whose sR geod flow is integrable.

space	common name	authors
$J^k = J^k(\mathbb{R}, \mathbb{R})$	Jet space	Felipe Perez-Monroy
$\mathbb{R}^2 \times \mathbb{S}^1$	'roto-translation group'	Citti, Scott Pauls
"	bicycling space	Ardentov, Bor, Sachkov, LeDonne, M-
$\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$	tricycling space	Perline-Tabachnikov (3 wks ago!!)
$\mathbb{R}^2 \times SO(3)$	rolling ball on plate	Jurdjevic
$\mathbb{R}^2 \times SL(2, \mathbb{R})$	rolling hyperb. plane on euc plane	"

and a candidate family

space	common name	authors
$M^k = \mathbb{R}^2 \times \mathbb{T}^k$	Monster Tower	M- , Zhitomirskii
"	or k-trailer space	Murray-Sastry; Jean, ...
"	or Semple Tower	Semple, Demailly, Colley-Kennedy
"	or k-fold Cartan prolongation	Cartan (E), Bryant

where $\mathbb{T}^k = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is the k-torus

Common properties.
a) Subriemannian (bracket generating!) of Rank 2.
b) Admit a sR submersion onto the Euclidean plane
c) have integrable sR geodesic flows

Problem 1: Classify the sR geometries having properties (a), (b) and (c).

Problem 2: Classify the Carnot groups having properties (a), (b) and (c).
Remark: (b) is implied by 'Carnot' so is unnneccessary

i.e. Problem 2 $\subseteq$ Problem 1

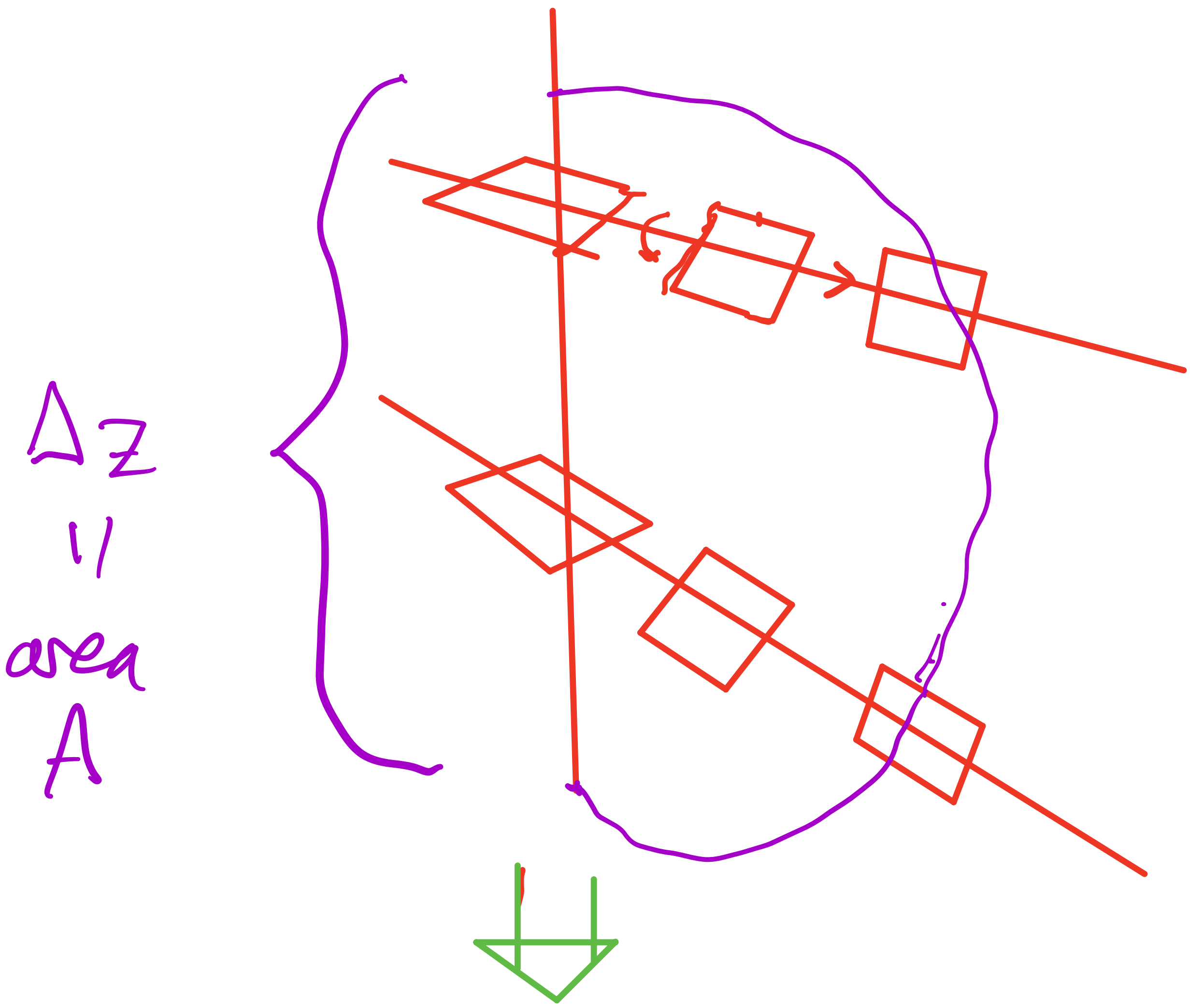
Problem 3. If a sR geometry is integrable then do all of its nilpotentizations (which are Carnot groups) also have integrable geodesic flows?

Model problem. $\boxed{\text{Heis group.}}$

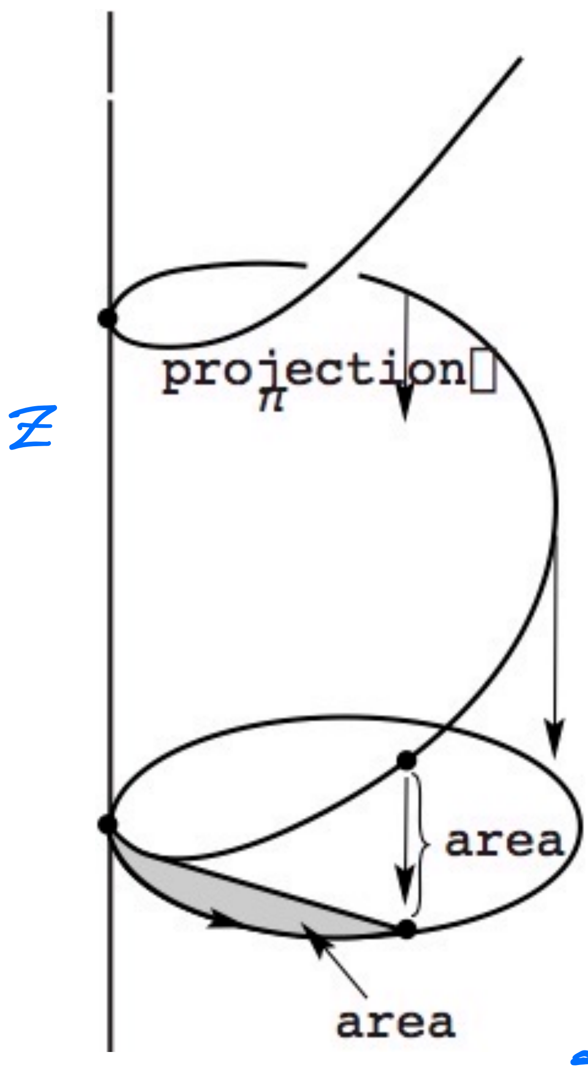
$Q = \mathbb{R}^3$.

$D =$ std "rot'l invariant" contact distribution

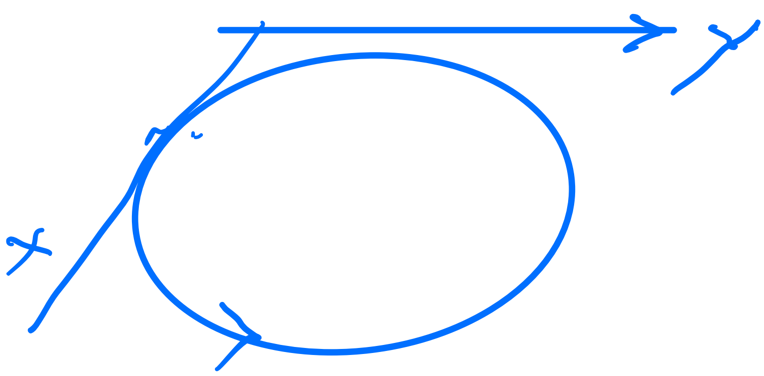
$= \{ dz - \frac{1}{2}(x\,dy - y\,dx) = 0 \}$



Heis geodesics...



[illegible] π



. . .

Project to

$\circlearrowright$ or $\longrightarrow$

circles or lines

Those corresp. to $\odot$ s & fail to minimize after one turn.

Only the horiz. lifts of $\longrightarrow$ s are globally minimizing or

$\boxed{\text{METRIC LINES.}}$

Describe

a) sR rank (2)

b) sR submersion

&

c) the sR Hamiltonian H

on → board

then to

Problem 4: For a given such geometry, classify the metric lines within that geometry.

solved for the bicycling space $\mathbb{R}^2 \times \mathbb{S}^1 = \mathbb{S}T\mathbb{R}^2$ (the 2nd entry in the table at the beginning of the talk) in Ardentov-Bor-LeDonne-M–Sachkov.

Thm: only the (lifts of lines) AND the "Euler soliton" are metric lines

Draw a picture , Richard? Why not?

J^2 = the Carnot group called 'Engel group" and is the Nilpotentization of the bicycling space (at any point) "so" admits elastica as projected geodesics. and 'thus" (cf Problem 3 above) this bicycling sol'n " $\implies$ " J^2 classification: lines + Euler solitons. (**Proof due to Ardentov-Sachkov): again:**

Bravo-Doddoli **ALMOST** fully solved Problem 4 for all of the jet spaces J^k , $k = 1, 2, \ldots$ (the first entry of the table at the beginning of the talk) 2023 UCSC thesis. Alejandro Bravo Doddoli is now at U of Mich, Ann Arbor.

Thm [Doddoli] Besides the lifts of Euc. lines and of the Euler soliton, there are, for each odd k, new metric lines which arise, and are a kind of hyperelliptic analogue of the Euler's soliton. Their planar projections are constructed from certain heteroclinic connections in degree one Hamiltonian systems of the form $\frac{1}{2}p^2 + \frac{1}{2}F(x)^2$ where $F(x)$ has degree k.

All geodesics for J^k are given in terms of solutions to a hyperelliptic ODE , the one induced by the H above.

Back to the main thread...

aiming for the Monster

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$\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$	tricycling space	Perline-Tabachnikov (3 wks ag
$\mathbb{R}^2 \times SO(3)$	rolling ball on plate	Jurdjevic
$\mathbb{R}^2 \times SL(2, \mathbb{R})$	rolling hyperb. plane on euc plane	"

conjectured integrable:

space	common name	authors
$M^k = \mathbb{R}^2 \times \mathbb{T}^k$	Monster Tower	M- , Zhitomirskii
"	or k-trailer space	Murray-Sastry; Jean, ...
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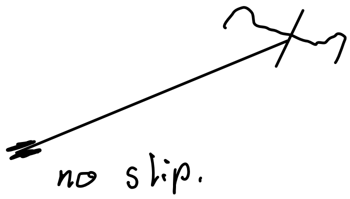
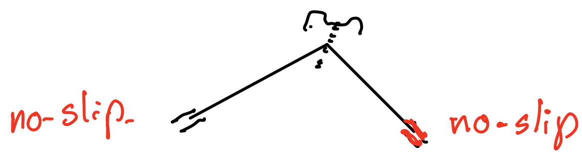


$\frac{d}{dt}$(back wheel) $\in$ line spanned by frame

$k=1$, $M_1 \simeq \mathbb{P}T\mathbb{R}^2 \simeq \mathbb{R}^2 \times S^1$

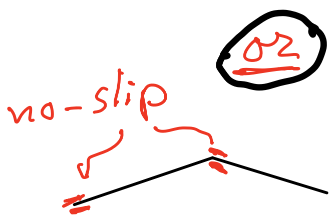
"bicycling"



$k=2$

Peg like-Tab

"tricycle"



truck w a trailer,

general k:

k trailers

(k=1)

from ``Leuven''
lecture notes,
Gary Kennedy

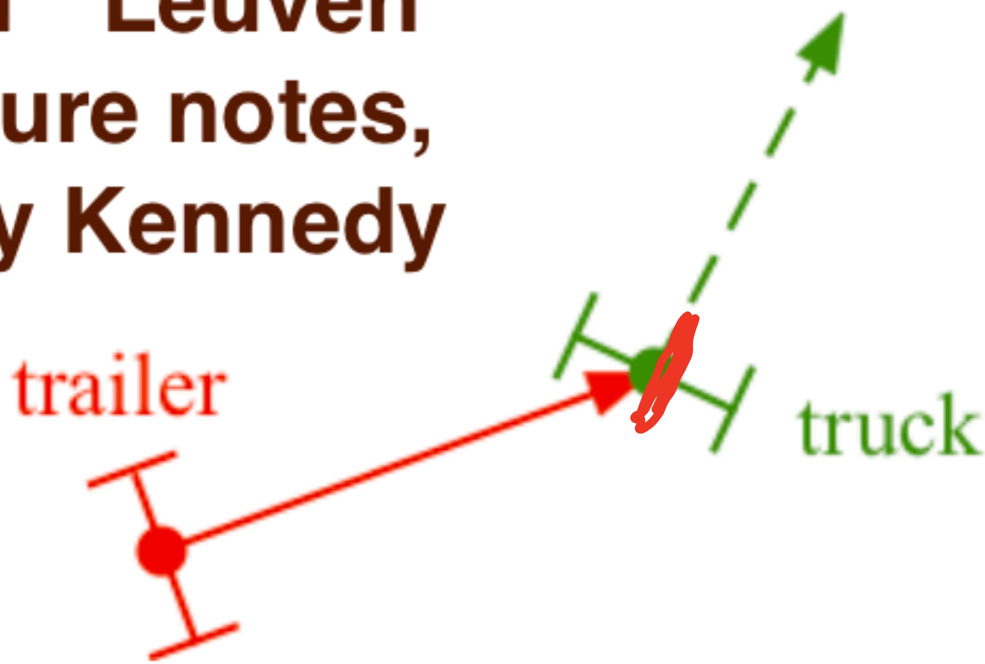


Figure 3: Truck and trailer.

(is really the
bicycle.)

(k=1)

from ``Leuven”

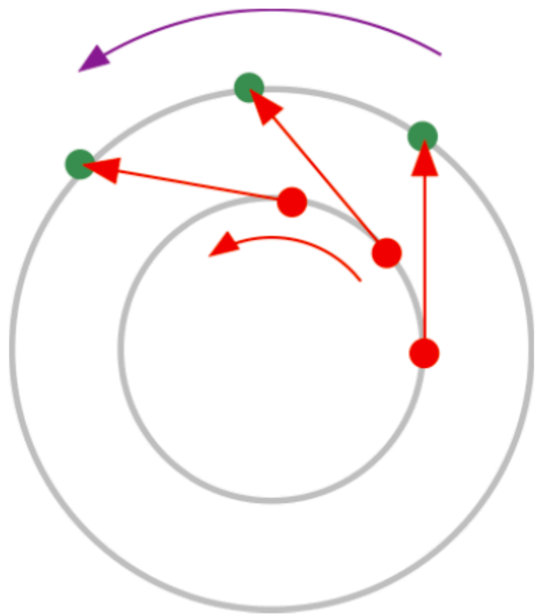


Figure 6: To move the trailer on a circle of radius r, drive the truck on a circle of radius $\sqrt{r^2+1}$.

from Leuven

bracket generating

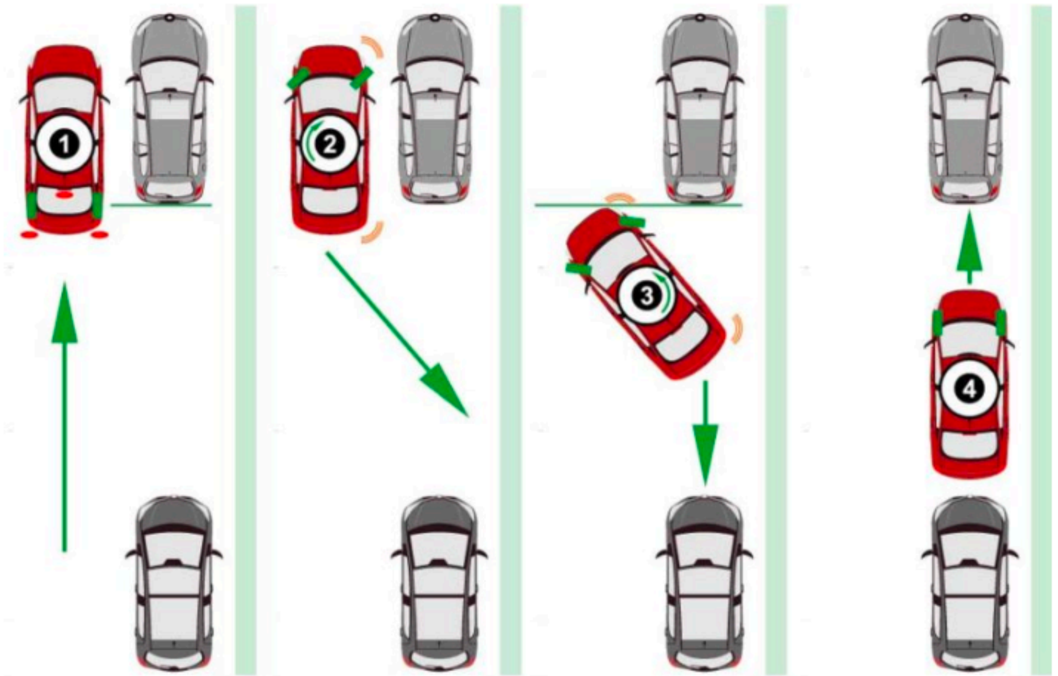


Figure 7: Parallel parking. (from officialdrivingschool.com)

controls: "Steer & Drive"

→ see E. Nelson,
``Tensor Analysis'',
for one of the best treatments

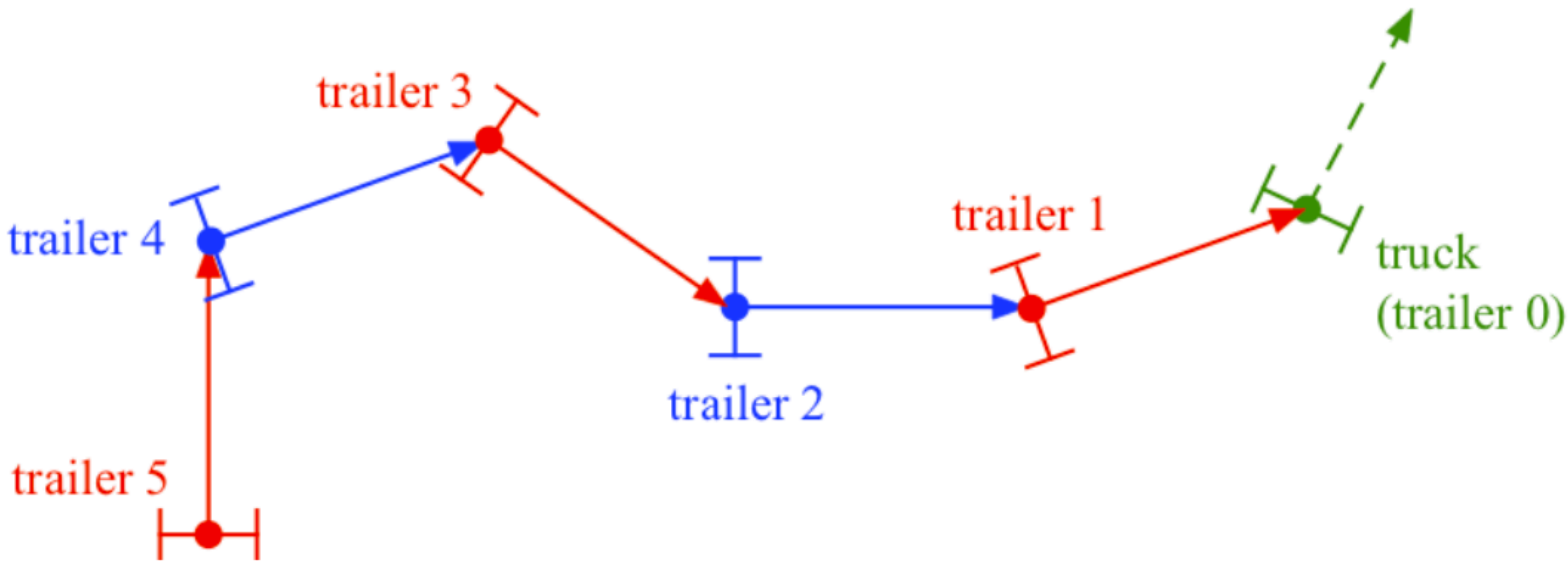


Figure 8: A train, made up of a truck pulling 5 trailers.

(M_5)

Put a sR metric on $D \subset TM_K$ by insisting length of a path is the length of the path swept out by the front wheel

equivalently

$$M_K \longrightarrow \mathbb{R}^2$$

k-trailer config. $\longrightarrow$ position of front wheel (front end of 1st line segment)

is a sR submersion

Another way to get the k-trailer space with its distribution: (projectively) prolong the plane k times

-result ``Monster tower'',
M- and Zhitomirskii

-``Semple tower''
the algebraic geometers:
Semple, Lejeune, Demailly,
Kennedy, Colley, ..

Prolongation:

``prolongation is just differentiating the equations you have and then adjoining those equations as new equations in the system.”

-Robert Bryant

there are different ways to PROLONG.

Affine:for functions on the line,
get the kth jet spaces J^k with its distribution and fibration structure
$R \to J^k \to J^{k-1}$.

Projectively: for curves in the plane
get the kth Monster Tower M_k with its rank 2 distribution and fibration structure
$P_1 \to M_k \to M_{k-1}$

Curve interpretations
graphs over $\mathbb{R} \subset \mathbb{R}^2$: J^k

let $k \to \infty$
in right way
integrable PDE ?

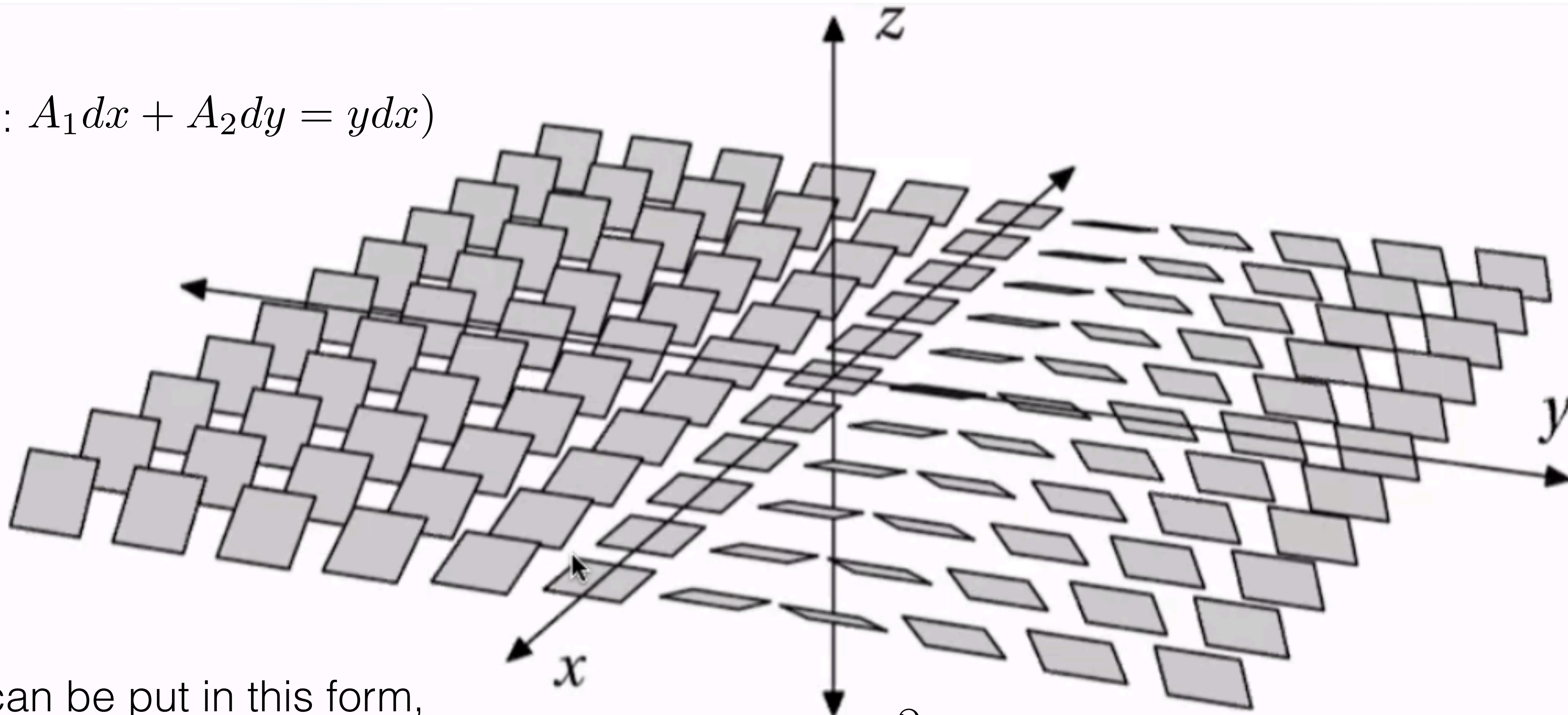
Fini !

...

Two-plane fields in 3-space: $\{\underline{dz - A_1(x,y)dx - A_2(x,y)dy} = 0\}$

`distribution', D

one-form, θ



(here: $A_1 dx + A_2 dy = y dx$)

D can be put in this form,

provided: the two-planes don't go vertical: $\frac{\partial}{\partial z} \notin D(x,y,z)$

and they are invariant under z-translations