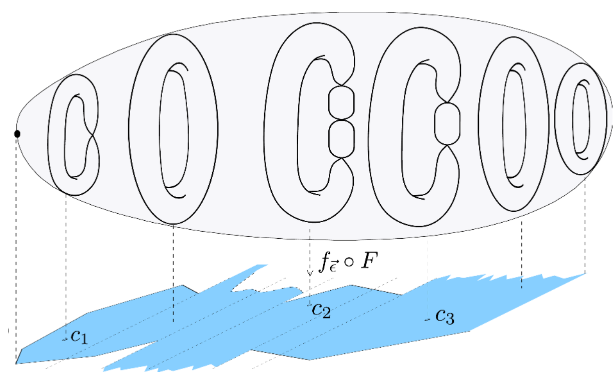
Conference

Integrable systems

CSF Ascona

June 19–24, 2016



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Integrable systems

Conference at the CSF Ascona, June 19-24, 2016

Organizers

Anton Alekseev (University of Geneva)
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Speakers

Anthony Bloch (Michigan)
Victor Buchstaber (Steklov, Moscow)
Pantelis Damianou (University of Cyprus)
Holger Dullin (University of Sydney)
Konstantinos Efstathiou (Groningen)
Francesco Fassò (Padova)
László Fehér (University of Szeged and Wigner RCP Budapest)
Anatoly Fomenko (Moscow State University)
Franois Gay-Balmaz (CNRS Paris)
Andrea Giacobbe (University of Catania)
Anton Izosimov (University of Toronto)
Igor Krichever (Columbia)
Franco Magri (Milano-Bicocca)
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Daisuke Tarama (Ritsumeikan University)
Sue Tolman (Urbana-Champaign)
Christophe Wacheux (Manchester)
Nguyen Tien Zung (Toulouse)

Invited Participants	**Posters presented by**
Jaume Alonso Fernandez	
Damien Bouloc	Damien Bouloc
Elena Bunkova	Elena Bunkova
Simone Chiarello	
Alina Dobrogowska	Alina Dobrogowska
Charalampos Evripidou	Charalampos Evripidou
Marta Farre	Marta Farre
Tatiana Fomenko	
Marine Fontaine	Marine Fontaine
Tamas F. Görbe	Tamas F. Görbe
Andrey Konyaev	Andrey Konyaev
Jeremy Lane	Jeremy Lane
Nikolay Martynchuk	Nikolay Martynchuk
Truong Hong Minh	
Jan-Cornelius Molnar	Jan-Cornelius Molnar
Florian Naef	
Cédric Oms	
Joseph Palmer	Joseph Palmer
Konrad Schöbel	Konrad Schöbel
Severin Schraven	
Gleb Smirnov	
Romero Solha	Romero Solha
Hendrik Süss	Hendrik Süss
Xiudi Tang	Xiudi Tang
Andreas Vollmer	Andreas Vollmer
Yannick Widmer	Yannick Widmer
Joachim Worthington	Joachim Worthington
Xiaomeng Xu	Xiaomeng Xu

Schedule

Breakfast is served between 7:00 – 10:00 am.

Monday	Tuesday	Wednesday	Thursday	Friday
9:00 – 9:15 **Welcome** 9:15 – 10:15 **Gay-Balmaz** 10:15 – 10:45 **Coffee break** 10:45 – 11:45 **Damianou** 11:50 – 12:20 **Efstathiou**	9:00 – 10:00 **Pelayo** 10:00 – 10:30 **Coffee break** 10:30 – 11:30 **Poster session** 11:45 – 12:15 **Tarama**	9:00 – 10:00 **Buchstaber** 10:00 – 10:30 **Coffee break** 10:30 – 11:30 **Panov** 11:45 – 12:15 **Izosimov**	9:00 – 10:00 **Dullin** 10:00 – 10:30 **Coffee break** 10:30 – 11:30 **Sabatini** 11:45 – 12:15 **Wacheux**	9:00 – 10:00 **Zung** 10:00 – 10:30 **Coffee break** 10:30 – 11:30 **Krichever** 11:45 – 12:15 **Hohloch**
Lunch	**Lunch**	**Lunch**	**Lunch**	**Lunch**
14:00 – 15:00 **Fasso** 15:00 – 15:30 **Coffee break** 15:30 – 16:30 **Feher** 16:45 – 17:45 **Magri**	14:00 – 15:00 **Fomenko** 15:00 – 15:30 **Coffee break** 15:30 – 16:30 **Tolman** 16:45 – 17:45 **Bloch**		14:00 – 15:00 **Miranda** 15:00 – 15:30 **Coffee break** 15:30 – 16:30 **Giacobbe** 16:45 – 17:45 **Mazzocco** 17:45 – 18:00 **– Award –**	
Dinner	**Dinner**	**Dinner**	**Dinner**	

Dinner starts between 19:00 – 19:30h.

Talks

(alphabetically ordered)

Symmetric Rigid Body Equations, Flows on Symmetric Matrices, and Optimal Control

Anthony M. Bloch*
Department of Mathematics
University of Michigan
Ann Arbor MI 48109

In this talk we consider certain integrable systems, their relationship to optimal control problems, and to the work of Moser on the geometry of quadrics. In particular we consider the rigid body problem and its symmetric representation and low rank formulation, a class of flows on symmetric matrices, and the geodesic flows on Stiefel manifolds. We also discuss the relationship with kinematic optimal control problems and the so-called Clebsch problem.

We discuss the general Clebsch problem and in particular how the optimal control problem for the rigid body equations gives rise to their symmetric representation. We also indicate how this leads to a natural discretization of these equations which is related to the Moser Veslov equations. We describe how this formulation is related to the Lax pair form. We also discuss the geometry of the flows on symmetric matrices and the integrability of this system.

Finally we relate these systems these systems to the Moser formulation of systems in Lax pair form where the matrix L is the Lax pair in a rank two perturbation of a fixed full rank matrix. We consider higher perturbations as well as the geodesic flow on the ellipsoid and the Neumann problem. The latter work is joint work with Gay-Balmaz and Ratiu.

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Polynomial Dynamical Systems and Hyperelliptic Functions of Genus 2

Victor M. Buchstaber
(Steklov Mathematical Institute of Russian Academy of Sciences)
buchstab@mi.ras.ru

Consider a non-singular hyperelliptic curve of genus g

$$V_\lambda = \left\{(x, y) \in \mathbb{C}^2 \, : y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \ldots + \lambda_{4g+2}\right\}.$$

According to Dubrovin–Novikov Theorem, the space of the universal bundle $\mathcal{U}_g$ of Jacobians $Jac\, V_\lambda$ is birationally equivalent to $\mathbb{C}^{3g}$. The talk develops of this result on the basis of the theory of sigma-function $\sigma(u;\lambda)$ (see [1, 2]), where $u = (u_1, \ldots, u_{2g-1})$, $\lambda = (\lambda_4, \ldots, \lambda_{4g+2})$. The focus will be on the polynomial dynamical systems on $\mathbb{C}^{3g}$, that are integrable in $\wp_\omega = \wp_\omega(u;\lambda)$ on $\mathcal{U}_g$, where $\omega = (j_1, \ldots, (2g-1)j_g)$, $j_1 \geqslant 0, \ldots, j_g \geqslant 0$, $j_1 + \cdots + j_g \geqslant 2$ and

$$\wp_\omega(u;\lambda) = -\frac{\partial^{j_1+\cdots+j_g}}{\partial_{u_1}^{j_1} \cdots \partial_{u_g}^{j_g}} \ln \sigma(u;\lambda).$$

The differential field $\mathcal{F}_g$ of genus g hyperelliptic functions is generated by the functions $\wp_{\omega_k}$, $k = 1, \ldots, g$, where $\omega_1 = (2, 0, \ldots, 0), \ldots, \omega_g = (1, 0, \ldots, 0, 2g-1)$. The function $2\wp_{\omega_1}$ is a g-zone solution of the KdV hierarchy and $\frac{\partial}{\partial u_1}\wp_{\omega_k} = \frac{\partial}{\partial u_{2k-1}}\wp_{\omega_1}$, $k = 2, \ldots, g$. Denote by $\mathcal{F}_g\wp \subset \mathcal{F}_g$ the subring generated by all functions $\wp_\omega$. The Lie algebra $\mathrm{Der}(\mathcal{F}_g\wp)$ of derivations of the ring $\mathcal{F}_g\wp$ is a $3g$-dimensional $\mathcal{F}_g\wp$-module with generators $L_{2q-1} = \frac{\partial}{\partial u_q}$, $q = 1, \ldots, g$ and $L_{2k-2}, k = 1, \ldots, 2g$ (see [3]). Consider in $\mathbb{C}^{3g}$ the coordinates $X_1 = (x_{2,0}, x_{3,0}, x_{4,0})$, $\ldots$, $X_g = (x_{1,2g-1}, x_{2,2g-1}, x_{3,2g-1})$.

Theorem 1. *Set $A_g = \mathbb{C}[X_1, \ldots, X_g]$. The ring homomorphism*

$$f \colon A_g \to \mathcal{F}_g\wp, \; f(X_k) = (\wp_{\omega_k}, \frac{\partial}{\partial u_1}\wp_{\omega_k}, \frac{\partial^2}{\partial u_1^2}\wp_{\omega_k})$$

is an isomorphism. It determines a Lie subalgebra $\mathcal{G}_g \subset \mathrm{Der}(A_g)$, which is a free $3g$-dimensional A_g-module with generators $\mathcal{L}_{2q-1} = \frac{\partial}{\partial u_q}$, $q = 1, \ldots, g$ and $\mathcal{L}_{2k-2}, k = 1, \ldots, 2g$, by $L_s f(P) = f(\mathcal{L}_s P), s = 1, \ldots, 3g$, for any $P \in A_g$.

In [4] the systems S_k in $\mathbb{C}^6$, determined by $\mathcal{L}_k$, $k = 1, \ldots, 6$, are described.

Theorem 2. *System S_1 has the form*

$$x_2' = x_3,\; x_3' = x_4,\; x_4' = 4(3x_2x_3 + z_5),\; z_4' = z_5,\; z_5' = z_6,\; z_6' = 4(2x_2z_5 + x_3z_4).$$

System S_3, given system S_1, is determined by the equations

$$\dot{x}_2 = z_5, \quad \dot{z}_4 = x_3z_4 - x_2z_5.$$

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[4] V. M. Buchstaber, *Polinomial dynamical systems and Korteweg-de Vries equation.*, arXiv:1605.04061 v1 [math.DS] 13 May 2016.

On integrable Toda and Volterra systems

Pantelis A. Damianou

University of Cyprus

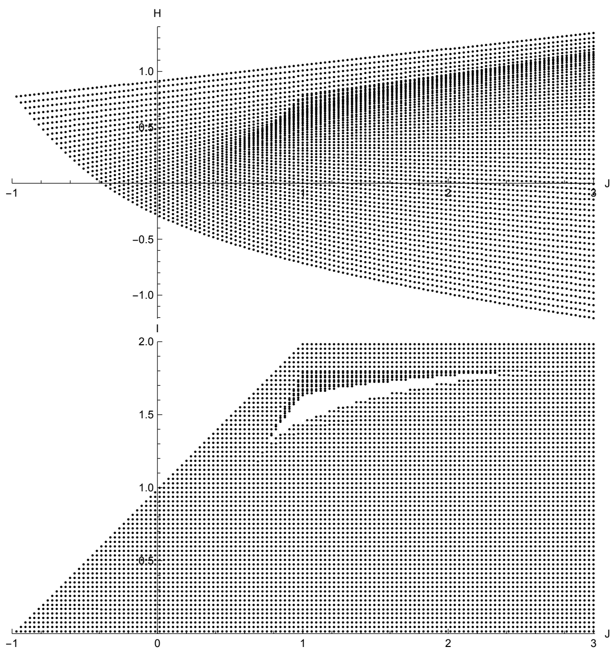
We review some recent (and some old) work on Toda and Volterra systems and we pose some open questions. After defining the classical Toda lattice we study its generalization to simple Lie algebras due to Bogoyavlenski. We review its well-known multi-hamiltonian structure and then we present some new results in the case of the exceptional Lie algebra of type G_2. We review the Kostant-Toda lattice and the full Kostant-Toda lattice and its symmetric counterparts. We then define a large family of Hamiltonian systems which interpolate between the classical Kostant-Toda lattice and the full Kostant-Toda lattice. There is one such system for every nilpotent ideal $\mathcal{I}$ in a Borel subalgebra $\mathfrak{b}_+$ of an arbitrary simple Lie algebra $\mathfrak{g}$. We define analogous systems in the case of Volterra lattice. We construct a large family of evidently integrable Hamiltonian systems which are generalizations of the KM system. The algorithm uses the root system of a complex simple Lie algebra. The Hamiltonian vector field is homogeneous cubic but in a number of cases a simple change of variables transforms such a system to a quadratic Lotka-Volterra system. We present in detail some examples in the case of A_n but the algorithm works in the case of an arbitrary simple Lie algebra. We finally present some recent work on Volterra lattices of full interaction.

SEMITORIC SYSTEMS WITH HYPERBOLIC SINGULARITIES

HOLGER R. DULLIN

School of Mathematics and Statistics, University of Sydney, Australia

Toric integrable systems with their momentum maps whose images are rational polytopes are well understood. More recently San Vu Ngoc and Alvaro Pelayo classified semitoric system in two degrees of freedom with additional singularities of focus-focus type. I will show that such a system can always be deformed such that the global S^1 action remains intact, but the focus-focus point is replaced by an elliptic-elliptic point and additional singularities, some of which are hyperbolic (joint work with Alvaro Pelayo [1]). This deformation is the Hamiltonian Hopf bifurcation, well known in dynamical systems. Then I will discuss a particular example which is a deformation of the spin-oscillator (joint work with Joachim Worthington [2]). The outer boundary of the classical polygonal invariant remains the same as in the original undeformed system, but inside the image of the momentum map, a hole appears, plus an additional partial polygonal invariant that fits in the hole. The quantum mechanics of the problem reveals the beautiful relation between the classical polygonal invariant and the quantum spectrum in the semiclassical limit, see the illustrations below. I will conclude with a discussion of polygonal invariants in other examples and discuss some open problems.



References

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[2] Holger R. Dullin and Joachim Worthington. The polygonal invariant of a deformed spin-oscillator with hyperbolic singularities. (in preparation, also see Joachim's poster)

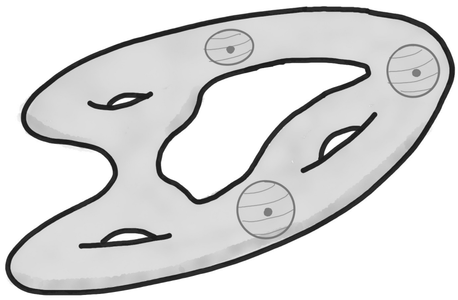
E-mail address: `holger.dullin@sydney.edu.au`.

GENERALIZED HAMILTONIAN MONODROMY AND CIRCLE ACTIONS

KONSTANTINOS EFSTATHIOU

Abstract

Standard Hamiltonian monodromy was introduced by Duistermaat as an obstruction to the existence of global action-angle coordinates in integrable Hamiltonian systems [1]. It refers to the monodromy of torus bundles that typically exist in such systems. Fractional Hamiltonian monodromy, introduced by Nekhoroshev, Sadovskií, and Zhilinskií in [5], generalizes standard monodromy by considering not only torus bundles but also more general fibrations with singular fibers.



In this talk we present results concerning generalized monodromy that were recently obtained in collaboration with Nikolay Martynchuk in [3] and [4]. It turns out that, in integrable Hamiltonian systems with a Hamiltonian circle action, generalized monodromy and, consequently, standard monodromy can be solely determined through a careful study of the fixed points of the circle action and their weights. A basic ingredient of this approach is the definition of generalized parallel transport of homology cycles introduced in [2]. These results will be demonstrated in several examples of integrable Hamiltonian systems.

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Address: Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands

E-mail address: K.Efstathiou@rug.nl

**Where has the monodromy gone
in the 3-dim Champagne Bottle?
(A case study in superintegrability)**

Francesco Fassò (Università di Padova)

Monodromy is an obstruction to the triviality of the fibration by the invariant tori of an integrable or superintegrable Hamiltonian system; it was discovered by N.N. Nekhoroshev and by J.J. Duistermaat in the 1970s.

The Champagne bottle (a point in a planar central force field shaped like the bottom of a wine bottle) is likely the simplest completely integrable system that exhibits monodromy (L. Bates, 1991). Like any planar central force field, the planar Champagne bottle is a subsystem of the system with the same central field, but in space. The spatial system is, actually, union of planar subsystems, and its invariant tori are those of the planar subsystems. The spatial system however cannot have monodromy, because the base of the fibration is simply connected.

In order to explain "where the monodromy went" in the 3-dim system (or, perhaps, "where it comes from" in the 2-dim subsystems) we describe the 3-dim system not as completely integrable, but as superintegrable, that is, in the realm of the theory developed by Nekhoroshev and Mischenko and Fomenko in the 1970s to describe systems which have more integrals of motion and more symmetry than what is needed for complete integrability. In superintegrable systems the invariant tori are isotropic, not Lagrangian, and there is also a coarser, natural foliation of the phase space that is given by a Casimir map and has coisotropic leaves. The geometry of the regular part of these foliations is well understood (Nekhoroshev 1972, Dazord and Delzant 1987). It is the consideration of the singularities of the coisotropic structure that allows to understand the origin of the monodromy in the planar subsystems.

This is a join work with Larry Bates (Calgary).

Compact forms of the Ruijsenaars–Schneider system

László Fehér

Department of Theoretical Physics, University of Szeged
and WIGNER RCP, RMKI, Budapest, Hungary
e-mail: lfeher@physx.u-szeged.hu

Abstract

In 1995 S. Ruijsenaars introduced an interesting real form of the trigonometric Ruijsenaars–Schneider (RS) system of n interacting particles on the circle for which the phase space of the relative motion is the projective space $\mathbb{CP}^{n-1}$ equipped with the Fubini–Study symplectic structure and the angular distance of the neighbouring particles is bounded from below by a parameter $0 < x < \pi/n$. In the first part of the talk, we explain how quasi-Hamiltonian reduction leads to a generalization of this model for any generic value of the parameter $0 < x < \pi$. It turns out that if x varies in a certain punctured interval around $c\pi/n$, where $c \in \{1, \ldots, n-1\}$ is coprime to n, then the phase space of the relative motion is again $\mathbb{CP}^{n-1}$ and the angular distance of the c-th neighbours is suitably constrained. These are called type (i) models, but type (ii) models also exist in which the particles can collide. In the second part of the talk, we give a direct construction of the type (i) models by first mapping the pertinent local RS system onto a dense open submanifold of $\mathbb{CP}^{n-1}$, and then showing that a suitable conjugate of the local Lax matrix extends to a smooth function on this compact phase space. Finally, we point out that the direct construction yields type (i) compactifications of the elliptic RS system, too.

References

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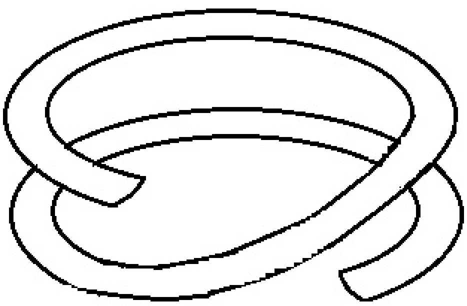
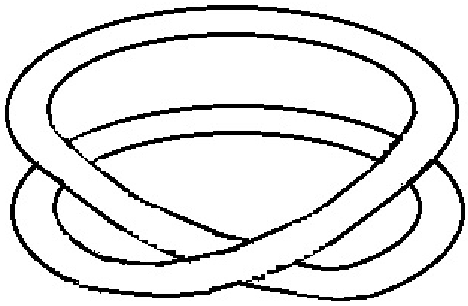
A.T.Fomenko
(Moscow State University)

Topology and Symmetries of the Billiards and Integrable Hamiltonian Systems in Mechanics

The results of this talk were recently obtained in collaboration with V.V.Fokicheva (Moscow State University). It turns out that two-dimensional locally flat topological billiard systems can be considered as the models for the integrable rigid body dynamics, with two degrees of freedom . Description of the Hamiltonian rigid body dynamics is a complex problem, which goes back to Euler and Lagrange. These systems are described in the six-dimensional phase space and have two integrals – the energy integral and the momentum integral. Of particular interest are the cases of rigid body dynamics, where there exists the additional integral, and where the Liouville integrability can be established. Because the solutions of many of such a systems are difficult to describe, the next step in their analysis is the calculation of topological invariants for integrable systems. More precisely, the so called "marked molecules" (one-dimensional graphs with "atom-vertices" and some numerical marks), which allow us to describe such a systems in the simple terms, and also allow us to set the Liouville equivalence between different integrable systems. Topological (generalized) billiard systems describe the motion of the material point on a locally plane domain, bounded by a system of smooth curves. The phase space is the four-dimensional manifold. Topological locally flat billiard systems can be integrable for a suitable choice of the boundary, for example, when the boundary consists of the arcs of the confocal ellipses, hyperbolas and parabolas. Some of these billiards cannot be embedded by a global isometry into Euclidean plane. Since such a billiard systems are Liouville integrable, they are classified by the marked molecules (which are also called as Fomenko-Zieschang invariants).In this talk, we simulate many cases of motion of a rigid body in 3-space by more simple topological integrable billiard systems. Namely, we set the Liouville equivalence between different systems by comparing the Fomenko-Zieschang invariants for the rigid body dynamics and for the topological billiard systems. For example, the Euler case can be simulated by the billiards for all values of energy integral. For many values of energy, such billiard simulation is done for the systems of the Lagrange top and Kovalevskaya top, then for the Zhukovskii gyrostat, for the systems by Goryachev-Chaplygin -Sretenskii, Clebsch, Sokolov, as well as expanding the classical Kovalevskaya top – Kovalevskaya- Yahia case.





On the symplectic geometry of the Flaschka transformation and the Toda lattice system

François Gay-Balmaz (joint work with A. Bloch and T. S. Ratiu)

The Toda lattice system and its many generalizations are fundamental integrable systems that have inspired an extraordinarily rich literature on its geometry and dynamics. A key advance in the study of this system was the introduction of the Flaschka map. Using the tools of cotangent bundle reduction in geometric mechanics, we show that the Flaschka map is the inverse of a momentum map. The diffeomorphic character of this momentum map is related to the existence of a Lie subalgebra which is a real polarization and, in addition, satisfies the so-called Pukanszky condition. We then show how this situation occurs for the generalized Toda lattice flows associated to semisimple Lie algebras, which generalize the Toda lattice flow on Jacobi matrices. We also apply our construction for certain coadjoint orbits of semidirect products of a Lie group with a vector space and isolate a class of coadjoint orbits that are symplectomorphic to magnetic cotangent bundles.

Quasi-periodicity of relative quasi-periodic tori

Andrea Giacobbe

Università degli Studi di Catania
Dipartimento di Matematica e Informatica

Abstract

Most well understood dynamical systems either admit invariant tori on which the dynamics is quasi-periodic or are "close" to having such a property [1]. In the Hamiltonian setting this property is direct consequence of what goes under the name of *complete integrability* (or Liouville-Arnol'd integrability), but quasi-periodicity is typically found also in many non-Hamiltonian systems (nonholonomic systems, for example).

One of the possible reasons for such a universal behavior could be the existence of a large group of symmetry together with a reasonably simple structure of the *relative dynamics* (i.e. the dynamics projected in the quotient under the group action). *Relative equilibria* and *relative periodic orbits* (i.e. orbits that project in the quotient onto an equilibrium or onto a periodic orbit) have been proven to be quasi-periodic [3, 4, 2]. The same statement becomes false when the relative dynamics is quasi-periodic with two frequencies or more [5].

We give conditions that yield quasi-periodicity of *relative quasi-periodic orbits* (meaning, orbits that project onto quasi-periodic orbits), analyze the associated geometry of the phase space, connect our analysis with the theory of reducibility for linear differential equations with quasi-periodic coefficients.

The work is in collaboration with Francesco Fassò and Luis Gracia-Naranjo.

References

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Compact semi-toric systems, Hamiltonian $\mathbb{S}^1$-actions and Integrable surgeries

Sonja Hohloch

(Antwerpen)

Roughly, a semi-toric system on a compact 4-dimensional manifold consists of two Poisson-commuting Hamiltonian flows one of which is periodic. Thus the flow parameters induce an $\mathbb{S}^1 \times \mathbb{R}$-action on the manifold. Under certain assumptions on the singularities, semi-toric systems have been classified by Pelayo & Vũ Ngọc by means of 5 invariants.

Every semi-toric system induces a Hamiltonian $\mathbb{S}^1$-action on the manifold by 'forgetting' the $\mathbb{R}$-valued flow parameter. Effective Hamiltonian $\mathbb{S}^1$-actions on compact 4-manifolds have been classified by Karshon by means of so-called 'labeled directed graphs'.

In the joint work [HSS] with S. Sabatini and D. Sepe, we linked Pelayo & Vũ Ngọc's classification of semi-toric systems to Karshon's classification of Hamiltonian $\mathbb{S}^1$-actions. More precisely, we showed that only 2 of the 5 invariants are necessary to deduce the Karshon graph of the underlying $\mathbb{S}^1$-action. In an ongoing work with S. Sabatini, D. Sepe and M. Symington, we study when/how one can 'lift' an effective Hamiltonian $\mathbb{S}^1$-action on a compact 4-manifold to a semi-toric system. We found precise conditions somewhat analogous to Karshon's lifting requirements. Our construction involves among others the generalization of nodal trades and blow-ups to the semi-toric world which we define by means of 'integrable surgeries'.

References

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Pentagrams, inscribed polygons, and Prym varieties

Anton Izosimov

The pentagram map is a discrete integrable system on the space of planar polygons introduced by R. Schwartz in 1992. The definition of the pentagram map is illustrated in Figure 1: the image of the polygon P under the pentagram map is the polygon P'.

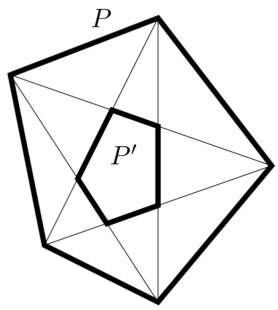

Figure 1: The pentagram map

Surprisingly, this simple map has deep connections with many different areas of mathematics such as integrable PDEs, cluster algebras etc.

The pentagram map is known to interact nicely with polygons inscribed in conic sections. In particular, it was observed in [1] and then rigorously proved in [2] that for inscribed polygons first integrals E_i, O_i of the pentagram map satisfy the relation $E_i = O_i$. The proof given in [2] is a rather hard calculation based on explicit formulas for the functions E_i, O_i.

In my talk, I will present a simple proof of the $E_i = O_i$ theorem, and discuss its various geometric consequences, in particular, the relation between inscribed polygons and Prym varieties. I will also pose some open questions.

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Igor Krichever

The parametric Riemann-Hilbert problem and its applications

In the talk a new approach to study a behavior of various objects (vector bundles, differentials ...) on smooth algebraic curves under degeneration to a singular stable curve will be presented.

Abel, Stäckel, and Kowalewski.

Franco Magri
Dipartimento di Matematica ed Applicazioni
Università di Milano Bicocca

Abstract. In the classical literature of Mechanics, one knows a few examples of integrable dynamical systems whose equations of motion can be written in the Abel's form

$$\frac{dx1}{\sqrt{R(x1)}} + \frac{dx2}{\sqrt{R(x2)}} = a1$$

$$\frac{x1\ dx1}{\sqrt{R(x1)}} + \frac{x2\ dx2}{\sqrt{R(x2)}} = a2$$

for a suitable choice of the coordinates x1 and x2, the function R being a polynomial function. The Kowalewski's top is one of these examples. In the talk I would like to discuss the conditions which allow to work out this kind of representation of the equations of motion. I will use the example of the Kowalewski's top to illustrate the use of these conditions.

COLLIDING HOLES IN RIEMANN SURFACES AND QUANTUM CLUSTER ALGEBRAS

MARTA MAZZOCCO[†]

Abstract. We investigate *bordered cusped Teichmüller spaces,* which are Teichmüller spaces of Riemann surfaces with at least one hole and at least one *bordered cusp* on its boundary. We show how bordered cusps arise when colliding holes in a Riemann surface. In the limit of two colliding holes (or colliding sides of the same hole), the geodesics that originally passed through the domain between colliding holes, which we call *chewing-gum,* become geodesic arcs between two bordered cusps decorated by horocycles. The lengths of these arcs are λ-lengths in Thurston–Penner terminology, or cluster variables by Fomin and Zelevinsky. We then consider classes of geodesic laminations comprising both closed curves in the interior of a Riemann surface and arcs passing between bordered cusps. We find the Poisson and quantum algebras of these laminations. We demonstrate that for any Riemann surface $\Sigma_{g,s,n}$ of genus g with $s \geq 1$ holes/orbifold points and $n \geq 1$ bordered cusps, we have geodesic laminations that are complete systems of arcs, or λ-lengths together with closed paths around the holes without bordered cusps or around orbifold points. From the physical point of view, our construction provides an explicit coordinatiaation of moduli spaces of open/closed string worldsheets and their quantization.

[†]Department of Mathematical Sciences, Loughborough University, LE11 3TU, United Kingdom. Email: m.mazzocco@lboro.ac.uk.

INTEGRABLE SYSTEMS ON SINGULAR SYMPLECTIC MANIFOLDS

EVA MIRANDA

Abstract. The starting point of this talk is an action-angle theorem for integrable systems on b-Poisson manifolds [KMS]. We will also mention some applications to KAM theory and the description of their cotangent models [KM].

Several Hamiltonian systems motivating this study come from celestial mechanics (e.g. the elliptic restricted 3-body problem) and their singularities ("collisions") can be described on a b^m-symplectic manifold or b^m-folded symplectic manifold [DKM]. Time permitting, we will discuss some open problems concerning these examples and new strategies to tackle them.

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Departament de Matemàtiques, Universitat Politècnica de Catalunya and Barcelona Graduate School of Mathematics BGSMath, EPSEB, Avinguda del Doctor Marañón 44–50, Barcelona

E-mail address: eva.miranda@upc.edu

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NEW CONSTRUCTIONS OF HAMILTONIAN-MINIMAL LAGRANGIAN SUBMANIFOLDS

TARAS PANOV

Hamiltonian minimality (*H-minimality*) for Lagrangian submanifolds is a symplectic analogue of minimality in Riemannian geometry. A Lagrangian immersion is called H-minimal if the variations of its volume along all Hamiltonian vector fields are zero.

A family of H-minimal Lagrangian submanifolds N in a complex space $\mathbb{C}^m$ can constructed from intersections of real or Hermitian quadrics. These intersection of quadrics are parametrised by convex simple polytopes, appear in the symplectic reduction construction of Hamiltonian toric manifolds, and are known in toric topology under the name *moment-angle manifolds*. The topology of moment-angle manifolds is complicated, but relatively well understood, and its knowledge can be used to describe new H-minimal Lagrangian submanifolds N with interesting topology. For example, starting from a polygon, one obtains as N a twisted product of a torus and a Riemannian surface of large genus.

The construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$ can be enhanced by applying it alongside with the symplectic reduction, which lead to new examples of H-minimal submanifolds in a projective space and other toric varieties.

Furthermore, manifolds N (both embedded and immersed) appear as degenerations of Liouville tori for some interesting Hamiltonian systems with polynomial integrals.

The talk is based on joint works with Andrey E. Mironov.

Department of Mathematics and Mechanics, Moscow State University
E-mail address: tpanov@mech.math.msu.su
URL: http://higeom.math.msu.su/people/taras/

Conference at CSF Ascona, Integrable Systems, June 19-24, 2016

SYMPLECTIC AND SPECTRAL THEORY OF INTEGRABLE SYSTEMS

ÁLVARO PELAYO

Abstract. I will report on recent progress on the symplectic and spectral geometry of classical and quantum finite dimensional integrable Hamiltonian systems. The talk will emphasize how to use symplectic and spectral techniques to solve inverse problems about quantum integrable systems.

12, 24 AND BEYOND: FROM COMBINATORICS TO SYMPLECTIC GEOMETRY AND BACK

SILVIA SABATINI

Reflexive polytopes were introduced by Batyrev [Bat] to construct possible pairs of mirror duals. Since their introduction, there has been extensive work to understand their combinatorial properties, as well as the connection to the geometry of the underlying toric varieties.

A particularly surprising phenomenon is the “12 and 24 formula”. Namely, reflexive polytopes Δ of dimension $n = 2$ and 3 satisfy the following striking identities:

- If $n = 2$ then
$$\sum_{e\in\Delta[1]} l(e) + \sum_{f\in\Delta^*[1]} l(f) = 12 \tag{0.1}$$
- If $n = 3$ then
$$\sum_{e\in\Delta[1]} l(e)l(e^*) = 24\,, \tag{0.2}$$

where $\Delta[1]$ (resp. $\Delta^*[1]$) denotes the edge set of Δ (resp. of its dual Δ^*), e^* the edge in $\Delta^*[1]$ dual to $e \in \Delta[1]$, and $l(e)$ the number of lattice points on e minus one. In [GHS] we generalise the formulas above to *Delzant reflexive polytopes* of any dimension n, obtaining
$$\sum_{e\in\Delta[1]} l(e) = 12 f_2 + (5-3n) f_1\,, \tag{0.3}$$
where f_i denotes the number of faces of Δ of dimension i.

Delzant reflexive polytopes are in one-to-one correspondence with *monotone symplectic toric manifolds*, which can be thought of as very special completely integrable systems satisfying $c_1 = [\omega]$, where c_1 denotes the first Chern class of the tangent bundle. Their classification and properties, as well as that of Delzant polytopes, is still under certain aspects a wide open problem.

We prove that (0.3) is a consequence of a much more general phenomenon which involves compact symplectic manifolds with much smaller symmetries admitting a *toric 1-skeleton*. For any such space we prove that the integral of c_1 on the toric 1-skeleton only depends on the even Betti number of the manifold. The proof uses in an essential way the ‘rigidity of the Hirzebruch genus’ and its behavior at -1. This result implies, in turn, bounds on the Betti numbers of some monotone symplectic manifolds.

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Mathematisches Institut, Universität zu Köln, Germany
E-mail address: `sabatini@math.uni-koeln.de`

Elliptic fibrations arising from free rigid bodies

Daisuke Tarama*

The free rigid body dynamics is a typical solvable example in analytical mechanics. The dynamics can essentially be described by Euler equation on $\mathfrak{so}(3) \cong \mathbb{R}^3$, which is completely integrable. The integral curve of this equation is in fact a smooth elliptic curve. It is also known that one can associate a Lax equation whose spectral curve is again a smooth elliptic curve.

Varying the parameters of the system, one obtains two elliptic fibrations over three-dimensional projective space. In this talk, the singular fibres and the monodromy of these fibrations, as well as their mutual relation, are analysed and the link to Birkhoff normal forms defined around equilibrium points will also be mentioned.

This talk is based on the collaborations [1, 2, 3] with Isao Naruki (Ritsumeikan University) and Jean-Pierre Françoise (Laboratoire J.-L. Lions, Universié P.-M. Curie).

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*Department of Mathematical Sciences, Ritsumeikan University. Partially supported by Grant-in-Aid for Young Scientists (B), JSPS KAKENHI Grant Number 26870289. E-mail: dtarama@fc.ritsumei.ac.jp

NON-HAMILTONIAN SYMPLECTIC CIRCLE ACTIONS WITH ISOLATED FIXED POINTS

SUSAN TOLMAN

Abstract. A circle action on a symplectic manifold is ”symplectic” if it preserves the symplectic form and ”Hamiltonian” if there exists a moment map. In the latter case, many invariants of the manifold are determined by the fixed set. Therefore, it is important to determine when symplectic actions are Hamiltonian. We answer a question posed by McDuff and Salamon by constructing a non-Hamiltonian symplectic circle action with exactly 32 fixed points on a closed-connected, six-dimensional symplectic manifold. Based in part on joint work with J. Watts

A semi-toric version of Atiyah - Guillemin & Sternberg theorem

Christophe Wacheux*

May 30, 2016

Abstract

For an integrable Hamiltonian system defined as a moment map $F = (f_1, .., f_n)$ on a (compact, connected) symplectic manifold (M^{2n}, ω), there exists a notion of non-degeneracy in the spirit of Morse theory. At non-degenerate critical points of F, there exists Darboux coordinates $(x_1, \xi_1, .., x_n, \xi_n)$ in which the moment map can be linearized, each component being of one of the following types

- elliptic: $f_i = x_i^2 + \xi_i^2$
- hyperbolic: $f_i = x_i \xi_i$
- focus-focus: $\begin{cases} f_i = x_i\xi_i + x_{i+1}\xi_{i+1} \\ f_{i+1} = x_i\xi_{i+1} - x_{i+1}\xi_i \end{cases}$
- regular (or transverse): $f_i = \xi_i$

Definition 0.1. *A* semi-toric system *is an integrable Hamiltonian system with only non-degenerate points, without hyperbolic components, and for which at least $n-1$ components have periodic flows, thus yieding a $\mathbb{T}^{n-1}$-action.*

A famous particular case of semi-toric systems are the *toric* systems, that yield a Hamiltonian $\mathbb{T}^n$-action.

Theorem 0.2 (Atiyah - Guillemin & Sternberg)**.** *Given a Hamiltonian $\mathbb{T}^k$-action on M with moment map J, we have that*

1. *The fibers of J are connected,*
2. *The image of M by J is a convex polytope.*

A theorem of Delzant states that for toric systems (the case $k = n$) under mild conditions, this *moment polytope* characterizes entierely the system up to $\mathbb{T}^n$-equivariant symplectomorphism. The moment polytope describing elegantly the dynamics of the system, these two results opened a whole new approach to try to classify and describe different integrable Hamiltonian systems in various settings.

When trying to extend it to semi-toric systems, we encounter several problems: the image of the moment map is not a convex polytope anymore, there is no reason for the fibers to be connected and the image of the moment map is not the only invariant anymore. Yet, in dimension $2n = 4$, a description of the image of the moment map and a classification "à la Delzant" for semi-toric systems was obtaind by San Vũ Ngọc and Pelayo.

In this talk, I will show how we can obtain a version of Atiyah - Guillemin & Sternberg theorem for semi-toric systems in all dimensions. This result, in turn, is of great promise in the view of a classification and description of semi-toric systems. The proof of this results relies on manipulation of local models for the moment map, and the stratification of semi-toric systems.

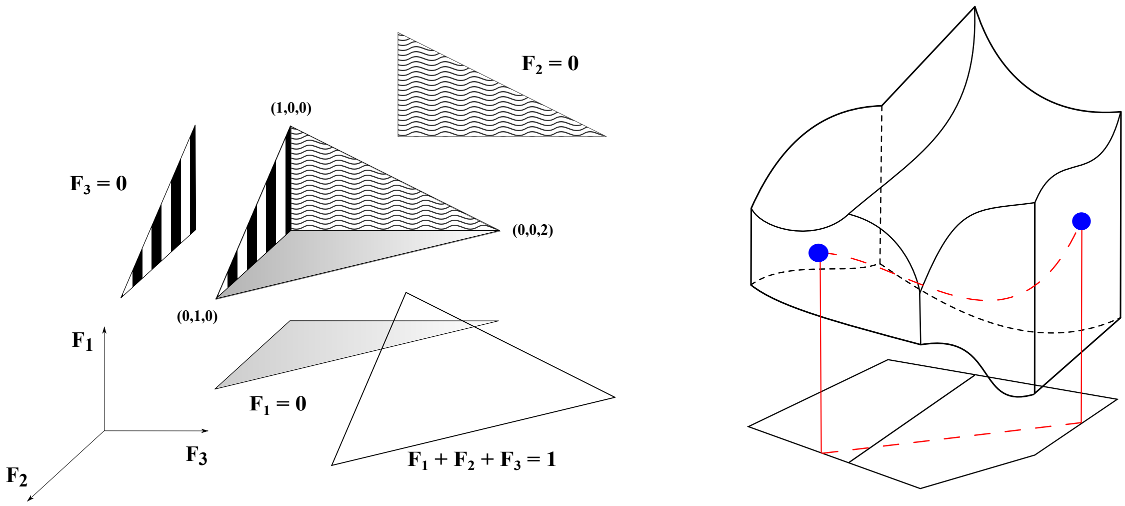

Figure 1: Left: an example of a toric system on $\mathbb{CP}^3$, Right: an example of a possible semi-toric system

*christophe.wacheux.math@gmail.com

Speaker : Nguyen Tien Zung

Title : Linearization of smooth integrable non-Hamiltonian systems

Abstract : In this talk, I'll discuss the problem of linearization of nondegenerate singularities of smooth integrable non-Hamiltonian systems. This problem is much more difficult than the analogous problems for smooth integrable Hamiltonian systems (Eliasson theorem) and for analytic integrable non-Hamiltonian systems (which I studied some years ago). New technical tools are required to solve the problem in the most general case.

This talk is based on joint work in progress with Marc Chaperon, Truong Hong Minh and Jiang Kai.

Posters

(alphabetically ordered)

Topology of Hyperbolic Actions of $\mathbb{R}^n$ on n-manifolds

Damien Bouloc
Institut de Mathématiques de Toulouse, Université Paul Sabatier

The study of nondegenerate actions of $\mathbb{R}^n$ on compact connected manifolds of dimension n has been initiated by Minh and Zung in their 2014 paper. In particular, they defined an invariant $t(\rho)$ of such an action ρ called the toric degree, and showed that the study of ρ can be reduced to the study of a torus action $\rho_{\mathbb{T}}$ of $\mathbb{T}^{t(\rho)}$ on M and of a reduced nondegenerate action $\rho_{\mathbb{R}}$ of $\mathbb{R}^{n-t(\rho)}$ on the quotient space $M/\rho_{\mathbb{T}}$ (satisfying $t(\rho_{\mathbb{R}}) = 0$). For this reason, it is interesting to study in more detail the *hyperbolic actions* of $\mathbb{R}^n$ on n-manifolds, that is the nondegenerate actions with toric degree equal to zero.

This poster presents some results about the hyperbolic actions of $\mathbb{R}^n$ on a connected compact n-manifold M. We give some examples in dimension 2 and 3 (it is still an open problem, for a fixed manifold of dimension $n \geq 3$, to find hyperbolic actions of $\mathbb{R}^n$ on M). We explain the behavior of the $\mathbb{R}$-action $t \mapsto \rho(-tv, .)$ induced by a fixed vector $v \in \mathbb{R}^n$ and how it is related to Morse functions. We also give some results about the numbers of fixed points and of hyperbolic domains (i.e. orbits of maximal dimension), with a particular attention to the case of surfaces.

The Structure of the Polynomial Lie Algebra of Differentiation of Abelian Functions of Genus 2

E. Yu. Bunkova

(Steklov Mathematical Institute of Russian Academy of Sciences)

bunkova@mi.ras.ru

We consider the fiber bundle whose base is the parameter space $\mathcal{B}$ of the family of non-degenerate curves of the form

$$V_\lambda = \{(x, y) \in \mathbb{C}^2 \, : y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}\}$$

and whose fibers are the Jacobi varieties of these curves.

The problem of constructing the Lie algebra of derivations of the field of fiberwise Abelian functions on the total space of this bundle was adressed in [1] and [2].

Consider the complex linear space $\mathbb{C}^6$ with coordinates $(x_2, x_3, x_4, z_4, z_5, z_6)$, and $\mathbb{C}^4$ with coordinates $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$. In [1] a homogeneous polynomial map $\pi \colon \mathbb{C}^6 \to \mathbb{C}^4$, was introduced. There is a diffeomorphism φ of bundles from the bundle described above to the bundle determined by this polynomial map with base $\mathcal{B}$. It can be described in terms of hyperelliptic functions (see [2]): $x_2 = \wp_{2,0}$, $x_3 = \wp_{3,0}$, $x_4 = \wp_{4,0}$, $z_4 = \wp_{1,3}$, $z_5 = \wp_{2,3}$, $z_6 = \wp_{3,3}$.

Consider the polynomial vector fields $\mathcal{L}_k, k = 0, 1, 2, 3, 4, 6$, on $\mathbb{C}^6$:

$$\begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_3 \end{pmatrix} = \begin{pmatrix} 2x_2 & 3x_3 & 4x_4 \\ x_3 & x_4 & R_5 \\ z_5 & z_6 & R_7 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_4} \end{pmatrix} + \begin{pmatrix} 4z_4 & 5z_5 & 6z_6 \\ z_5 & z_6 & R_7 \\ Q_7 & \mathcal{L}_1 Q_7 & \mathcal{L}_1 \mathcal{L}_1 Q_7 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_4} \\ \frac{\partial}{\partial z_5} \\ \frac{\partial}{\partial z_6} \end{pmatrix},$$

where $R_5 = 4(3x_2x_3 + z_5)$, $R_7 = 4(2x_2z_5 + x_3z_4)$, $Q_7 = x_3z_4 - x_2z_5$. The polynomial vector fields $\mathcal{L}_2$, $\mathcal{L}_4$, $\mathcal{L}_6$ are described in [2]. The fields $\mathcal{L}_k$ are linearly independent outside their discriminant variety $\Delta \subset \mathbb{C}^6$ and tangent to this variety. We have $\mathcal{B} = \mathbb{C}^4 \backslash \pi(\Delta)$.

The fields $\varphi^* \mathcal{L}_k$ generate the Lie algebra of derivations of the field of fiberwise Abelian functions.

In the talk we will describe the structure of this Lie algebra.

This work was supported in part by the program "Fundamental Problems of Nonlinear Dynamics" of the Russian Academy of Sciences.

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"Multiparameter bi-Hamiltonian structures and related integrable systems"

Alina Dobrogowska

Institute of Mathematics, University of Białystok
Ciołkowskiego 1M
15-245 Białystok
Poland
alina.dobrogowska@uwb.edu.pl

We construct R operators on the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ or more generally Hilbert–Schmidt operators $\mathcal{L}^2$ in Hilbert space. These operators are related to a multiparameter deformation given by a sequence of parameters $\alpha = \{a_1, a_2, \dots\}$. We determine for which choices of parameters R operators are R-matrices. We also construct the Lax pair for the corresponding Hamilton equations.

Families of superintegrable systems

Damianou Pantelis, **Evripidou Charalampos**, Kassotakis Pavlos
(University of Cyprus)

In this poster I will present an infinite family of superintegrable Hamiltonian systems. The systems we consider are of Lotka Volterra type of maximal interaction described by the differential equations

$$\dot{a}_i = a_i \sum_{j=1}^{n} C^{(k)}_{i,j} a_j, \quad 0 \leq i, j \leq n,$$

where $C^{(k)}$ is the matrix

$$C^{(k)} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 0 & 1 & \cdots & 1 & 1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \cdots & 1 & 1 & 1 & \ddots & -1 & -1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & -1 \\ 1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & -1 & -1 & -1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & -1 & -1 & -1 & \cdots & 1 & 0 \end{pmatrix}.$$

The superscript k is the minimum j such that $C^{(k)}_{j+1,n} = 1$ and can take the values $k = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$ for $n \in \mathbb{N}$. For each such n and k we show that the corresponding system has $n - k - 1$ independent first integrals. The $k + 1$ of these integrals are polynomial, obtained from a Lax pair of the system, while the rest are rational. The extreme cases $k = 0$ and $k = \left\lfloor \frac{n-1}{2} \right\rfloor$, give rise to well known results.

The inverse problem of the calculus of variations and applications.

Marta Farré Puiggalí
Instituto de Ciencias Matemáticas (ICMAT)

Abstract:
The inverse problem of the calculus of variations consists in determining whether a given system of second order differential equations is equivalent to some regular Lagrangian system. I will explain some results and applications of the inverse problem to hamiltonization of nonholonomic systems and stabilization of controlled Lagrangian systems.

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Explicit Symmetry breaking for Hamiltonian systems with symmetries

Marine Fontaine (The University of Manchester)

Using the Marle-Guillemin-Sternberg Normal Form Theorem, we give a geometric approach to study the phenomenon of forced symmetry breaking in equivariant Hamiltonian systems. We study the particular case where the perturbed system still has some symmetries. We give an estimate for the number of equilibria that will persist under this perturbation, provided we verify some non-degeneracy conditions. Some preliminary results for persistence of relative equilibria are also treated.

Lax representation of the hyperbolic van Diejen system with two coupling parameters

Tamás F. Görbe

Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
website: www.staff.u-szeged.hu/∼tfgorbe
e-mail: tfgorbe@physx.u-szeged.hu

Abstract

In his 1994 thesis, Jan Felipe van Diejen proved the quantum integrability of the hyperbolic Ruijsenaars-Schneider model attached to the BC_n root system. This led to explicit formulas for a complete set of Poisson commuting functions in the classical limit, but a Lax matrix generating these Hamiltonians as its spectral invariants was lacking ever since*. In a recent joint work with B.G. Pusztai [arXiv:1603.06710], we constructed a Lax pair for the classical hyperbolic BC_n system with two independent couplings. We showed that the dynamics can be solved by a projection method and worked out the asymptotic form of the solutions. The equivalence of the first integrals provided by the eigenvalues of our Lax matrix and van Diejen's commuting Hamiltonians was also demonstrated.

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*Except for the 1-coupling cases obtained from the standard A_m models by 'folding'.

Singular points of Nijenhuis operators and linearization

Andrei Konyaev, Moscow State University

May 28, 2016

Nijenhuis operators are operator fields with vanishing Nijenhuis torsion. We define the notion of singular points for operator fields. We show that the tangent space in singular has a natural structure of so-called left-symmetric algebras (LSA). These algebras naturally appear in variety of settings. For example, one class of these algebras is known as Novikov algebras. To study the linearization problem we introduce the notion of non-degenerate LSAs and study the question of existence of such algebras in two-dimensions. To do so we classify all real LSAs in dimension two and for each algebra we solve the question of its non-degeneracy.

Results on the Topology of Gelfand-Zeitlin Systems
Jeremy Lane (University of Toronto)

In 1983, Guillemin and Sternberg constructed completely integrable systems on $U(n)$ and $SO(n)$ coadjoint orbits which they called *Gelfand-Zeitlin systems* for their relation to Gelfand-Zeitlin bases in representation theory. These systems are constructed from of non-Abelian Hamiltonian symmetries of the coadjoint orbits which are maximal in the sense that they are multiplicity-free. Using the description of $U(n)$ coadjoint orbits as congruency classes of Hermitian matrices, Guillemin and Sternberg proved that the image of these Gelfand-Zeitlin systems are convex polytopes. The description of these convex polytopes and the associated Lagrangian torus fibrations of the coadjoint orbits was employed recently by Milena Pabiniak to prove tight lower bounds on the Gromov width of $U(n)$ and $SO(n)$ coadjoint orbits.

Motivated by Pabiniak's work, and an interest in the topological structure of integrable systems, this work explores the topological properties of momentum maps *constructed by Thimm's trick*, of which the Gelfand-Zeitlin systems on $U(n)$ and $SO(n)$ coadjoint orbits are an example. Suppose M is a connected symplectic manifold equipped with a Hamiltonian action of a compact connected group G. Using results of various authors on proper torus momentum maps, we give a short proof that a proper momentum map constructed by Thimm's trick on M has convex image and connected fibres. Our topological approach to this problem allows us to prove connectedness even for fibres where the momentum map is not a smooth function. For Gelfand-Zeitlin systems (where M is multiplicity-free), we can also prove that these 'marginal' fibres are smooth submanifolds. The structure of these integrable systems near the marginal fibres remains somewhat mysterious.

As an example, we consider a one-parameter family of coadjoint orbits of the exceptional Lie group G_2. These coadjoint orbits are 5 dimensional projective subvarieties of $\mathbb{CP}^{13}$ with degree 18, that come equipped with a multiplicity-free action of the subgroup $SU(3) \leq G_2$ that was described by Chris Woodward. We construct a Gelfand-Zeitlin system on these orbits and give a complete description of it's Gelfand-Zeitlin polytope, as well as the associated Lagrangian torus fibration of an open dense submanifold. Following the approach of Pabiniak (used earlier by Traynor and Schlenk), this data gives a tight lower bound on the Gromov width of these coadjoint orbits that agrees with the upper bounds obtained by Caviedes Castro and results of Loi-Zuddas.

FRACTIONAL MONODROMY AND BEYOND

NIKOLAY MARTYNCHUK

The Euler number of a principal circle bundle can be defined in several equivalent ways:

1: as a self-intersection number,
2: as an obstruction to the existence of a global section,
3: via the classifying space $BU(1)$.

Each of these ways has a natural extension to Seifert fibrations and they all give the same result. In our recent work with K. Efstathiou [3] we have shown that the Euler number of a principal circle bundle can be also defined

4: via the parallel transport of homology cycles.

In this poster we present an extension of **4** to Seifert fibrations. Surprisingly, such an extension allows to compute standard (in the sense of Duistermaat [1]) and fractional (in the sense of K. Efstathiou and H. W. Broer [2]) monodromy in various integrable Hamiltonian system.

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E-mail address: N.Martynchuk@rug.nl

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands

Analytic extensions of frequencies of integrable PDEs and applications

Jan-Cornelius Molnar

May 27, 2016

The KdV2 equation and the mKdV equations on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, are Hamiltonian PDEs

$$\partial_t u = \partial_x \partial_u H^{kdv2}, \qquad \partial_t v = \partial_{\overline{v}} H^{mkdv},$$

with Hamiltonians

$$H^{kdv2}(u) = \frac{1}{2}\int_{\mathbb{T}} ((\partial_x^2 u)^2 + 10u(\partial_x u)^2 + 5u^4)\,\mathrm{d}x,$$
$$H^{mkdv}(v) = \int_{\mathbb{T}} (v\partial_x^3 \overline{v} - 3v^2\overline{v}\partial_x \overline{v})\,\mathrm{d}x.$$

They are both completely integrable in the strongest possible sense, meaning that they admit global Birkhoff coordinates

$$\Phi\colon L^2 \to h^{1/2}, \quad u \mapsto (z_n)_{n\in\mathbb{Z}}, \qquad \Psi\colon L^2 \to \ell^2, \quad v \mapsto (\zeta_n)_{n\in\mathbb{Z}},$$

so that H^{kdv2} and H^{mkdv}, respectively, when restricted to appropriate subspaces of sufficient regularity, are real analytic functions of the action $I_n = z_n z_{-n}$ and $J_n = \zeta_n\overline{\zeta_n}$, respectively. In these coordinates, the Hamiltonian systems for mKdV and KdV2 are given by

$$\partial_t z_n = \omega_n^{kdv2} z_n, \qquad \omega_n^{kdv2} = \partial_{I_n} H^{kdv2},$$
$$\partial_t \zeta_n = \omega_n^{mkdv}\zeta_n, \qquad \omega_n^{mkdv} = \partial_{J_n} H^{mkdv},$$

where ω_n^{kdv2} and ω_n^{mkdv}, respectively, is called the nth KdV2 and mKdV frequency, respectively.

In form of a case study for these two equations, we discuss a novel approach of representing frequencies of integrable PDEs which allows to extend them analytically to spaces of low regularity and to study their asymptotics. Applications include properties of the actions to frequencies map as well as wellposedness results in spaces of low regularity. This is joint work with Thomas Kappeler.

The combinatorial invariant of semitoric manifolds

Joseph Palmer

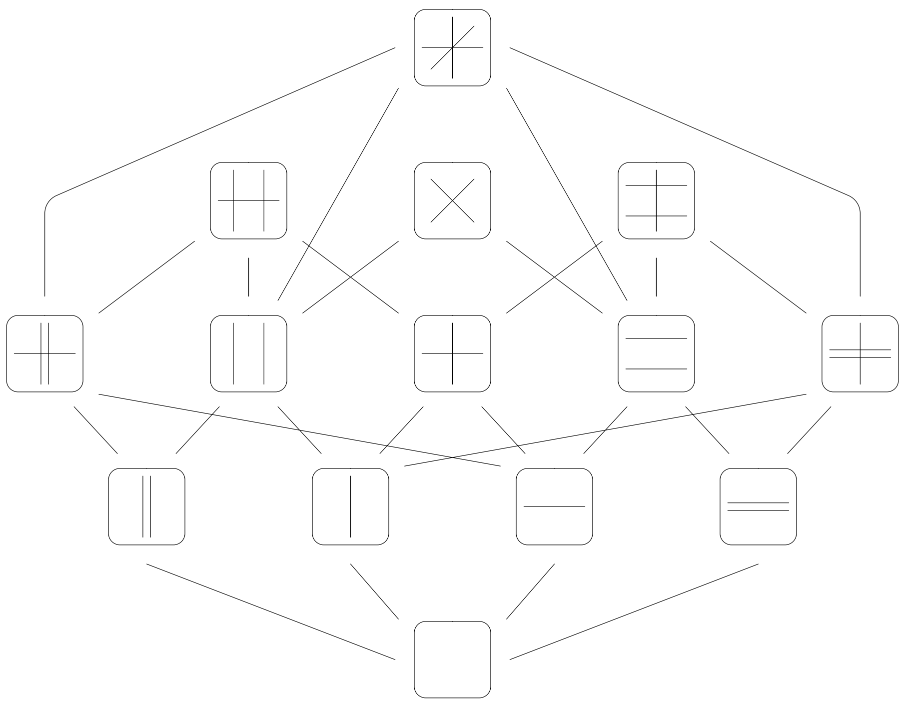
We introduce a new combinatorial symplectic invariant of compact symplectic semitoric 4-manifolds, the semitoric helix. It encodes information about the singular affine structure in a neighborhood of the set of elliptic singular points of the system by correcting for the effects of the Duistermaat monodromy from the nodal singular points. The semitoric helix is particularly well-behaved with respect to semitoric blowups/downs and we use this new invariant paired with a novel algebraic technique to study how semitoric manifolds transform with respect to these operations.

This work is joint with Álvaro Pelayo and Daniel M. Kane.

An Algebraic Geometric Classification of Superintegrable Systems in the Euclidean Plane

Konrad Schöbel

We prove that the set of second order maximally superintegrable systems in the complex Euclidean plane carries a natural structure of a projective variety by deriving the corresponding system of homogeneous algebraic equations. We then solve these equations explicitly and give a detailed analysis of the algebraic geometric structure of the corresponding projective variety. This avoids the need for computer algebra and considerably simplifies the classification of superintegrable systems. In particular, a unique completely decomposable ternary cubic as well as a planar line triple arrangement is associated to every superintegrable system, providing intrinsic geometric as well as algebraic labelling schemes for superintegrable systems and their normal forms under isometries. This joint work with Jonathan Kress initiates a programme to "algebro-geometrise" the classification of superintegrable systems and their applications to special functions.



Focus-focus singularities in real geometric quantisation

Romero Solha (UFMG)

Joint work with Eva Miranda (UPC) and Francisco Presas (ICMAT)

Abstract

This poster address the focus-focus contribution to geometric quantisation, with real polarisations given by integrable systems. Near a singular Bohr–Sommerfeld focus-focus fibre of an integrable system in dimension four, the zeroth and first cohomology groups computing geometric quantisation are trivial, but the second cohomology group is not. The latter is actually infinite dimensional, contrary to previous expectations.

The fact that both hyperbolic and focus-focus type of singularities provide infinite dimensional contributions to real geometric quantisation (even though only Bohr–Sommerfeld focus-focus fibres appear, whereas all hyperbolic singularities contribute) raises doubt about its applicability to Theoretical Physics. One could also conjecture that stable equilibria give no contribution to geometric quantisation (as it is the case for elliptic singularities), whilst unstable equilibria give infinite dimensional contributions (even if the singularity is degenerate, but possibly assuming a Bohr–Sommerfeld condition).

ALGEBRAIC COMPLEXITY-ONE SPACES AND SEMITORIC SYSTEMS

HENDRIK SÜSS (JOINT WORK WITH CHRISTOPHE WACHEUX)

Definition 1 (Karshon-Tolman)**.** A symplectic manifold M^{2n} with a Hamiltonian action of $(S^1)^{n-1}$ is called a complexity-one space.

These spaces were classified by Karshon in the 4-dimensional case in terms of labelled graphs and in higher dimensions by Karshon and Tolman. For those complexity-one spaces which arise from a smooth projective variety with an algebraic torus action there also exists an alternative classification by piecewise linear decompositions of the Duistermaat-Heckman function. More precisely, consider the pushforward of the Liouville measure via the moment map. This gives a measure on the moment polytope P of the torus action, called the Duistermaat-Heckman measure. The density function $\rho : P \to \mathbb{R}$ of this measure is piecewise linear and concave. Now, algebraic complexity-one spaces M correspond to certain collections of piecewise linear and concave functions $\rho_1 \dots, \rho_r$ on some polytope P, where $\rho = \sum_i \rho_i$ holds for the Duistermaat-Heckman function of M.

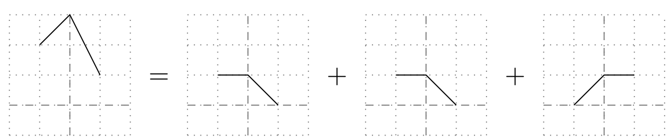


Figure 1. Duistermaat-Heckman decomposition for the blowup of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ in $(0,0)$, $(0,1)$ and (∞,∞).

In the 4-dimensional case we can compare the description by Duistermaat-Heckman decompositions with Karshon's labelled graphs and show how to obtain one from the other.

In dimension four Hohloch, Sabatini, Sepe and Symington determined for which complexity-one spaces the moment map of the torus action can be extended in a well-behaved way to an integrable systems—a so-called semitoric system. They formulated a criterion in terms of Karshon's graph. By using our translation and existing results of Ilten and Süß on toric degenerations of algebraic complexity-one spaces we obtain the following reformulation.

Corollary 2. *A 4-dimensional complexity-one space admits a semi-toric system if and only if it admits an equivariant degeneration to a toric variety.*

This observation motivated our ongoing project where we are studying the existence of semitoric systems on algebraic complexity-on spaces in higher dimensions. Here, the situation becomes more involved and the statement of the corollary does not generalise directly.

Stability of volume forms on noncompact fiber bundles

Xiudi Tang
joint with Álvaro Pelayo

May 28, 2016

Abstract

The goal of this paper is to generalize the Moser stability theorem for volume forms on smooth manifolds to smooth fiber bundles with noncompact fibers over a closed manifold. Let F be a noncompact manifold, $f\colon F \to \mathbb{R}$ an exhausion function and let B be a closed manifold. Consider a fiber bundle $\pi\colon M \to B$ with fiber F and structure group G, which consists of maps preserving f and ends of F, and there is an arbitrarily high regular level set of f, whose connected components are invariant under G. If ω_p and τ_p are volume forms on the fiber $\pi^{-1}(p)$ depending smoothly on $p \in B$, and satisfy some additional conditions, we will prove that there exists a volume preserving diffeomorphism φ of M such that $\left(\varphi|_{\pi^{-1}(p)}\right)^* \omega_p = \tau_p$. If $B = \{p\}$ is a point this recovers the Moser stability theorem (for volume forms on compact F) and its generalization to Greene and Shiohama for noncompact F. To bridge the gap between base $\{p\}$ and a general base B and to construct φ we use a result of Edward Bueler on Hodge theory for weighted noncompact manifolds.

Determining Killing tensors of higher rank

Andreas Vollmer*

Consider a 4-dimensional manifold endowed with a (pseudo-)Riemannian metric. Constants of motion are smooth functions on the phase space that remain unchanged along solutions of the geodesic equation.
Killing tensors are in 1-to-1 correspondence with constants of motion that are homogeneous polynomials in the momenta (i.e. the fiber coordinates of the cotangent space). The most basic examples of Killing tensors are the metric and the (0,1)-tensors corresponding to Killing vector fields.

Killing tensors play a crucial role in physics, as for instance in the classical Kepler problem. Another famous example are Kerr space-times, for which a non-trivial second-rank Killing tensor has been found by B. Carter in 1968.

The poster presents an approach how to prove non-existence of non-trivial Killing tensors of a given rank, or to find them if they exist. We use this method to prove that Weyl metrics (static and axially symmetric vacuum metrics) have no non-trivial Killing tensors of cubic rank. In addition, we demonstrate how the method can be used on a computer in a self-contained way in case of explicitly given metrics (a first implementation of this computer-based approach was given in [1] by B. Kruglikov & V. Matveev).

The poster will also include a similar application in sub-Riemannian geometry (joint work with B. Kruglikov and G. Lukes-Gerakopoulos). Moreover, it will give an outlook on research perspectives, e.g. in the context of C-metrics and Ricci solitons.

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* Istituto Nazionale di Alta Matematica "F. Severi" / Politecnico di Torino
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On the Birkhoff Normal Form of the mKdV, NLS and sine-Gordon hierarchies

Yannick Widmer

May 29, 2016

The mKdV equation on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ admits two representations as a Hamiltonian PDE

$$\partial_t u = \partial_x \partial K(u) = -\mathrm{i}(\partial_2 S)(u, u),$$

with corresponding Hamiltonians

$$K(u) = \frac{1}{2}\int_{\mathbb{T}} (\partial_x u)^2 + u^4 \, \mathrm{d}x, \qquad S(v_1, v_2) = \mathrm{i}\int_{\mathbb{T}} (v_1 \partial_x^3 v_2 - 3v_1^2 v_2 \partial_x v_2) \, \mathrm{d}x.$$

Both representations admit an infinite sequence of recursively defined pairwise Poisson commuting integrals. In the first case, this sequence is referred to as the *mKdV hierarchy* with $K_1 = \frac{1}{2}\int_{\mathbb{R}} u^2 \, \mathrm{d}x$, $K_2 = K$, $\ldots$ and in the second case as the *NLS hierarchy* $S_1 = \int_{\mathbb{T}} v_1 v_2 \, \mathrm{d}x$,

$$S_2 = \mathrm{i}\int_{\mathbb{T}} v_2 \partial_x v_1 \, \mathrm{d}x, \qquad S_3 = \int_{\mathbb{T}} (\partial_x v_1 \partial_x v_2 + v_1^2 v_2^2) \, \mathrm{d}x, \qquad S_4 = S, \quad \ldots$$

Note that S_3 is the NLS Hamiltonian. Our main result is, that *every* Hamiltonian PDE in the mKdV hierarchy is contained in the NLS hierarchy. More precisely, denoting by Y_{K_m} the mth Hamiltonian vector field of the mKdV hierarchy and by X_{S_n} the nth Hamiltonian vector field of the NLS hierarchy, then for each $m \geqslant 1$,

$$X_{S_{2m}}(u, u) = (Y_{K_m}(u), Y_{K_m}(u)), \qquad \forall\, u \in H^m(\mathbb{T}, \mathbb{C}) \tag{1}$$

The proof relies in a crucial on the fact that both hierarchies are completely integrable in the strongest possible sense, meaning that there exist global Birkhoff coordinates

$$\Psi \colon L^2 \to h^{1/2}, \quad u \mapsto (\zeta_n)_{n\in\mathbb{Z}}, \qquad \Phi \colon L^2 \to \ell^2, \quad v \mapsto (z_n)_{n\in\mathbb{Z}},$$

in which the transformed Hamiltonians, when restricted to subspaces of sufficient regularity, are real analytic functions of the actions $J_n = \zeta_n \overline{\zeta_n}$ and $I_n = z_n \overline{z_n}$, respectively. In these coordinates, the corresponding Hamiltonian systems are given by

$$\partial_t \zeta_n = \eta_{n,m}\zeta_n, \quad \eta_{n,m} \coloneqq \partial_{J_n} K_m, \qquad \partial_t z_n = \omega_{n,m} z_n, \quad \omega_{n,m} \coloneqq \partial_{I_n} S_m.$$

As an application of (1) we compare the solution curves of the vectorfields $X_{S_{2m}}$ and Y_{K_m} in their corresponding Birkhoff coordinates and show that

$$\eta_n^m = (-1)^m \omega_{n,2m}.$$

This is joint work with Jan Molnar.

In a related work we show that the sine-Gordon equation can be embedded in the phase space of mKdV in a way that it's transformed Hamiltonian is also a function of the actions alone.

Moment Maps in Perturbed Semitoric Integrable Systems

Holger Dullin, ***Joachim Worthington*** *(University of Sydney)*

In their 2012 paper, Pelayo and Vũ Ngọc studied the dynamics of semitoric systems. In particular, they considered coupled spin-oscillator systems near focus-focus singularities. Semi-global symplectic invariants were calculated for the semitoric system with a focus-focus singularity.

D. and Pelayo showed that one can modify generic semitoric integrable systems around a focus-focus singular point to create a Hamiltonian Hopf bifurcation. This creates a hyperbolic singular point.

In this poster we will shed some light on interpreting the change in the symplectic invariants as a system undergoes this bifurcation. We will look at a particular example which has been studied in the two forementioned papers. We show that the polygonal invariants are conserved in the perturbed system, and calculate a new action which is created by the formation of the singular point.

Stokes phenomenon, dynamical r-matrices and quantum groups

Xiaomeng Xu
Université de Genéve

Abstract

Our study lies at the crossroads of Stokes phenomenon, dynamical r-matrices and quantum groups. In particular, the main results in this poster are

- A construction of a gauge transformation between the standard classical r-matrix and the Alekseev-Meinrenken dynamical r-matrix, using the Stokes data of a certain irregular Riemann-Hilbert problem. Geometrically, this gauge transformation is interpreted as a generalization of a symplectic neighborhood version of the Ginzburg-Weinstein linearization theorem.
- An extension to the quantum analogue of the Stokes phenomenon, and its relation with the Yang-Baxter equation. Furthermore, we prove that the quantum Stokes factors satisfy a quantum isomonodromic equation, which is a quantization of the isomonodromic equation of Boalch, Jimbo, Miwa and Ueno.