Integrations of CA

$$\{CA\} \overset{\text{Roytenberg}}{\underset{\text{Thm}}{\longleftrightarrow}} \left\{\begin{matrix}\text{deg } 2 \text{ symplectic} \\ \text{dg-manifold}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{deg } 2 \\ \text{dg-man}\end{matrix}\right\}$$

$$\{\text{Lie } 2\text{-gpd}\} \xrightarrow{\text{Lie}} \left\{\begin{matrix}\text{deg } 2 \\ \text{dg-man}\end{matrix}\right\}$$

Q: what about the symplectic form?

Def

Let $G_\bullet$ be a Lie n-groupoid

- $(\Omega^\bullet(G_\bullet), \delta, d)$ double complex.
- An m-shifted k-form
$$\alpha_\bullet = \sum_{i=0}^{m} \alpha_i \in \Omega^{m+k-i}(G_i).$$
we say α is closed if $(\delta \pm d)\alpha_\bullet = 0$
- The tangent Complex $(\tau_\bullet^G, \partial)$ ← Complex of v.b. over G_0.

$$\tau_\ell^G = \begin{cases} \mathrm{Ker}(Tp_{-\ell}^{-\ell})|_{G_0} & \text{if } \ell < 0 \\ TG_0 & \text{if } \ell = 0 \\ 0 & \ell > 0 \end{cases} \qquad \partial = (-1)^\ell Td_{-\ell}$$

- (Getzler) For an m-shifted 2-form $\omega_\bullet$, the IM-Pairing at $x \in G_0$, $u \in \tau_{-\ell}^G|_x$, $v \in \tau_{-m+\ell}^G|_x$

$$\lambda^\omega(u,v) = \sum_{\sigma \in sh(\ell, m-\ell)} |\sigma|\, \omega_m\left(T(s_{\sigma(m-1)} \cdots s_{\sigma(m-\ell)})u, T(s_{\sigma(m-\ell-1)} \cdots s_{\sigma(0)})v\right)$$

(If $\delta\omega_m = 0 \Rightarrow \lambda^{\omega}(\partial v, v) + (-1)^{|u|}\lambda^{\omega}(v, \partial v) = 0$)

So $\lambda^{\omega,\#} : \tau^G_\bullet \longrightarrow \tau^{*G}[m]$ cochain map.

- An m-shifted symplectic Lie n-groupoid $(G_\bullet, \omega_\bullet)$

  Lie n-gpd $G_\bullet$ + close m-shifted 2-form $\omega_\bullet$

s.t $\lambda^{\omega,\#}$ is a quasi-isomorphism.

Thm (Cabrera - del Hoyo)

$G_\bullet$ Lie n-gpd with $\mathrm{Lie}(G_\bullet) = (\mathcal{M}, Q)$

Then $VE : (\Omega^\bullet(G_\bullet), \delta, d) \longrightarrow (\Omega^\bullet_\bullet(\mathcal{M}), \mathcal{L}_Q, d)$

Main

Given a 2-shifted symplectic Lie 2-gpd $(G_\bullet, \omega_\bullet)$ we say that integrates the CA defined by $(\mathcal{M}, Q, \overline{\omega})$ if

$$\mathrm{Lie}(G_\bullet) = (\mathcal{M}, Q) \text{ and } VE(\omega_\bullet) = \overline{\omega}$$

Examples

① Quadratic Lie algebras $(\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$

Let G be any Lie group s.t. Ⓐ $\mathrm{Lie}(G) = \mathfrak{g}$

Ⓑ $\langle , \rangle$ is Ad-invariant

$\Rightarrow (N_\bullet G, \omega_2 + \omega_1)$ is 2-shifted symplectic Lie 1-gp

$$\omega_2 = \frac{-1}{2}\langle pr_1^* \theta^l, pr_2^* \theta^r\rangle \in \Omega^2(\underset{\overset{\shortparallel}{N_2 G}}{G\times G})$$

$$\omega_1 = \frac{-1}{12}\langle \theta^l, [\theta^l, \theta^l]\rangle \in \Omega^3(\underset{\overset{\shortparallel}{N_1 G}}{G})$$

Cartan 3-form

(B) $\Longrightarrow (\delta \pm d)(\omega_1 + \omega_2) = 0$

$$\tau^{N.G} = 0 \to \overset{-1}{\mathfrak{g}} \to 0$$

$\langle\cdot,\cdot\rangle$ non-deg $\Longrightarrow$ $\underset{\overset{\shortparallel}{\langle\cdot,\cdot\rangle^\flat}}{\lambda^{\omega_1\#}} : \mathfrak{g} \to \mathfrak{g}^*$

$$VE(\omega_2 + \omega_1) = VE(\omega_2) + \underbrace{VE(\omega_1)}_{}\overset{=0}{} = \langle\cdot,\cdot\rangle$$

Rmk

- $[\omega_2 + \omega_1] \in H^4(BG)$ first pontryagin class $\left(\frac{1}{8\pi^2}\right)$
- Connection with chern-simons theory
- shifted lagrangians (Alvarez - Bursztyn - _)
- Polyakov-Wiegmann, Brylinski, Weinstein ....

84'

Goldstone fields in 2 dim with multivalued actions.

If one looks into 3d Chern-simons on Disk $\times [0,1]$. Δ

then $\{A, \tilde{\omega}\} = \dfrac{\text{flat connections}^{\text{Disk}}}{\text{gauge s.t. } \mathrm{id}_\partial}$ is ∞-dim, symplectic and produces a Lie 2-gp.

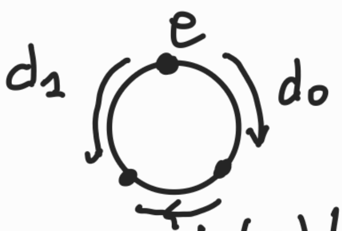
Let G be a connected and simply connected integration $\mathfrak{g}$.

$\mathbb{G}_\bullet$ Lie 2-gp

$\mathbb{G}_0 = pt$

$\mathbb{G}_1 = P_e G$, path on G starting at e of fixed Sobolev class r.

$\mathbb{G}_2 = \Omega G$, based loops on G of Sobolev class r.



$$d_2(\tau)(t) = \tau\left(\frac{1+t}{3}\right) \cdot \tau(1/3)^{-1}$$

$(\mathbb{G}_\bullet, \tilde{\omega})$ 2-shifted symplectic

$$(\tilde{\omega}_\tau)(a,b) = \int_{S^1} \langle \hat{a}'(t), \hat{b}(t) \rangle dt$$

$$\hat{a}(t) = L_{\tau(t)^{-1}} a(t).$$

Thm (— - Zhu)

The evaluation map $ev_\bullet : \mathbb{G}_\bullet \longrightarrow N_\bullet G$

$ev_1(\gamma) = \gamma(1)$, $ev_2(\tau) = (\tau(2/3)\tau(1/3)^{-1}, \tau(1/3))$

is an hypercover and

$$\tilde{\omega} - ev_\bullet^*(\omega_2 + \omega_1) = (\delta \pm d)\, \omega^P$$

1-shifted 2-form

$$\omega^P_\gamma(u,v) = \frac{1}{2}\int_0^1 \langle \hat{u}'(t), \hat{v}(t) \rangle - \langle \hat{u}(t), \hat{v}'(t) \rangle \, dt$$

② Doubles of Lie bialgebroids

Let $(A \to M, [\cdot,\cdot], \rho, d_{A^*})$ be a Lie bialgebroid

$$\Big\downarrow \text{[Mackenzie-Xu]}$$

$(G \rightrightarrows M, \pi)$ Poisson gpd

$$\updownarrow$$

$\pi \in \mathfrak{X}^2_{mul}(G)$ $\quad [\pi,\pi]=0$

Assume

$$\begin{tikzcd} \Gamma \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G^* \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Double symplectic gpd

$\omega_\Gamma \in \Omega^2(\Gamma)$

Lu-Weinstein

$\longrightarrow$

Mackenzie

$$\Big\downarrow (\mathrm{Lie})^2$$

$$\begin{tikzcd} T^*A \arrow[r, Rightarrow] \arrow[d, Rightarrow] & A^* \arrow[d, Rightarrow] \\ A \arrow[r, Rightarrow] & M \end{tikzcd}$$

Double Lie algebroid.

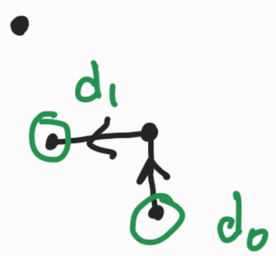
Thm (Mehta - Tang)

Given a full Double symplectic gpd then $(\overline{W}_\bullet \Gamma, \overline{W}\omega_\Gamma)$ is a 2-shifted symplectic Lie 2-gpd integrating the Double $A \oplus A^*$.

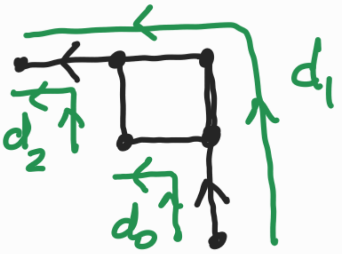
What is $\overline{W}_\bullet \Gamma$?

$\overline{W}_0 \Gamma = M$

$\overline{W}_1 \Gamma = G \times_M G^*$



$\overline{W}_2 \Gamma = G \times_M \Gamma \times_M G^*$



and $\overline{W}.\omega_\Gamma = p_\Gamma^* \omega_\Gamma \in \Omega^2(\overline{W}_2\Gamma)$

Examples of double symplectic

1 - Lie bialgebras (Lu-Weinstein)

Rmk $(\overline{W}.\Gamma, \overline{W}.\omega_\Gamma)$ and $(N.D, \omega_2+\omega_1)$ are symplectic Morita equivalent

2 - Cotangent of gpds (Mackenzie)

$$\begin{array}{ccc} T^*G & \rightrightarrows & A^* \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

3 - Transitive Courant (Alvarez)

Q: If $L_+ \oplus L_- = E = L'_+ \oplus L'_-$

How relate $\overline{W}\Gamma$ and $\overline{W}\Gamma'$?

Alvarez - Krishna - Ronchi

③ Exact CA.

M manifold $\rightsquigarrow$ $M \times M \rightrightarrows M$ pair gpd

$\mathrm{Lie}(M^2) = TM$

then
$$\begin{array}{ccc} T^*(M^2) & \rightrightarrows & T^*M \\ \downarrow\downarrow & & \downarrow\downarrow \\ M^2 & \rightrightarrows & M \end{array}$$
Double symplectic

$$\overline{W}\, T^*M^2 \equiv M \times (T^*M^2 \times_M T^*M) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} M \times T^*M \rightrightarrows M$$

Rmk $\overline{W}\, T^*M^2 \longrightarrow pt$ is a weak equivalence !

$$\mathrm{Lie}(\overline{W}\, T^*M^2) = TM \oplus T^*M = (T^*[2]T[1]M, \underset{\substack{\| \\ \theta^i \frac{\partial}{\partial x^i} + P_i \frac{\partial}{\partial \xi_i}}}{d^{cot}_{dR}}, \omega_{can})$$

What about $H \in \Omega^3_{cl}(M)$?

$$(\overline{W}(T^*M^2), \overline{W}\omega_{can} + \delta H)$$

$$\mathrm{Lie}(\overline{W}\, T^*M^2) = (T^*[2]T[1]M, \underset{\substack{\# \\ }}{d^{cot}_{dR}}, \omega_{can} + \hat{H})$$

$$Q^H = \theta^i \frac{\partial}{\partial x^i} + (P_i + H_{ijk}\theta^j\theta^k)\frac{\partial}{\partial \xi_i}$$

Rmk
one can also consider other integrations of $(T[1]M, d_{dR})$ like $\Pi_1(M)$.

$$(T^*[2]T[1]M, Q^H, \omega_{can})$$

Li-bland Severa $\Big\downarrow \tilde{P}_i = P_i - \frac{1}{3} H_{ijk}\theta^j\theta^k$

$$(T^*[2]T[1]M, d^{cot}_{dR}, \omega_H)$$

gauge trans $\Big\downarrow \mathcal{L}_Q{}^H$

$$(T^*[2]T[1]M, d^{cot}_{dR}, \omega_{can} + \hat{H}) = \mathrm{Lie}$$

④ Cotangent of Lie 2-gpds (— - Ronchi)

If $G_\bullet$ Lie 2-gpd as we introduced $T^{2*}G_\bullet$ and show that it is 2-shifted symplectic Lie 2-gpd.

and if $\mathrm{Lie}(G_\bullet) = (\mathcal{M}, Q)$ then $\mathrm{Lie}(T^{2*}G_\bullet, \omega_{can})$ and $(T^*[2]\mathcal{M}, Q^{cot}, \omega_{can})$ are quasi-isomorphic as symplectic dg-manifolds.