Stationary coupled KdV systems and their Stäckel representations

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Some historical developments

- Various invariant reductions of soliton hierarchies, like stationary flows or restricted flows, lead to finite-dimensional systems integrable in the Liouville sense.
- Theory of stationary flows of KdV hierarchy and their finite-gap solutions was developed in 1970s (Dubrovin, Novikov, Its, Matveev).
- Bogoyavlenskii and Novikov (1976) proved that these flows can be represented by finite-dimensional Hamiltonian systems.
- Later a bi-Hamiltonian formulation for the stationary KdV flows was presented by means of the degenerate Poisson tensors and the Liouville integrability of stationary KdV flows was established It was observed that there are two Hamiltonian finite-dimensional representations of the stationary flows connected by Miura maps (Antonowicz, Fordy and Rauch-Wojciechowski (1987))
- The idea in a sense reverse, that starting from a particular family of Stäckel systems one can reconstruct the related soliton hierarchies was for the first time explored - for the case of cKdV hierarchy - by Błaszak and Marciniak (2006) and in Blaszak and Marciniak (2008) and then - for cHD - in Marciniak and Blaszak (2010)

Definition of the n-th stationary N-field cKdV system

Consider the N-field cKdV hierarchy $\boldsymbol{u}_{t_r} = \boldsymbol{K}_r[\boldsymbol{u}]$, $r = 1, 2, \ldots$, $\boldsymbol{u} = (u_0, u_1, \ldots, u_{N-1})^T$. The stationary condition $\boldsymbol{K}_{n+1} = 0$ provides N independent constraints on the infinite-dimensional (functional) manifold $\mathcal{F}$ on which the cKdV hierarchy is defined, reducing it to the finite-dimensional submanifold, n-th stationary manifold:

$$\mathcal{M}_n = \{[\boldsymbol{u}] \in \mathcal{F} \mid \boldsymbol{K}_{n+1} = 0\}, \qquad \dim \mathcal{M}_n = 2n + N.$$

Definition

The n-th stationary cKdV system consists of the first n evolution equations from the cKdV hierarchy together with its $(n+1)$-th stationary flow:

$$\boldsymbol{u}_{t_1} = \boldsymbol{K}_1, \qquad \boldsymbol{u}_{t_2} = \boldsymbol{K}_2, \qquad \ldots, \qquad \boldsymbol{u}_{t_n} = \boldsymbol{K}_n, \qquad \boldsymbol{K}_{n+1} = 0.$$

The cKdV case is presented in the preprint: `arXiv:2305.02282`.

The KdV case is presented in the preprint: `arXiv:2204.10632`.

Results

Main results

- We show that every N-field stationary cKdV system can be written, after a careful reparametrization of jet variables, as a classical separable Stäckel system on $N+1$ different ways.
- For each of these $N+1$ parametrizations we present an explicit map between the jet variables and the separation variables of the system.
- We show that each pair of Stäckel representations of the same stationary cKdV system, when considered in the phase space extended by Casimir variables, is connected by an appropriate Miura map.
- It leads immediately to $(N+1)$-Hamiltonian formulation for the stationary cKdV system.

Side results

- We present a novel formula for co-symmetries of the cKdV hierarchy.
- We present novel formulas for (integrated) kernels of all $N+1$ Hamiltonian operators $\mathbb{B}_i$ of the cKdV hierarchy.

We do not study the higher vector fields $K_{n+2}, K_{n+3}, \ldots$ on $\mathcal{M}_n$.

Coupled KdV hierarchy (Alonso, Antonowicz, Fordy)

Generated by the energy-dependent Schrödinger spectral problem:

$$\psi_{xx} + \mathbb{Q}\psi = 0, \quad \psi_{t_k} = \frac{1}{2}\mathbb{P}_k\psi_x - \frac{1}{4}(\mathbb{P}_k)_x\psi, \qquad k = 1, 2, \ldots,$$

$$\mathbb{Q} = -\lambda^N + \sum_{i=0}^{N-1} u_i\lambda^i, \quad u_i = u_i(x, t_1, t_2, \ldots), \quad u_N \equiv -1.$$

The compatibility conditions $(\psi_{xx})_{t_n} = (\psi_{t_n})_{xx}$ give the (infinite) cKdV hierarchy

$$\mathbb{Q}_{t_k} = (\mathbb{P}_k)_x\mathbb{Q} + \frac{1}{2}\mathbb{P}_k\mathbb{Q}_x + \frac{1}{4}(\mathbb{P}_k)_{3x} \equiv J\mathbb{P}_k, \qquad k = 1, 2, \ldots \tag{1}$$

Here $\mathbb{P}_k = \left[\lambda^{k-1}\mathcal{P}\right]_{\geqslant 0}$, where $\mathcal{P} = \sum_{i=0}^{\infty} P_i\lambda^{-i}$ is s.t.

$$J\mathcal{P} \equiv \mathcal{P}_x\mathbb{Q} + \frac{1}{2}\mathcal{P}\mathbb{Q}_x + \frac{1}{4}\mathcal{P}_{3x} = 0.$$

$$J \equiv \sum_{i=0}^{N} J_i\lambda^i, \qquad J_i := \frac{1}{4}\delta_{i0}\partial_x^3 + \frac{1}{2}(u_i\partial_x + \partial_x u_i)$$

- $N = 1$, Korteweg–de Vries (KdV) $\quad \mathbb{Q} = -\lambda + u_0$
- $N = 2$, Dispersive Water Waves (DWW) $\quad \mathbb{Q} = -\lambda^2 + u_1\lambda + u_0$

Coupled KdV hierarchy

Comparing in (1) the coefficients at various powers of λ we obtain the standard form of the coupled KdV hierarchy as an infinite suit of N-component evolution equations:

$$\boldsymbol{u}_{t_k} = \boldsymbol{K}_k[\boldsymbol{u}], \qquad k = 1, 2, \ldots,$$

defined on an infinite-dimensional functional (smooth) manifold $\mathcal{F}$. Coordinates on $\mathcal{F}$ are given by jet variables $[\boldsymbol{u}] := (\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots)$, with the (field) vector $\boldsymbol{u} := (u_0, \ldots, u_{N-1})^T$.

Some remarks:

- The construction works since $P_i = P_i[\boldsymbol{u}]$ in (1) actually do not depend on k.
- All the vector fields $\boldsymbol{K}_k[\boldsymbol{u}]$ mutually commute
- All the vector fields $\boldsymbol{K}_k[\boldsymbol{u}]$ are thus tangent to the stationary manifold $\mathcal{M}_n$.

Definition of $\mathcal{J}$

We need the following (symmetric) bi-linear operators

$$\mathcal{J}(f,g) := \sum_{i=0}^{N} \mathcal{J}_i(f,g)\lambda^i, \qquad \mathcal{J}_i(f,g) := -\frac{1}{16}\delta_{i0}\left(f_{xx}g + fg_{xx} - f_x g_x\right) - \frac{1}{4}u_i fg,$$

Thus

$$\mathcal{J}(f,g) \equiv -\frac{1}{16}(f_{xx}g + fg_{xx} - f_x g_x) - \frac{1}{4}\mathbb{Q}fg$$

It can be proved that

$$\mathcal{J}_i(f,g)_x = -\frac{1}{4}\left(fJ_i g + gJ_i f\right), \qquad \mathcal{J}(f,f)_x = -\frac{1}{2}fJf.$$

Algebraic recursion

The hierarchy generating polynomial $\mathcal{P} = \sum_{i=0}^{\infty} P_i \lambda^{-i}$ satisfy (integrated) equation

$$\frac{1}{2}\mathcal{P}(\mathcal{P})_{xx} - \frac{1}{4}(\mathcal{P})_x^2 + \mathbb{Q}\mathcal{P}^2 = -4\lambda^N, \qquad P_0 = 2.$$

Proposition

The coefficients P_i satisfy the recursive formula

$$P_k = P_k[\boldsymbol{u}] = -\sum_{j=1}^{k-1} \mathcal{J}_N(P_j, P_{k-j}) - \sum_{i=0}^{N-1} \sum_{j=0}^{i+k-N} \mathcal{J}_i(P_j, P_{i-j+k-N}), \qquad k = 1, 2, \ldots.$$

This (new, although known for $N = 1$ (S. I. Al'ber, 1979)) formula is purely differential-algebraic.

Multi-hamiltonian structure

All the N-component evolution equations from the cKdV hierarchy are Hamiltonian with respect to $N+1$ Hamiltonian operators $\mathbb{B}_i$:

$$\boldsymbol{u}_{t_k} = \boldsymbol{K}_k = \mathbb{B}_0 \boldsymbol{\gamma}_k = \ldots = \mathbb{B}_m \boldsymbol{\gamma}_{m-N} = \ldots = \mathbb{B}_N \boldsymbol{\gamma}_{k-N}, \quad k = 1, 2, \ldots$$

where $\boldsymbol{u} = (u_0, \ldots, u_{N-1})^T$ and $\boldsymbol{\gamma}_k = (P_k, \ldots, P_{N+k-1})^T$.

The operators $\mathbb{B}_m$ have the explicit form:

$$\mathbb{B}_m = \left(\begin{array}{ccc|ccc} & & J_0 & & & \\ & \cdot^{\cdot^{\cdot}} & \vdots & & 0 & \\ J_0 & \cdots & J_{m-1} & & & \\ \hline & & & -J_{m+1} & \cdots & -J_N \\ & 0 & & \vdots & \cdot^{\cdot^{\cdot}} & \\ & & & -J_N & & \end{array}\right)$$

Multi-hamiltonian structure

Any two consecutive Hamiltonian operators $\mathbb{B}_m$ define (the same) hereditary recursion operator $\mathbb{R}$ through

$$\mathbb{R} := \mathbb{B}_{m+1}\mathbb{B}_m^{-1}, \qquad m = 0, 1, \dots, N-1,$$

given explicitly by

$$\mathbb{R} = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & -J_0 J_N^{-1} \\ \hline 1 & & & -J_1 J_N^{-1} \\ & \ddots & & \vdots \\ & & 1 & -J_{N-1} J_N^{-1} \end{array}\right),$$

so that

$$\boldsymbol{K}_{r+1} = \mathbb{R}^r \boldsymbol{K}_1 \qquad \text{and} \qquad \boldsymbol{\gamma}_{r+1-N} = (\mathbb{R}^\dagger)^r \boldsymbol{\gamma}_{1-N}, \qquad r = 1, 2, \dots,$$

where $\boldsymbol{K}_1 = \boldsymbol{u}_x$ and $\boldsymbol{\gamma}_{1-N} = (0, \dots, P_0)^T$. However the above method requires integrating the nonlocal operator while our formula gives us an explicit (although recursive) form of all P_k that are obtained by purely differential operations.

Example: DWW hierarchy

Assume $N = 2$ and denote $\boldsymbol{u} := (u_0, u_1)^T = (u, v)^T$. We get DWW hierarchy, with the first flows given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \boldsymbol{K}_1 \equiv \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \boldsymbol{K}_2 \equiv \begin{pmatrix} \frac{1}{2} v u_x + u v_x + \frac{1}{4} v_{3x} \\ u_x + \frac{3}{2} v v_x \end{pmatrix},$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_3} = \boldsymbol{K}_3 \equiv \begin{pmatrix} \frac{3}{8} v^2 u_x + \frac{3}{2} u v v_x + \frac{3}{2} u u_x + \frac{1}{4} u_{3x} + \frac{3}{8} v v_{3x} + \frac{9}{8} v_x v_{2x} \\ \frac{3}{2} v u_x + \frac{3}{2} u v_x + \frac{15}{8} v^2 v_x + \frac{1}{4} v_{3x} \end{pmatrix},$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_4} = \boldsymbol{K}_4 \equiv \begin{pmatrix} \star \\ \frac{15}{8} v^2 u_x + \frac{15}{4} u v v_x + \frac{3}{2} u u_x + \frac{1}{4} u_{3x} + \frac{35}{16} v^3 v_x + \frac{5}{8} v v_{3x} + \frac{5}{4} v_x v_{2x} \end{pmatrix},$$

where

$$\star = \frac{3}{2} u^2 v_x + \frac{5}{16} v^3 u_x + \frac{15}{8} u v^2 v_x + \frac{9}{4} u v u_x + \frac{3}{8} v u_{3x} + \frac{9}{8} u_{2x} v_x + \frac{5}{4} u_x v_{2x} + \frac{5}{8} u v_{3x} + \frac{15}{32} v^2 v_{3x} + \frac{45}{16} v v_x v_{2x} + \frac{15}{16} v_x^3 + \frac{1}{16} v_{5x}$$

This hierarchy is three-hamiltonian with

$$\mathbb{B}_0 = \begin{pmatrix} -\frac{1}{2} v \partial_x - \frac{1}{2} \partial_x v & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad \mathbb{B}_1 = \begin{pmatrix} \frac{1}{4} \partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x u & 0 \\ 0 & \partial_x \end{pmatrix},$$

$$\mathbb{B}_2 = \begin{pmatrix} 0 & \frac{1}{4} \partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x u \\ \frac{1}{4} \partial_x^3 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x u & \frac{1}{2} v \partial_x + \frac{1}{2} \partial_x v \end{pmatrix}.$$

Example: DWW hierarchy

The cosymmetries $\gamma_k = (P_k, P_{k+1})^T$ are given by

$$P_0 = 2, \qquad P_1 = v, \qquad P_2 = u + \frac{3}{4}v^2, \qquad P_3 = \frac{3}{2}uv + \frac{5}{8}v^3 + \frac{1}{4}v_{2x},$$

$$P_4 = \frac{3}{4}u^2 + \frac{15}{8}uv^2 + \frac{1}{4}u_{2x} + \frac{35}{64}v^4 + \frac{5}{8}vv_{2x} + \frac{5}{16}v_x^2,$$
$$P_5 = \frac{15}{8}u^2v + \frac{35}{16}uv^3 + \frac{5}{8}vu_{2x} + \frac{5}{8}u_xv_x + \frac{5}{8}uv_{2x} + \frac{63}{128}v^5 + \frac{35}{32}v^2v_{2x} + \frac{35}{32}vv_x^2 + \frac{1}{16}v_{4x},$$
$$\vdots$$

while the recursion operator attains the form

$$\mathbb{R} = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1} \\ 1 & v + \frac{1}{2}v_x\partial_x^{-1} \end{pmatrix}.$$

Characterisation of kernels of $\mathbb{B}_m$ (generic case)

We need to define some auxiliary functions:

$$f_{k,m}(\boldsymbol{\xi}) := \sum_{i=0}^{k-1} \sum_{j=i+1}^{k} \mathcal{J}_i(\xi_{m+j-k}, \xi_{m+i-j+1}),$$

$$g_{k,m}(\boldsymbol{\xi}) := -2 \sum_{i=k}^{N} \sum_{j=k}^{i} \mathcal{J}_i(P_{j-k}, \xi_{m+i-j+1}),$$

where $k \in \{1, \dots, N\}$, $m \in \{0, \dots, N\}$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)^T$ is an arbitrary covector with $\xi_i = \xi_i[\boldsymbol{u}]$.

Proposition

For fixed $m \in \{0, 1, \dots, N\}$, $\boldsymbol{\xi} \in \ker \mathbb{B}_m$, (that is $\mathbb{B}_m \boldsymbol{\xi} = 0$), if and only if

$$f_{k,m}(\boldsymbol{\xi}) = c_k \qquad \text{for} \qquad 1 \leqslant k \leqslant m,$$
$$g_{k,m}(\boldsymbol{\xi}) = c_k \qquad \text{for} \qquad m+1 \leqslant k \leqslant N,$$

where $c_1, \dots, c_N$ are arbitrary constants.

Hamiltonian foliation of the stationary manifold $\mathcal{M}_n$

For the m-th Hamiltonian representation $\mathbb{B}_m \boldsymbol{\gamma}_{n+1-m} = 0$ of the $(n+1)$-th stationary cKdV flow $\boldsymbol{K}_{n+1} = 0$ on the stationary manifold $\mathcal{M}_n$ the covector $\boldsymbol{\gamma}_{n+1-m}$ belongs to the kernel of the respective Hamiltonian operator $\mathbb{B}_m$.

Hence, by setting $\boldsymbol{\xi} = \boldsymbol{\gamma}_{n+1-m}$, where $\xi_i = P_{n-m+i}$, we find the integrated form of the $(n+1)$-th stationary cKdV flow:

$$\begin{aligned} f_k &:= f_{k,m}(\boldsymbol{\gamma}_{n+1-m}) = c_k \qquad \text{for} \qquad 1 \leqslant k \leqslant m, \\ g_k &:= g_{k,m}(\boldsymbol{\gamma}_{n+1-m}) = c_k \qquad \text{for} \qquad m+1 \leqslant k \leqslant N, \end{aligned} \tag{$\star$}$$

where $c_1, \ldots, c_N$ are (arbitrary) integration constants.
The leaves of this foliation are given by

$$\mathcal{M}^{\boldsymbol{c}}_{n,m} := \{[\boldsymbol{u}] \in \mathcal{M}_n \,|\, \text{s.t. } (\star)\},$$

so that for each m:

$$\mathcal{M}_n \equiv \bigcup_{\boldsymbol{c} \in \mathbb{R}^N} \mathcal{M}^{\boldsymbol{c}}_{n,m}.$$

We will refer to this foliation (one for each m) as *Hamiltonian foliation* of $\mathcal{M}_n$. Two such foliations for two different m are transversal to each other.

Stäckel foliation

The $(n+1)$-th stationary flow $K_{n+1}=0$ of the cKdV hierarchy can be written as

$$\mathbb{Q}_{t_{n+1}} \equiv (\mathbb{P}_{n+1})_x\mathbb{Q} + \frac{1}{2}\mathbb{P}_{n+1}\mathbb{Q}_x + \frac{1}{4}(\mathbb{P}_{n+1})_{3x} \equiv J\mathbb{P}_{n+1} = 0.$$

The stationary condition can be integrated:

$$\mathcal{J}(\mathbb{P}_{n+1},\mathbb{P}_{n+1}) \equiv -\frac{1}{8}\mathbb{P}_{n+1}(\mathbb{P}_{n+1})_{xx} + \frac{1}{16}(\mathbb{P}_{n+1})_x^2 - \frac{1}{4}\mathbb{Q}\mathbb{P}_{n+1}^2 = C(\lambda),$$

where $C(\lambda)$ is an appropriate polynomial in λ with coefficients being integration constants that follow from the next proposition.

Proposition

We find that

$$\mathcal{J}(\mathbb{P}_{n+1},\mathbb{P}_{n+1}) \equiv \lambda^{2n+N} + \sum_{k=0}^{n+N-1} h_k\lambda^k, \qquad (2)$$

where

$$h_k = \sum_{i=0}^{N}\sum_{j=i}^{k}\mathcal{J}_i(P_{n-k+j},P_{n+i-j}).$$

For $n+N \leq k < 2n+N$ the coefficients h_k vanish and $h_{2n+N}=1$.

Stäckel foliation

Theorem

Let us fix $m \in \{0, \ldots, N\}$. Then, the set of solutions of the system of equations

$$\begin{aligned} h_k &= c_{k+1} \qquad k = 0, \ldots, m-1 \\ h_k &= c_{k-n+1}, \qquad k = n+m, \ldots n+N-1. \end{aligned} \tag{$\star$}$$

where all c_k vary over $\mathbb{R}$, coincide with the stationary manifold $\mathcal{M}_n$.

When we fix the values of all c_i then the equations above define a a particular leaf of a $2n$-dimensional foliation of the stationary manifold $\mathcal{M}_n$. This foliation is parameterized by the vector $\boldsymbol{c} \equiv (c_1, \ldots, c_N)$. Therefore, for each $m \in \{0, 1, \ldots, N\}$, we define

$$\bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m} := \{[\boldsymbol{u}] \in \mathcal{M}_n \,|\, \text{s.t. } (\star)\},$$

and then $\mathcal{M}_n$ is foliated into $\bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m}$:

$$\mathcal{M}_n \equiv \bigcup_{\boldsymbol{c} \in \mathbb{R}^N} \bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m}.$$

We call this foliation the m-th Stäckel foliation of $\mathcal{M}_n$. Two Stäckel foliations with different m are mutually transversal.

Proposition

The Hamiltonian foliation $\mathcal{M}^{\boldsymbol{c}}_{n,m}$ and the Stäckel foliation $\bar{\mathcal{M}}^{\boldsymbol{c}}_{n,m}$ coincide.

Geometry of the first n cKdV flows on the stationary manifold $\mathcal{M}_n$

- the $2n+N$-dimensional stationary manifold $\mathcal{M}_n = \{[\boldsymbol{u}] \in \mathcal{F} \mid \boldsymbol{K}_{n+1} = 0\}$ is - for each $m \in \{0, 1, \ldots, N\}$ - foliated into $\mathcal{M}^{\boldsymbol{c}}_{n,m}$:

$$\mathcal{M}_n \equiv \bigcup_{\boldsymbol{c} \in \mathbb{R}^N} \mathcal{M}^{\boldsymbol{c}}_{n,m}.$$

- Two foliations with two different m are mutually transversal.
- Flows of

$$\boldsymbol{u}_{t_1} = \boldsymbol{K}_1, \qquad \boldsymbol{u}_{t_2} = \boldsymbol{K}_2, \qquad \ldots, \qquad \boldsymbol{u}_{t_n} = \boldsymbol{K}_n$$

  are not only tangent (which is obvious) to $\mathcal{M}_n$ but they are also tangent to every leaf of any of the the above foliations.
- For each $m \in \{0, 1, \ldots, N\}$ these flows can be parametrized as (flat) Stäckel systems on $\mathcal{M}^{\boldsymbol{c}}_{n,m}$.
- Ther exist a (non-canonical) Miura map connecting any two Stäckel systems with two different m - after extending both Stäckel systems to the whole stationary manifold $\mathcal{M}_n$.

Let me now focus on two last statements.

Stäckel systems of Benenti type

They are Hamiltonian systems generated by separation curves of the form

$$\sigma(\lambda)+\sum_{k=1}^{n} H_k\lambda^{n-k}=\lambda^m\mu^2, \qquad m\in\mathbb{Z},$$

where $\sigma(\lambda)$ is a Laurent polynomial. Taking n copies we derive the separation relations:

$$\sigma(\lambda_i)+\sum_{k=1}^{n} H_k\lambda_i^{n-k}=\lambda_i^m\mu_i^2, \qquad i=1,\ldots,n,$$

Solving the above w.r.t. H_k we obtain n quadratic in momenta (Stäckel) Hamiltonians:

$$H_k=\frac{1}{2}\boldsymbol{\mu}^T A_k G_m\boldsymbol{\mu}+V_k, \qquad k=1,\ldots,n,$$

on the phase space T^*Q spanned by the coordinates $(\boldsymbol{\lambda},\boldsymbol{\mu})$ where $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)^T$ are local coordinates on the configuration space Q while $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_n)^T$ are the momentum coordinates. Then G_m are treated as contravariant metrics on the configuration space Q. Explicitly

$$G_m=2\,\mathrm{diag}\left(\frac{\lambda_1^m}{\Delta_1},\ldots,\frac{\lambda_n^m}{\Delta_n}\right), \qquad \Delta_i=\prod_{j\neq i}(\lambda_i-\lambda_j).$$

Stäckel systems of Benenti type

Further, all A_k are $(1,1)$-Killing tensors for all the metrics G_m and they are given by

$$A_k = (-1)^{k+1}\operatorname{diag}\left(\frac{\partial s_k}{\partial \lambda_1}, \ldots, \frac{\partial s_k}{\partial \lambda_n}\right), \qquad k = 1, \ldots, n,$$

where s_k denotes the elementary symmetric polynomials in variables λ_i of degree k.

$(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are separation coordinates for all n Stäckel Hamiltonians H_k.

Stäckel systems in Vieté coordinates

We make an extensive use of the so-called Vieté (canonical) coordinates:

$$q_i = (-1)^i s_i, \qquad p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i} \mu_k}{\Delta_k}, \qquad i = 1, \dots, n.$$

where s_k are the elementary symmetric polynomials in λ_i and $\Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j)$.

Let $\boldsymbol{p} = (p_1, \dots, p_n)^T$ and $\boldsymbol{q} = (q_1, \dots, q_n)^T$.

The Stäckel Hamiltonians take in Vieté coordinates the following form:

$$H_k = \frac{1}{2} \boldsymbol{p}^T A_k G_m \boldsymbol{p} + V_k, \qquad k = 1, \dots, n,$$

where G_m represents the contravariant metric and the respective Hamiltonian evolution equations are

$$\boldsymbol{q}_{t_k} = \{\boldsymbol{q}, H_k\}, \qquad \boldsymbol{p}_{t_k} = \{\boldsymbol{p}, H_k\},$$

where $\{\cdot, \cdot\} = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$.

Main theorem

Theorem

For a given $m \in \{0, 1, \ldots, N\}$, the transformation between the jet variables $[\mathbf{u}]$ on the stationary manifold $\mathcal{M}_n$ and the Viète coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{c})$ given by

$$q_i = \frac{1}{2} P_i, \qquad p_i = \frac{1}{2} \sum_{j=1}^{n} \left(G_m^{-1}\right)_{ij} (P_j)_x, \qquad i = 1, \ldots, n,$$

and

$$c_i = h_{i-1}, \qquad i = 1, \ldots, m, \qquad c_i = h_{n+i-1}, \qquad i = m+1, ..., N,$$

maps (after the identification $t_1 \equiv x$) the r-th flow $\boldsymbol{u}_{t_r} = \boldsymbol{K}_r$ of the stationary cKdV system onto the Stäckel system defined by the following separation curve

$$\lambda^{2n+N-m} + \sum_{k=1}^{N-m} c_{m+k} \lambda^{n+k-1} + \sum_{k=1}^{n} H_k \lambda^{n-k} + \sum_{k=1}^{m} c_k \lambda^{k-m-1} = \lambda^m \mu^2 \tag{3}$$

Note that the covariant metric G_m^{-1} in the second part of the above map must be first calculated in jet variables using the first part of the map.

Some remarks

- The Stäckel Hamiltonians H_k calculated from (3) are now defined on the extended phase space with coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{c})$ and are given as before, i.e. $H_k = \frac{1}{2}\boldsymbol{\mu}^T A_k G_m \boldsymbol{\mu} + V_k$ (or in Vieté coordinates by $H_k = \frac{1}{2}\boldsymbol{p}^T A_k G_m \boldsymbol{p} + V_k$) and with the potentials V_k given by

$$V_k(\boldsymbol{q}, \boldsymbol{c}) = \mathcal{V}_k^{(2n+N-m)} + \sum_{i=1}^{N-m} c_{m+i}\mathcal{V}_k^{(n+i-1)} + \sum_{i=1}^{m} c_k \mathcal{V}_k^{(i-m-1)}.$$

- The so called elementary separable potentials $\mathcal{V}_k^{(\alpha)}$ can be explicitely constructed from the formula (Blaszak 2011)

$$\mathcal{V}^{(i)} = R^i \mathcal{V}^{(0)}, \qquad \mathcal{V}^{(i)} = \left(\mathcal{V}_1^{(i)}, \ldots, \mathcal{V}_n^{(i)}\right)^T, \qquad \mathcal{V}^{(0)} = (0, \ldots, 0, -1)^T,$$

where

$$R = \begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{q_n} \\ 1 & 0 & 0 & \vdots \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -\frac{q_{n-1}}{q_n} \end{pmatrix}.$$

- The proof of this theorem goes by comparing the Lax formulation of the stationary cKdV system and the Lax formulation of the corresponding Stäckel system, as found in Błaszak, Domański (2019). See the article for details.

The second stationary DWW system, $n = N = 2$

The stationary system consists of two evolution equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \boldsymbol{K}_1 \equiv \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \boldsymbol{K}_2 \equiv \begin{pmatrix} \frac{1}{2} v u_x + u v_x + \frac{1}{4} v_{3x} \\ u_x + \frac{3}{2} v v_x \end{pmatrix}$$

and of the stationary flow $\boldsymbol{K}_3 = 0$,

$$\begin{aligned} u_{3x} &= -6 u u_x + \frac{15}{2} v^2 u_x + 3 u v v_x + \frac{45}{4} v^3 v_x - \frac{9}{2} v_x v_{2x} \\ v_{3x} &= -6 v u_x - 6 u v_x - \frac{15}{2} v^2 v_x, \end{aligned}$$

which defines the stationary manifold $\mathcal{M}_2$ parameterized by the jet variables $[\boldsymbol{u}] = (u, u_x, u_{2x}, v, v_x, v_{2x})$.
The vector fields on $\mathcal{M}_2$ attains the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} -v u_x - \frac{1}{2} u v_x - \frac{15}{8} v^2 v_x \\ u_x + \frac{3}{2} v v_x \end{pmatrix}.$$

Three representations of the stationary system of DWW ($N = 2$) for $n = 2$

The stationary system consists of two evolution equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} \frac{vu_x}{2} + uv_x + \frac{v_{3x}}{4} \\ u_x + \frac{3vv_x}{2} \end{pmatrix}$$

and the stationary flow

$$\begin{pmatrix} \frac{3v^2u_x}{8} + \frac{3}{2}uvv_x + \frac{3uu_x}{2} + \frac{u_{3x}}{4} + \frac{3}{8}vv_{3x} + \frac{9}{8}v_xv_{2x} \\ \frac{3vu_x}{2} + \frac{3uv_x}{2} + \frac{15v^2v_x}{8} + \frac{v_{3x}}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and according to theorem above it has three Stäckel representations given by the following three spectral curves:

- $m = 0$: $\quad \lambda^6 + c_1\lambda^3 + c_2\lambda^2 + H_1\lambda + H_2 = \mu^2$
- $m = 1$: $\quad \lambda^5 + c_1\lambda^2 + \tilde{c}_1\lambda^{-1} + H_1\lambda + H_2 = \lambda\mu^2$
- $m = 2$: $\quad \lambda^4 + \tilde{c}_2\lambda^{-1} + \tilde{c}_1\lambda^{-2} + H_1\lambda + H_2 = \lambda^2\mu^2$

The generic spectral curve

The generic (see (2)) spectral curve attains the form

$$\lambda^6 + h_3\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0 = \lambda^{2m}\mu^2$$

which yields the following h_k on $\mathcal{M}_2$

$$\begin{aligned}
h_3 &= -\frac{3}{2}uv - \frac{1}{4}v_{2x} - \frac{5}{8}v^3,\\
h_2 &= -\frac{9}{8}uv^2 - \frac{1}{4}u_{2x} - \frac{3}{4}u^2 - \frac{1}{2}vv_{2x} - \frac{5}{16}{v_x}^2 - \frac{15}{64}v,\\
h_1 &= -\frac{1}{8}vu_{2x} + \frac{1}{8}u_xv_x - \frac{1}{8}uv_{2x} - \frac{3}{4}uv^3 - \frac{3}{4}u^2v - \frac{9}{32}v^2v_{2x} - \frac{9}{64}v^5,\\
h_0 &= -\frac{3}{32}v^2u_{2x} + \frac{3}{16}vu_xv_x - \frac{3}{16}uvv_{2x} - \frac{3}{16}u{v_x}^2 - \frac{9}{64}uv^4 - \frac{3}{8}u^2v^2 + \frac{1}{16}{u_x}^2 - \frac{1}{8}uu_{2x}\\
&\quad - \frac{1}{4}u^3 - \frac{9}{64}v^3v_{2x}.
\end{aligned}$$

First representation $m = 0$

Putting $h_3 = c_2$ and $h_2 = c_1$ we obtain the foliation of $\mathcal{M}_2$ into leaves $\mathcal{M}^{\mathfrak{c}}_{2,0}$. Solving these relations,

$$u_{2x} = 8c_2 v - 4c_1 - 3u^2 + \frac{15}{2}uv^2 + \frac{65}{16}v^4 - \frac{5}{4}v_x^2, \qquad v_{2x} = -4c_2 - 6uv - \frac{5}{2}v^3,$$

we arrive at the curve for the leaves $\mathcal{M}^{\mathfrak{c}}_{2,0}$. It has the form

$$\lambda^6 + c_2\lambda^3 + c_1\lambda^2 + H_1\lambda + H_2 = \mu^2$$

while H_i are

$$\begin{aligned} H_1 &= \frac{3}{8}u^2 v + \frac{5}{16}uv^3 + \frac{1}{8}u_x v_x + \frac{7}{128}v^5 + \frac{5}{32}vv_x^2 + \frac{1}{2}c_2 u + \frac{1}{8}c_2 v^2 + \frac{1}{2}c_1 v, \\ H_2 &= +\frac{1}{8}u^3 + \frac{3}{32}u^2v^2 - \frac{5}{128}uv^4 + \frac{3}{16}vu_x v_x - \frac{1}{32}uv_x^2 + \frac{1}{16}u_x^2 - \frac{15}{512}v^6 \\ &\quad + \frac{15}{128}v^2v_x^2 - \frac{1}{4}c_2 uv + \frac{1}{2}c_1 u - \frac{3}{16}c_2 v^3 + \frac{3}{8}c_1 v^2. \end{aligned}$$

Transformation between jet and Viète coordinates

The (inverse) map between jet and Viète coordinates is given by

$$u = 2q_2 - 3q_1^2, \qquad u_x = 4p_1 - 8p_2 q_1, \qquad v = 2q_1, \qquad v_x = 4p_2$$

and

$$u_{2x} = 16c_2 q_1 - 4c_1 - 20p_2^2 - 52q_1^4 + 96q_2 q_1^2 - 12q_2^2, \qquad v_{2x} = -4c_2 + 16q_1^3 - 24q_2 q_1.$$

The (Stäckel) Hamiltonians in Viéte coordinates have the form

$$H_1 = p_2^2 q_1 + 2p_1 p_2 + q_1^5 - 4q_2 q_1^3 + 3q_2^2 q_1 - c_2 q_1^2 + c_1 q_1 + c_2 q_2,$$
$$H_2 = p_2^2 q_1^2 + 2p_1 p_2 q_1 - p_2^2 q_2 + p_1^2 + q_2 q_1^4 - 3q_2^2 q_1^2 + q_2^3 - c_2 q_2 q_1 + c_1 q_2.$$

Miura maps between different Stäckel representations of (the same) stationary cKdV system

Theorem

The following Miura map on the stationary manifold $\mathcal{M}_n$

$$\begin{aligned}
&\boldsymbol{q} = \bar{\boldsymbol{q}} \\
&\boldsymbol{p} = \left(R^T\right)^m \bar{\boldsymbol{p}}, \qquad \left[\left(R^T\right)^m\right]_{ij} = \mathcal{V}_j^{(n-i+m)}(\bar{\boldsymbol{q}}), \quad i,j = 1,...,n, \\
&c_i = \bar{H}_{m-i+1}(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{c}}), \qquad i = 1,\ldots,m, \\
&c_i = \bar{c}_i, \quad i = m+1,\ldots,N
\end{aligned}$$

transforms the Stäckel system defined by the curve (i.e with $m=0$)

$$\lambda^{2n+N} + \sum_{k=1}^{N} c_k \lambda^{n+k-1} + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2$$

to the Stäckel system generated by the curve (with $m \in \{1,\ldots,N\}$)

$$\bar{\lambda}^{2n+N-m} + \sum_{k=1}^{N-m} \bar{c}_{m+k} \bar{\lambda}^{n+k-1} + \sum_{k=1}^{n} \bar{H}_k \bar{\lambda}^{n-k} + \sum_{k=1}^{m} \bar{c}_k \bar{\lambda}^{k-m-1} = \bar{\lambda}^m \bar{\mu}^2,$$

Miura maps between different Stäckel representations of (the same) stationary cKdV system

The inverse of the above Miura map is given by

$$
\begin{aligned}
\bar{\boldsymbol{q}} &= \boldsymbol{q} \\
\bar{\boldsymbol{p}} &= \left(R^T\right)^{-m} \boldsymbol{p}, \qquad \left[\left(R^T\right)^m\right]_{ij} = \mathcal{V}_j^{(n-i+m)}(\boldsymbol{q}), \quad i,j = 1, ..., n, \\
\bar{c}_i &= H_{n-i+1}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{c}), \qquad i = 1, \ldots, m \\
\bar{c}_i &= c_i, \qquad i = m+1, \ldots, N.
\end{aligned}
$$

and it transforms the Stäckel system defined by the second curve back to the Stäckel system generated by the first curve.

It is easy to obtain a Miura map between two Stäckel representations with two arbitrary (dfferent) m.

Thus, all the Stäckel representation of (the same) stationary cKdV system, considered in the extended phase space $\mathcal{M}_n$, are equivalent, and the equivalence is given by a Miura map similar to the one given above.

(N+1)-Hamiltonian formulation for stationary cKdV system

Below I present the result only for $m = 0$ but we have the general formulas for arbitrary m. The formulas are given in separable coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c})$ associated with the system.

Corollary

The Stäckel system defined by the curve (i.e with $m = 0$)

$$\lambda^{2n+N} + \sum_{k=1}^{N} c_k \lambda^{n+k-1} + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2$$

is $(N+1)$-Hamiltonian, with the (degenerated) Hamiltonian operators π_r, $r = 0, \ldots, N$ given by

$$\pi_r = \sum_{i=1}^{n} \lambda_i^r \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i} + \sum_{j=1}^{r} X_j \wedge \frac{\partial}{\partial c_{r-j+1}}, \quad X_j = \pi_0 dH_j, \tag{4}$$

Note that π_0 is canonical and that the N functions $H_{n-r+1}, \ldots, H_n, c_{r+1}, \ldots, c_N$ are Casimir functions for π_r.

Example: three-Hamiltonian formulation for the system $n = N = 2$

Consider again the third DWW system, i.e. the Stäckel system generated by the above curve with $n = 2$, $N = 2$:

$$\lambda^4 + c_2\lambda^3 + c_1\lambda^2 + H_1\lambda + H_2 = \mu^2.$$

The operators π_r on $\mathcal{M} = R^6$ are

$$\pi_0(\lambda,\mu,c) = \begin{pmatrix} \begin{matrix} 0 & I \\ -I & 0 \end{matrix} & 0 & 0 \\ * & & \end{pmatrix}, \quad \pi_1(\lambda,\mu,c) = \begin{pmatrix} \begin{matrix} 0 & \Lambda \\ -\Lambda & 0 \end{matrix} & X_1 & 0 \\ * & & \end{pmatrix},$$

$$\pi_2(\lambda,\mu,c) = \begin{pmatrix} \begin{matrix} 0 & \Lambda^2 \\ -\Lambda^2 & 0 \end{matrix} & X_2 & X_1 \\ * & & \end{pmatrix},$$

where $I = \mathrm{diag}(1,1)$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$

Example: three-Hamiltonian formulation for the system $n = N = 2$

The corresponding the bi-Hamiltonian chains attain the form:

$$
\begin{array}{lll}
\pi_0 dc_1 = 0 & \pi_0 dc_1 = 0 & \pi_1 dc_2 = 0 \\
\pi_0 dH_1 = X_1 = \pi_1 dc_1 & \pi_0 dH_1 = X_1 = \pi_2 dc_2 & \pi_1 dc_1 = X_1 = \pi_2 dc_2 \\
\pi_0 dH_2 = X_2 = \pi_1 dH_1 & \pi_0 dH_2 = X_2 = \pi_2 dc_1 & \pi_1 dH_1 = X_2 = \pi_2 dc_1 \\
0 = \pi_1 dH_2 & 0 = \pi_2 dH_1 & 0 = \pi_2 dH_1
\end{array} .
$$

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