Geometric constructions of integrable birational maps

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Discretization in
Geometry and Dynamics
SFB Transregio 109

History of Kahan discretization. 1: Kahan

- W. Kahan. *Unconventional numerical methods for trajectory calculations* (Unpublished lecture notes, 1993).

$$\dot{x} = Q(x) + Bx + c \quad \rightsquigarrow \quad (\widetilde{x} - x)/\epsilon = Q(x, \widetilde{x}) + B(x + \widetilde{x})/2 + c,$$

where $B \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *quadratic* form, and $Q(x, \widetilde{x}) = (Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x}))/2$ is the corresponding symmetric *bilinear* function. Thus,

$$\dot{x}_k \rightsquigarrow (\widetilde{x}_k - x_k)/\epsilon, \quad x_k^2 \rightsquigarrow x_k\widetilde{x}_k, \quad x_jx_k \rightsquigarrow (x_j\widetilde{x}_k + \widetilde{x}_jx_k)/2.$$

Linear w.r.t. $\widetilde{x}$, therefore defines a *rational* map $\widetilde{x} = \Phi_f(x, \epsilon)$. Obvious symmetry: $x \leftrightarrow \widetilde{x}$, $\epsilon \mapsto -\epsilon$, therefore Φ_f *reversible*:

$$\Phi_f^{-1}(x, \epsilon) = \Phi_f(x, -\epsilon).$$

In particular, Φ_f is *birational*, and $\deg \Phi_f = \deg \Phi_f^{-1} = n$.

History. 2: Hirota-Kimura

- R. Hirota, K. Kimura. *Discretization of the Euler top.* J. Phys. Soc. Japan **69** (2000) 627–630,
- K. Kimura, R. Hirota. *Discretization of the Lagrange top.* J. Phys. Soc. Japan **69** (2000) 3193–3199.

Renewed interest:

- T. Ratiu. Talk at the Oberwolfach Workshop “Geometric Integration”, March 2006. Claims: HK-type discretizations integrable for *Clebsch system* (true) and for *Kovalevsky top* (wrong).

History. 3: Team Berlin

- M. Petrera, A. Pfadler, Yu. S. *On integrability of Hirota-Kimura type discretizations. Experimental study of the discrete Clebsch system.* Exp. Math., 2009, **18**, 223–247.
- M. Petrera, A. Pfadler, Yu. S. *On integrability of Hirota-Kimura type discretizations.* RCD, 2011, **16**, 245–289.

Integrability for (besides Euler top and Lagrange top):

- reduced Nahm equations,
- three-wave interaction system,
- periodic Volterra chain of period $N = 3, 4$,
- dressing chain with $N = 3$,
- system of two interacting Euler tops,
- Kirchhof case of rigid body in an ideal fluid,
- Clebsch case of rigid body in an ideal fluid.

History. 4: Team Norway-Australia-New Zealand

- E. Celledoni, R.I. McLachlan, B. Owren, G.R.W. Quispel. *Geometric properties of Kahan's method.* J. Phys. A, 2013, **46**, 025201.

Theorem. *Let* $f(x) = J\nabla H(x)$*, with* $J \in so(n)$*, Hamilton function* $H : \mathbb{R}^n \to \mathbb{R}$ *of* $\deg = 3$*. Then* $\Phi_f(x, \epsilon)$ *admits a rational integral:*

$$\widetilde{H}(x, \epsilon) = H(x) + \frac{\epsilon}{3}(\nabla H(x))^{\mathrm{T}} \left(I - \frac{\epsilon}{2} f'(x)\right)^{-1} f(x),$$

and an invariant volume form

$$\frac{dx_1 \wedge \ldots \wedge dx_n}{\det\left(I - \frac{\epsilon}{2} f'(x)\right)}.$$

Degree of denominator $\det(I - \frac{\epsilon}{2} f'(x))$ *is n, degree of numerator of* $\widetilde{H}(x, \epsilon)$ *is* $n + 1$*.*

Example 1: Geometry of Kahan discretization of 2D Hamiltonian systems

Let $n = 2$ and let $H(x, y)$ be a polynomial with $\deg H = 3$.
Consider $f(x, y) = J\nabla H(x, y)$, with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. According to theorem by Celledoni et al., Φ_f is a birational planar map with an invariant measure and an integral $\Rightarrow$ completely integrable.
Integral:

$$\widetilde{H}(x, y, \epsilon) = \frac{C(x, y, \epsilon)}{D(x, y, \epsilon)},$$

where $\deg C = 3$, $\deg D = 2$. Level sets:

$$\mathcal{E}_\lambda = \big\{(x, y) : C(x, y, \epsilon) - \lambda D(x, y, \epsilon) = 0\big\},$$

a *pencil of cubic curves*, characterized by its nine *base points*.
On each invariant curve, Φ_f induces a translation (respective to the addition law on this curve).

Complexification, projectivization

Pencil

$$\bar{\mathcal{E}}_\lambda = \Big\{[x:y:z] \in \mathbb{CP}^2 \ : \ \bar{C}(x,y,z,\epsilon) - \lambda z \bar{D}(x,y,z,\epsilon) = 0\Big\}.$$

spanned by two curves,

$$\bar{\mathcal{E}}_0 = \Big\{[x:y:z] \in \mathbb{CP}^2 \ : \ \bar{C}(x,y,z,\epsilon) = 0\Big\},$$

assumed nonsingular, and

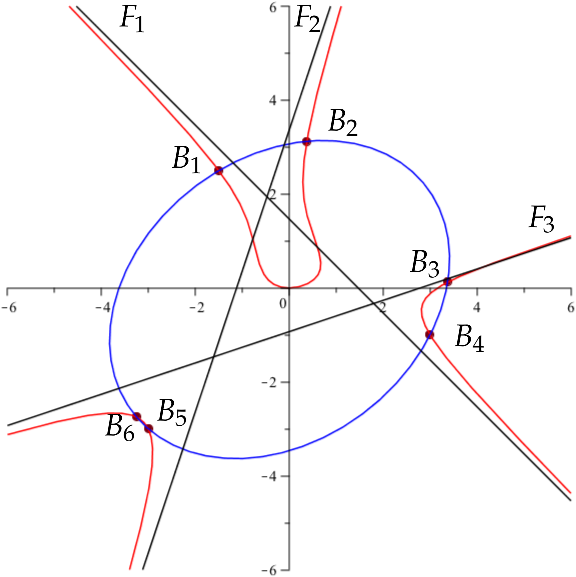
$$\bar{\mathcal{E}}_\infty = \Big\{[x:y:z] \in \mathbb{CP}^2 \ : \ z\bar{D}(x,y,z,\epsilon) = 0\Big\}$$

reducible, consisting of conic $\{\bar{D}(x,y,z,\epsilon) = 0\}$ and the line at infinity $\{z = 0\}$. Three base points at infinity:

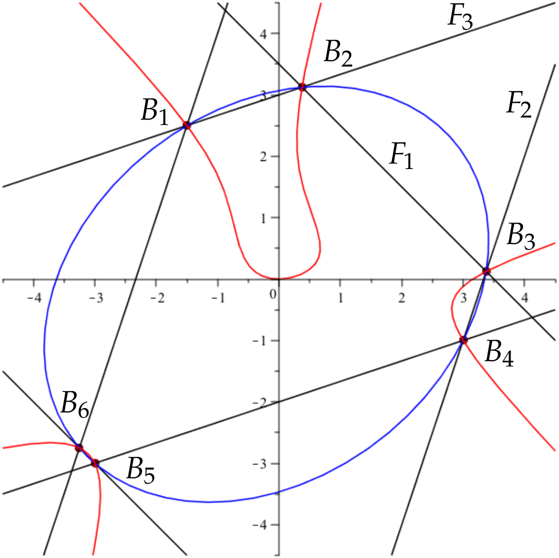
$$\{F_1, F_2, F_3\} = \bar{\mathcal{E}}_0 \cap \{z = 0\},$$

and six (finite) base points $\{B_1, \ldots, B_6\} = \bar{\mathcal{E}}_0 \cap \{\bar{D} = 0\}$.

Illustration



Observation



Main result

- M. Petrera, J. Smirin, Yu. S. *Geometry of the Kahan discretizations of planar quadratic Hamiltonian systems*. Proc. R. Soc. A **476** (2019) 20180761

Theorem. *A pencil of elliptic curves consists of invariant curves for Kahan's discretization of a planar quadratic Hamiltonian vector field iff the hexagon through the six finite base points has three pairs of parallel sides which pass through the three base points at infinity.*

Main tool: Manin involutions for cubic curves ...

Definition. Consider a nonsingular cubic curve $\bar{\mathcal{E}}$ in $\mathbb{CP}^2$.

- For a point $P_0 \in \bar{\mathcal{E}}$, the *Manin involution* $I_{\bar{\mathcal{E}},P_0} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is defined as follows:
  - For $P \neq P_0$, the point $\bar{P} = I_{\bar{\mathcal{E}},P_0}(P)$ is the unique third intersection point of $\bar{\mathcal{E}}$ with the line $(P_0 P)$;
  - For $P = P_0$, the point $\bar{P} = I_{\bar{\mathcal{E}},P_0}(P)$ is the unique second intersection point of $\bar{\mathcal{E}}$ with the tangent line to $\bar{\mathcal{E}}$ at $P = P_0$.
- For two distinct points $P_0, P_1 \in \bar{\mathcal{E}}$, the *Manin transformation* $M_{\bar{\mathcal{E}},P_0,P_1} : \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is defined as

$$M_{\bar{\mathcal{E}},P_0,P_1} = I_{\bar{\mathcal{E}},P_1} \circ I_{\bar{\mathcal{E}},P_0}.$$

With a natural addition law on $\bar{\mathcal{E}}$:

$$I_{\bar{\mathcal{E}},P_0}(P) = -(P_0 + P), \quad M_{\bar{\mathcal{E}},P_0,P_1}(P) = P + P_0 - P_1.$$

... and for cubic pencils

Definition. Consider a pencil $\mathfrak{E} = \{\bar{\mathcal{E}}_\lambda\}$ of cubic curves in $\mathbb{CP}^2$ with at least one nonsingular member.

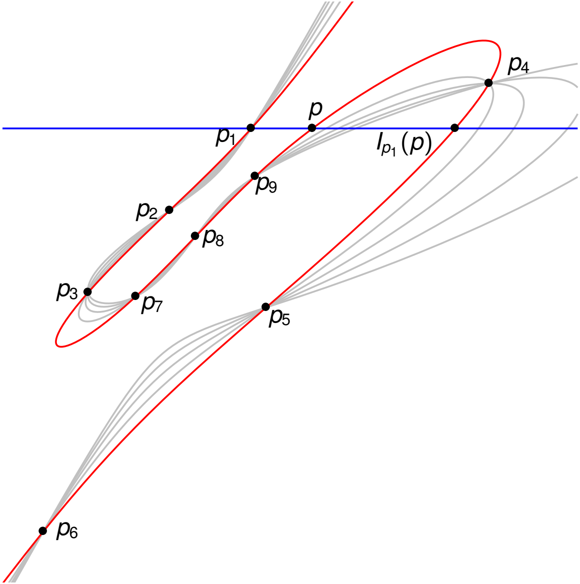
• Let B be a base point of the pencil. The *Manin involution* $I_{\mathfrak{E},B} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is a birational map defined as follows. For any $P \in \mathbb{CP}^2$, not a base point of $\mathfrak{E}$, let $\bar{\mathcal{E}}_\lambda$ be the unique curve of $\mathfrak{E}$ through P. Set

$$I_{\mathfrak{E},B}(P) = I_{\bar{\mathcal{E}}_\lambda,B}(P).$$

• Let B_1, B_2 be two distinct base points of the pencil. The *Manin transformation* $M_{\mathfrak{E},B_1,B_2} : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is a birational map defined as

$$M_{\mathfrak{E},B_1,B_2} = I_{\mathfrak{E},B_2} \circ I_{\mathfrak{E},B_1}.$$

Manin involutions for cubic pencils



Direct statement. Proof.

One shows that Kahan map Φ_f is a Manin transformation in six different ways:

$$\begin{aligned}
\Phi_f &= I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},F_1} = I_{\mathfrak{E},F_1} \circ I_{\mathfrak{E},B_4} \\
&= I_{\mathfrak{E},B_5} \circ I_{\mathfrak{E},F_2} = I_{\mathfrak{E},F_2} \circ I_{\mathfrak{E},B_2} \\
&= I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},F_3} = I_{\mathfrak{E},F_3} \circ I_{\mathfrak{E},B_6}.
\end{aligned}$$

Thus (on any invariant curve of $\mathfrak{E}$):

$$F_1 - B_1 = B_2 - F_2 = F_3 - B_3 = B_4 - F_1 = F_2 - B_5 = B_6 - F_3,$$

and

$$F_1 + F_2 + F_3 = O.$$

As a consequence, e.g.:

$$B_1 + B_2 = F_1 + F_2 = -F_3 \quad \Rightarrow \quad B_1 + B_2 + F_3 = O.$$

Thus, line $(B_1 B_2)$ passes through F_3. $\square$

Inverse statement. Proof.

Prescribe arbitrary nine coefficients of the side lines of the hexagon (three slopes μ_1, μ_2, μ_3 and six heights $b_1, \ldots, b_6$):

$$\begin{array}{ll}(B_1B_2): y = \mu_3 x + b_1, & (B_4B_5): \; y = \mu_3 x + b_4, \\ (B_2B_3): y = \mu_1 x + b_2, & (B_5B_6): \; y = \mu_1 x + b_5, \\ (B_3B_4): y = \mu_2 x + b_3, & (B_6B_1): \; y = \mu_2 x + b_6.\end{array}$$

This defines nine points $B_1, \ldots, B_6$ and F_1, F_2, F_3, therefore the pencil $\mathfrak{E}$ of cubic curves with those nine base points. Set

$$\begin{aligned}\Phi &= I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},F_1} = I_{\mathfrak{E},F_1} \circ I_{\mathfrak{E},B_4} \\ &= I_{\mathfrak{E},B_5} \circ I_{\mathfrak{E},F_2} = I_{\mathfrak{E},F_2} \circ I_{\mathfrak{E},B_2} \\ &= I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},F_3} = I_{\mathfrak{E},F_3} \circ I_{\mathfrak{E},B_6}.\end{aligned}$$

This is a birational map of $\mathbb{CP}^2$ **of degree 2**. Check that this is a Kahan discretization of $f = J\nabla H$ with $\deg H = 3$.

Inverse statement. Proof.

Explicit expression:

$$H(x,y) =$$
$$\frac{2\mu_{12}}{b_{14}\mu_{23}\mu_{13}}\Big(\tfrac{1}{3}(\mu_3 x - y)^3 + \tfrac{1}{2}(b_1+b_4)(\mu_3 x - y)^2 + b_1 b_4(\mu_3 x - y)\Big)$$
$$-\frac{2\mu_{23}}{b_{25}\mu_{12}\mu_{13}}\Big(\tfrac{1}{3}(\mu_1 x - y)^3 + \tfrac{1}{2}(b_2+b_5)(\mu_1 x - y)^2 + b_2 b_5(\mu_1 x - y)\Big)$$
$$+\frac{2\mu_{13}}{b_{36}\mu_{12}\mu_{23}}\Big(\tfrac{1}{3}(\mu_2 x - y)^3 + \tfrac{1}{2}(b_3+b_6)(\mu_2 x - y)^2 + b_3 b_6(\mu_2 x - y)\Big),$$

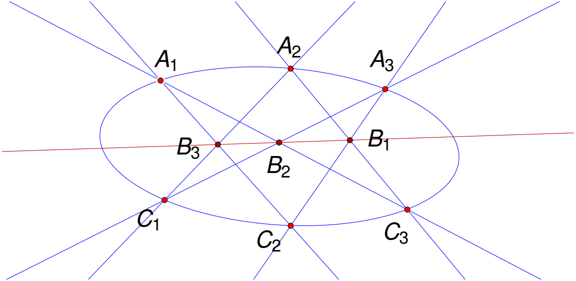
where $b_{ij} = b_i - b_j$, $\mu_{ij} = \mu_i - \mu_j$.

Geometry implies dynamics! $\square$

Projective generalization

Pascal configuration: six points A_1, A_2, A_3, C_1, C_2, C_3 on a conic $\mathcal{C}$, and three intersection points on a line ℓ:

$$B_1 = (A_2C_3)\cap(A_3C_2), \quad B_2 = (A_3C_1)\cap(A_1C_3), \quad B_3 = (A_1C_2)\cap(A_2C_1).$$



Construction

Consider the pencil $\mathfrak{E}$ of cubic curves passing through the nine points A_i, C_i, B_i. It contains a reducible cubic $\mathcal{C} \cup \ell$, as well as two triples of lines,

$$(A_1C_2) \cup (A_2C_3) \cup (A_3C_1) \text{ and } (A_2C_1) \cup (A_3C_2) \cup (A_1C_3).$$

Theorem [Yu. S.' 2020]. *The map*

$$\begin{aligned} \Phi &= I_{\mathfrak{E},A_1} \circ I_{\mathfrak{E},B_1} = I_{\mathfrak{E},B_1} \circ I_{\mathfrak{E},C_1} \\ &= I_{\mathfrak{E},A_2} \circ I_{\mathfrak{E},B_2} = I_{\mathfrak{E},B_2} \circ I_{\mathfrak{E},C_2} \\ &= I_{\mathfrak{E},A_3} \circ I_{\mathfrak{E},B_3} = I_{\mathfrak{E},B_3} \circ I_{\mathfrak{E},C_3} \end{aligned}$$

is a birational map **of degree 2** *leaving each curve of the pencil $\mathfrak{E}$ invariant (thus with a rational integral of motion of* $\deg = 3$*).*

An early example

- R. Penrose, C. Smith. *A quadratic mapping with invariant cubic curve*. Math. Proc. Camb. Phyl. Soc. **89** (1981), 89–105:

$$\Phi : \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_0(x_0 + ax_1 + a^{-1}x_2) \\ x_1(x_1 + ax_2 + a^{-1}x_0) \\ x_2(x_2 + ax_0 + a^{-1}x_1) \end{bmatrix}$$

with

$$A_1 = [0 : 1 : -a], \quad C_1 = [0 : a : -1], \quad B_1 = [0 : 1 : -1]$$

(and others cyclically). Upon a projective transformation sending B_1, B_2, B_3 to infinity, get a Kahan discretization of a Hamiltonian vector field with $H(x, y) = xy(1 - x - y)$ with the time step $\epsilon = (a - 1)/(a + 1)$.

Example 2: $(\gamma_1, \gamma_2, \gamma_3)$-family

A family of quadratic planar systems parametrized by $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{\ell_1^{\gamma_1 - 1} \ell_2^{\gamma_2 - 1} \ell_3^{\gamma_3 - 1}} J \nabla H,$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H(x, y) = (\ell_1(x, y))^{\gamma_1} (\ell_2(x, y))^{\gamma_2} (\ell_3(x, y))^{\gamma_3},$$

and $\ell_i(x, y) = a_i x + b_i y$ linear forms.

Origin: reduced Nahm equations for symmetric monopoles [N. Hitchin, N. Manton, M. Murray' 1995]

- Tetrahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$:

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = x^2 - y^2, \end{cases} \qquad H_1(x, y) = x\big(y^2 - \frac{1}{3}x^2\big).$$

- Octahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$:

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = x^2 - 2y^2, \end{cases} \qquad H_2(x, y) = x^2\big(y^2 - \frac{1}{4}x^2\big).$$

- Icosahedral symmetry, $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$:

$$\begin{cases} \dot{x} = 2xy - 2x^2, \\ \dot{y} = 2xy - y^2, \end{cases} \qquad H_3(x, y) = x^2y^3\big(- \frac{2}{3}x + \frac{1}{2}y\big).$$

In all three cases level curves $H_i(x, y) = c$ are of genus $g = 1$.

Discretization

Kahan-Hirota-Kimura discretizations are integrable [M. Petrera, A. Pfadler, Yu. S.' 2011]:

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = x^2 - y^2, \end{cases} \quad \rightsquigarrow \quad \begin{cases} (\widetilde{x} - x)/\epsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\epsilon = \widetilde{x}x - \widetilde{y}y, \end{cases}$$

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = x^2 - 2y^2, \end{cases} \quad \rightsquigarrow \quad \begin{cases} (\widetilde{x} - x)/\epsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\epsilon = \widetilde{x}x - 2\widetilde{y}y, \end{cases}$$

$$\begin{cases} \dot{x} = 2xy - 2x^2, \\ \dot{y} = 2xy - y^2, \end{cases} \quad \rightsquigarrow \quad \begin{cases} (\widetilde{x} - x)/\epsilon = (\widetilde{x}y + x\widetilde{y}) - 2\widetilde{x}x, \\ (\widetilde{y} - y)/\epsilon = (\widetilde{x}y + x\widetilde{y}) - \widetilde{y}y. \end{cases}$$

In all three cases, the map admits an invariant pencil of elliptic curves, of degrees 3, 4, and 6, respectively.

Break of homogeneity? $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1)$: no problem

According to theorem by Celledoni et al., for *any* $H(x, y)$ with $\deg H = 3$ (also non-homogeneous), Kahan discretization of the Hamiltonian system with H is integrable.

Break of homogeneity? $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$: no!

Consider $(\dot{x}, \dot{y})^{\mathrm{T}} = x^{-1} J \nabla H$ with $H(x, y) = x^2(y^2 - \frac{1}{4}x^2 - \frac{1}{2}b)$,

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = b + x^2 - 2y^2. \end{cases}$$

One can show: Kahan discretization

$$\begin{cases} (\widetilde{x} - x)/\epsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\epsilon = b + x\widetilde{x} - 2y\widetilde{y} \end{cases}$$

is non-integrable! However, an *adjusted Kahan discretization*

$$\begin{cases} (\widetilde{x} - x)/\epsilon = \widetilde{x}y + x\widetilde{y}, \\ (\widetilde{y} - y)/\epsilon = b + x\widetilde{x} - (2 - \epsilon^2 b)y\widetilde{y} \end{cases}$$

is integrable!

Break of homogeneity? $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$: no!

Consider system

$$\begin{aligned} \dot{x} &= 2xy - 2x^2 + c, \\ \dot{y} &= 2xy - y^2. \end{aligned}$$

It is of the form $(\dot{x}, \dot{y})^{\mathrm{T}} = (xy + c)^{-1} J \nabla H$ with

$$H(x, y) = (xy + c)^2 \Big(-\frac{2}{3} xy + \frac{1}{2} y^2 + \frac{1}{3} c \Big).$$

One can show: Kahan discretization

$$\begin{aligned} (\widetilde{x} - x)/\epsilon &= (\widetilde{x} y + x \widetilde{y}) - 2x\widetilde{x} + c, \\ (\widetilde{y} - y)/\epsilon &= (\widetilde{x} y + x \widetilde{y}) - y\widetilde{y} \end{aligned}$$

is non-integrable! However, an *adjusted Kahan discretization*

$$\begin{aligned} (\widetilde{x} - x)/\epsilon &= (1 + \epsilon^2 c)(\widetilde{x} y + x \widetilde{y}) - (2 - \epsilon^2 c) x\widetilde{x} + c - \epsilon^2 c (2 + \epsilon^2 c) y \widetilde{y}, \\ (\widetilde{y} - y)/\epsilon &= (\widetilde{x} y + x \widetilde{y}) - (1 + \epsilon^2 c) y\widetilde{y} \end{aligned}$$

is integrable again!

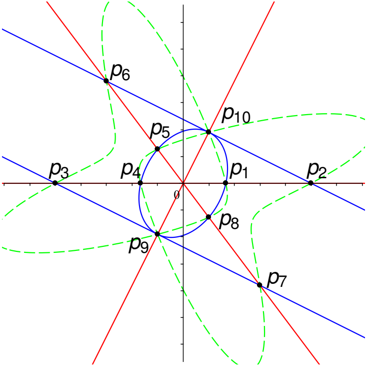
How to find repairing adjustments?

General answer: **only geometry knows**! It reveals solutions, when asked properly.

• M. Petrera, Yu. S., R. Zander.
How one can repair non-integrable Kahan discretizations.
J. Phys. A: Math. Theor., 2020, **53**, 37LT01, 7 pp.

• M. Schmalian, Yu. S., Yu. Tumarkin.
How one can repair non-integrable Kahan discretizations. II. A planar system with invariant curves of degree 6.
Math. Phys. Anal. Geom., 2021, **24**:40,19 pp.

Geometry of discretization for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 2)$



Invariant pencil $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$ of quartic curves with two double points, featuring three reducible quartics:

- red: two simple lines and one double line $(p_9 p_{10})$,
- blue: conic $(p_1 p_4 p_5 p_8 p_9 p_{10})$ and two lines $(p_2 p_6 p_{10}), (p_3 p_7 p_9)$.
- four lines $(p_9 p_8 p_2)$, $(p_9 p_4 p_6)$, $(p_{10} p_1 p_7)$, $(p_{10} p_5 p_3)$.

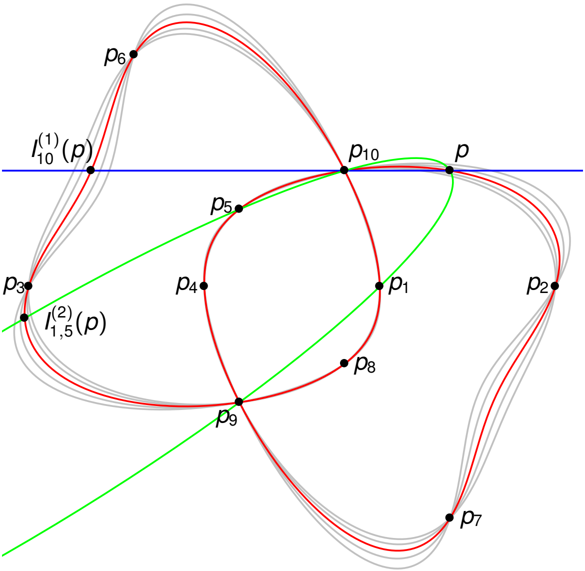
Involutions for quartic pencils with two double points

• M. Petrera, Yu. S., K. Wei, R. Zander.
Manin involutions for elliptic pencils and discrete integrable systems. Math. Phys. Anal. Geom., 2021, **24**:6, 26 pp.

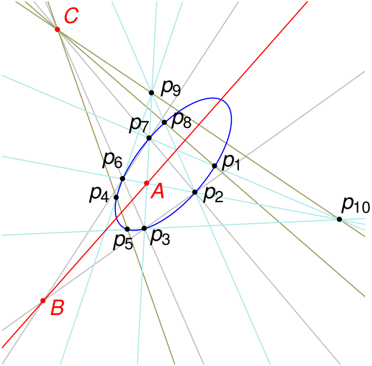
Manin involutions for $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$:

- $I_k^{(1)}$, $k \in \{9, 10\}$: $I_k^{(1)}(p)$ is the third intersection point of the quartic through p with the line (pp_k).
- $I_{i,j}^{(2)}$, $i, j \in \{1, \ldots, 8\}$: $I_{i,j}^{(2)}(p)$ is the sixth intersection point of the quartic through p with the conic through p_9, p_{10}, p_i, p_j, p.

Involutions for quartic pencils with two double points



Generalized geometry: projectively symmetric pencils



Geometry of base points of a *projectively symmetric quartic pencil with two double points* $\mathfrak{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$, featuring two reducible curves:

- conic $(p_1 \ldots p_8)$ and double line $(p_9 p_{10})$,
- four lines $(p_9 p_8 p_2)$, $(p_9 p_4 p_6)$, $(p_{10} p_1 p_7)$, $(p_{10} p_5 p_3)$.

Quadratic Manin maps for projectively symmetric quartic pencils

Theorem [M. Petrera, Yu. S., K. Wei, R. Zander' 2021].

1. *The projective involution σ can be represented as*

$$\sigma = I^{(2)}_{1,8} = I^{(2)}_{2,7} = I^{(2)}_{3,6} = I^{(2)}_{4,5}.$$

2. *The map*

$$\Phi \;=\; I^{(1)}_{9} \circ \sigma \;=\; \sigma \circ I^{(1)}_{10} \;=\; I^{(2)}_{i,k} \circ I^{(2)}_{j,k},$$

$(i,j) \in \{(1,2),(2,3),(3,4),(5,6),(6,7),(7,8)\}$ *and* $k \in \{1,\ldots,8\}$ *distinct from* i,j*, is a birational map* **of degree 2**, *leaving all curves of the pencil* $\mathfrak{E}$ *invariant (i.e., with a rational integral of* $\deg = 4$*).*

Example 3: Zhukovsky-Volterra gyrostat

ZV(β_1, β_2):

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3 - \beta_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1 + \beta_1 x_3, \\ \dot{x}_3 = \alpha_3 x_1 x_2 + \beta_2 x_1 - \beta_1 x_2. \end{cases}$$

Integrable if

- either $\beta_2 = 0$ (and α_i arbitrary),
- or $\alpha_1 + \alpha_2 + \alpha_3 = 0$,

with integrals of motion

$$\begin{aligned} H_3(x) &= \alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2), \\ H_2(x) &= \alpha_3 x_1^2 - \alpha_1 x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2). \end{aligned}$$

Kahan discretization: integrable for $\beta_2 = 0$ ...

dZV(β_1, β_2) [M. Petrera, A. Pfadler, Yu. S.' 2011]:

$$\begin{cases} \widetilde{x}_1 - x_1 = \varepsilon\alpha_1(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3) - \varepsilon\beta_2(\widetilde{x}_3 + x_3), \\ \widetilde{x}_2 - x_2 = \varepsilon\alpha_2(\widetilde{x}_3 x_1 + x_3\widetilde{x}_1) + \varepsilon\beta_1(\widetilde{x}_3 + x_3), \\ \widetilde{x}_3 - x_3 = \varepsilon\alpha_3(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2) + \varepsilon\beta_2(\widetilde{x}_1 + x_1) - \varepsilon\beta_1(\widetilde{x}_2 + x_2). \end{cases}$$

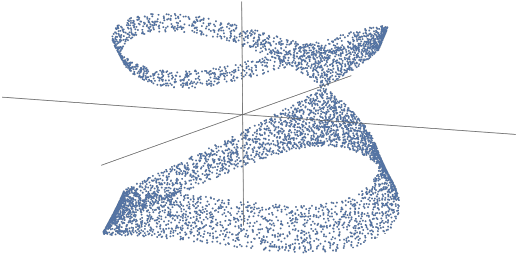
Integrable if $\beta_2 = 0$, with two integrals

$$\begin{aligned} \mathcal{H}_2(x;\varepsilon) &= \frac{\alpha_3 x_1^2 - \alpha_1 x_3^2 - 2\beta_1 x_1 + \dfrac{\beta_1^2}{\alpha_3}}{1 - \varepsilon^2\alpha_3\alpha_1 x_2^2}, \\ \mathcal{H}_3(x;\varepsilon) &= \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2\beta_1 x_1 - \dfrac{\beta_1^2}{\alpha_2}}{1 - \varepsilon^2\alpha_1\alpha_2 x_3^2}. \end{aligned}$$

... and non-integrable for $\beta_2 \neq 0$

Non-integrable if $\beta_2 \neq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, with only one integral

$$\mathcal{H}_3(x;\varepsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2) + \dfrac{\beta_2^2}{\alpha_1} - \dfrac{\beta_1^2}{\alpha_2}}{1 - \varepsilon^2 \alpha_1 \alpha_2 x_3^2}.$$



Integrable adjustment

$$\begin{cases} \widetilde{x}_1 - x_1 = \varepsilon\alpha_1(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3) - \varepsilon\beta_2(\widetilde{x}_3 + x_3), \\ \widetilde{x}_2 - x_2 = \varepsilon\alpha_2(\widetilde{x}_3 x_1 + x_3\widetilde{x}_1) + \varepsilon\beta_1(\widetilde{x}_3 + x_3), \\ \widetilde{x}_3 - x_3 = \varepsilon\alpha_3(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2) - \varepsilon\beta_2\dfrac{\alpha_2 + \alpha_3}{\alpha_1}(\widetilde{x}_1 + x_1) - \varepsilon\beta_1(\widetilde{x}_2 + x_2) \\ \qquad\qquad -\varepsilon^2\beta_1(\alpha_2 + \alpha_3)(x_1\widetilde{x}_3 - \widetilde{x}_1 x_3) - \varepsilon^2\beta_2\alpha_2(x_2\widetilde{x}_3 - \widetilde{x}_2 x_3). \end{cases}$$

This map possesses (for arbitrary α_i) two integrals of motion, $\mathcal{H}_3(x;\varepsilon)$ as above and

$$\mathcal{H}_2(x;\varepsilon) = \frac{\alpha_3 x_1^2 - \alpha_1 x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2) + \dfrac{\beta_2^2}{\alpha_1} - \dfrac{\beta_1^2}{\alpha_2}}{1 - \varepsilon^2\alpha_1\alpha_2 x_3^2}.$$

If $\alpha_1 + \alpha_2 + \alpha_3 = 0$, get integrable discretization of ZV(β_1, β_2).

Generators of a separable pencil of quadrics

Consider a *separable pencil of quadrics* in $\mathbb{P}^3$:

$$\mathcal{P}_\mu = \{X_1 X_2 - \mu X_3 X_4 = 0\},$$

where X_j are four independent linear forms on $\mathbb{C}^4$. All $\mathcal{P}_\mu$ pass through the *base set* consisting of four lines

$$\{X_1 = X_3 = 0\} \cup \{X_1 = X_4 = 0\} \cup \{X_2 = X_3 = 0\} \cup \{X_2 = X_4 = 0\}.$$

Two straight line generators of $\mathcal{P}_\mu$ through $[X_1 : X_2 : X_3 : X_4]$:

$$\ell_1(X) = \Big\{ [X_1 : tX_2 : tX_3 : X_4] : t \in \mathbb{P}^1 \Big\},$$

and

$$\ell_2(X) = \Big\{ [tX_1 : X_2 : tX_3 : X_4] : t \in \mathbb{P}^1 \Big\}.$$

Involutions defined by a second pencil of quadrics

Let

$$\mathcal{Q}_\lambda = \{Q_0(x) - \lambda Q_\infty(x) = 0\}$$

be a second pencil of quadrics. Space $\mathbb{P}^3$ is foliated by elliptic curves $\mathcal{E}_{\mu\lambda} = \mathcal{P}_\mu \cap \mathcal{Q}_\lambda$. For any $X \in \mathbb{P}^3$ not from base sets of both pencils, determine μ and λ such that $X \in \mathcal{E}_{\mu\lambda}$. Set

$$\begin{aligned} i_1(X) &= \text{the second intersection point of } \ell_1(X) \text{ with } \mathcal{Q}_\lambda, \\ i_2(X) &= \text{the second intersection point of } \ell_2(X) \text{ with } \mathcal{Q}_\lambda. \end{aligned}$$

Birational involutions on $\mathbb{P}^3$ of $\deg = 5$ in general; under certain geometric conditions, of $\deg = 3$. Leave $\mathcal{P}_\mu$ and $\mathcal{Q}_\lambda$ invariant.

3D generalization of QRT maps

Definition.

• *3D generalization of a QRT map:*

$$g = i_2 \circ i_1 .$$

• *3D generalization of a QRT root,* for the case, when $\mathcal{P}_\mu$, $\mathcal{Q}_\lambda$ are invariant under a linear projective map σ on $\mathbb{P}^3$:

$$f = i_2 \circ \sigma = \sigma \circ i_1 ,$$

so that $g = f \circ f$.

These birational maps have, by construction, integrals of motion

$$\frac{X_1 X_2}{X_3 X_4} = \mu \quad \text{and} \quad \frac{Q_0(x)}{Q_\infty(x)} = \lambda.$$

Discrete time Zhukovsky-Volterra gyrostat as 3D QRT

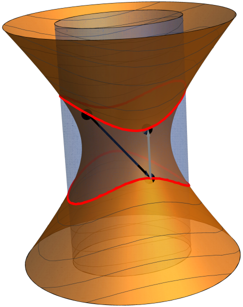
Theorem [J. Alonso, Yu. S., K. Wei' 2022]. *Set*

$$\begin{cases} X_1 = \sqrt{\alpha_1}x_2 - \sqrt{\alpha_2}x_1 - \Big(\dfrac{\beta_1}{\sqrt{\alpha_2}} + \dfrac{\beta_2}{\sqrt{\alpha_1}}\Big)x_4, \\ X_2 = \sqrt{\alpha_1}x_2 + \sqrt{\alpha_2}x_1 + \Big(\dfrac{\beta_1}{\sqrt{\alpha_2}} - \dfrac{\beta_2}{\sqrt{\alpha_1}}\Big)x_4, \\ X_3 = x_4 - \varepsilon\sqrt{\alpha_1\alpha_2}x_3, \\ X_4 = x_4 + \varepsilon\sqrt{\alpha_1\alpha_2}x_3. \end{cases}$$

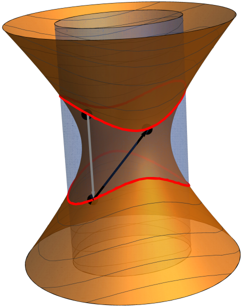
Further, set $Q_\lambda(x) = Q_0(x) - \lambda Q_\infty(x)$, where $Q_0(x) = X_3X_4$,

$$Q_\infty(x) = \alpha_3x_1^2 - \alpha_1x_3^2 - 2(\beta_1x_1 + \beta_2x_2)x_4 + \left(\frac{\beta_2^2}{\alpha_1} - \frac{\beta_1^2}{\alpha_2}\right)x_4^2.$$

Then $\mathcal{P}_\mu$, $\mathcal{Q}_\lambda$ are symmetric w.r.t. $\sigma : x_3 \to -x_3$, and the 3D QRT root $f = \sigma \circ i_1 = i_2 \circ \sigma$ is the adjusted Kahan discretization of $ZV(\beta_1, \beta_2)$.



$f = \sigma \circ i_1$



$f = i_2 \circ \sigma$