Integrable Systems: from Poisson to Nijenhuis
Lecture 2

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Plan: continue travelling from Poisson to Nijenhuis

- Generic and singular points
- Linearisation at singular points
- Left-symmetric algebras
- Linearisation and non-degeneracy problems
- Some classification results
- Non-degeneracy of the diagonal algebra
- Conservation laws and symmetries
- Integrable systems related to $\mathfrak{gl}$-regular Nijenhuis operators

Poisson Geometry: generic and singular points

Let P be a Poisson tensor on M and $p \in$ M.

The algebraic type of $P(p)$ is characterised by a single number, $\operatorname{rank} P(p)$.

- $p \in$ M is *generic* if the algebraic type of P remains the same in a neighbourhood of p, that is, $\operatorname{rank} P(x)$ is locally constant, $x \in U(p)$.
- the set of generic points is open and dense in M
- $p_0 \in$ M is *singular* if it is not generic, i.e., $\operatorname{rank} P$ drops at the point p_0 (as compared to neighboring points)
- the extreme case (most singular point): $P(p_0) = 0$

Nijenhuis Geometry: generic and singular points

Let L be a Nijenhuis operator on M and $x \in \mathrm{M}$.

The algebraic type of $L(p)$ is characterised by the Segre characteristic which includes the information about multiplicities of each eigenvalue together with the sizes of Jordan blocks for each of them.

- $p \in \mathrm{M}$ is *generic* if the algebraic type of L remains the same in a neighbourhood of p, that is, the Segre characteristic of L is locally constant (the multiplicities of the eigenvalues and the sizes of Jordan blocks do not change)
- the set of generic points is open and dense in M
- $p_0 \in \mathrm{M}$ is *singular* if it is not generic, i.e., the Segre characteristic of L changes at the point p_0
- the extreme case (most singular point): $L(p_0) = \lambda\,\mathrm{Id}$.
  We refer to p_0 as a singular point of scalar type.

Poisson geometry: linearisation

Let $P = \left(P^{ij}(x)\right)$ be a Poisson structure, $x \in \mathbb{R}^n$. If $P(0) = 0$, then

$$P(x) = 0 + P_1(x) + P_2(x) + P_3(x) + \dots$$

where the entries of $P_k(x)$ are homogeneous polynomials in $x^1, \dots, x^n$ of degree k.

The linear part $P_{\mathrm{lin}} = P_1 = \left(p^{ij}_k x^k\right)$ satisfies the Jacobi identity and hence is a Poisson structure itself, which is naturally considered as the linearisation of P (at the singular point).

P_{lin} is a Lie-Poisson structure, i.e., p^{ij}_k form the structure tensor of a certain Lie algebra. Conversely, if $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}^*$ carries a natural Poisson structure (Lie-Poisson tensor)

$$P_{\mathfrak{g}} = \left(c^k_{ij} x_k\right)$$

Linearisation of a Poisson structure $=$ Lie algebra

or, more or less equivalently,

Linear Poisson structures $=$ Lie-Poisson structures

Nijenhuis geometry: linearisation

Let $L = \left(L^i_j(x)\right)$ be a Nijenhuis operator, $x \in \mathbb{R}^n$ and $x = 0$ be a singular point of *scalar type*, that is, $L(0) = \lambda \cdot \mathrm{Id}$.
Assume (w.l.o.g.) that $\lambda = 0$. Then locally:

$$L(x) = 0 + L_1(x) + L_2(x) + L_3(x) + \dots$$

where the entries of $L_k(x)$ are homogeneous polynomials in $x^1, \dots, x^n$ of degree k.

Proposition (Definition)

The linear part $L_{\mathrm{lin}} = L_1 = \left(l^i_{jk} x^j\right)$ is itself a Nijenhuis operator that is called the *linearisation* of L at the point $x = 0$.

Question. Are there any special properties of Nijenhuis operators $L_{\mathrm{lin}} = \left(l^i_{jk} x^j\right)$ whose components are linear in local coordinates?

Answer. The corresponding tensor l^i_{jk} defines a structure of a left-symmetric algebra $(\mathfrak{a}_L, *)$ on $T_p\mathrm{M}$, and the converse is also true.

Left-symmetric algebras

Recall that an algebra $\mathfrak{a}$, in a very general context, is a vector space with a bilinear operation $* : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$. Different types of algebras: commutative, associative, unital, Lie, Frobenius, etc.

Left-symmetric algebras can be understood as a generalisation of associative algebras. Namely, the associativity condition[1]

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = 0$$

is replaced with a weaker condition as follows.

Definition

An algebra $(\mathfrak{a}, *)$ is called *left-symmetric* if:

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = \eta * (\xi * \zeta) - (\eta * \xi) * \zeta,$$

for all $\xi, \eta, \zeta \in \mathfrak{a}$.

In particular, every associative algebra is left-symmetric. But there are many other examples.

[1] The left-hand side $\mathcal{A}(\xi, \eta, \zeta) = \xi * (\eta * \zeta) - (\xi * \eta) * \zeta$ is called *associator*.

More examples

Example

Consider the two-dimensional algebra $\mathfrak{a} = \mathrm{Span}(e_1, e_2)$ with the relations

$$\begin{aligned} e_1 * e_1 &= e_2 \\ e_1 * e_2 &= 0 \\ e_2 * e_1 &= -e_1 \\ e_2 * e_2 &= -2e_2 \end{aligned}$$

This algebra is left-symmetric, but not associative.

Example

Consider functions f on the real line $\mathbb{R}$ with coordinate x and introduce the following operation

$$f * g = f\, g_x,$$

where g_x denotes the derivative in x. The associator

$$\begin{aligned} f * (g * h) - (f * g) * h = f * (g\, h_x) - (f\, g_x) * h = \\ = f\, g_x h_x + fg\, h_{xx} - f\, g_x h_x = fg\, h_{xx}. \end{aligned}$$

is symmetric w.r.t. f and g. Obviously, this operation is not associative, but left-symmetric.

More examples

Example

Let M be a manifold with a flat symmetric connection ∇ and define the operation on vector fields as

$$\xi * \eta = \nabla_\xi \eta.$$

The associator takes the form

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_{\nabla_\xi \eta} \zeta.$$

The condition $\mathcal{A}(\xi, \eta, \zeta) = \mathcal{A}(\eta, \xi, \zeta)$ takes the form

$$\begin{aligned} \nabla_\xi \nabla_\eta \zeta - \nabla_{\nabla_\xi \eta} \zeta - \nabla_\eta \nabla_\xi \zeta + \nabla_{\nabla_\eta \xi} \zeta &= \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{(\nabla_\eta \xi - \nabla_\xi \eta)} \zeta = \\ &= \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi, \eta]} \zeta = 0. \end{aligned}$$

Relationship between LSAs and Nijenhuis operators.

Theorem (Winterhalder)

An operator of the form $L = \left(l^i_{jk} x^j \right)$ is Nijenhuis if and only if l^i_{jk} form the structure constants of a left-symmetric algebra, i.e., the operation

$$\xi * \eta = \sum_{i,j,k} l^i_{jk} \xi^j \eta^k e_i, \quad \xi = \xi^j e_j,\, \eta = \eta^k e_k,$$

defines the structure of a left symmetric algebra on the vector space $\mathfrak{a}$ with a basis $e_1, \dots, e_n$.

Linearisation of a Nijenuis structure $\simeq$ left-symmetric algebra

or, more or less equivalently,

Nijenhuis operators with linear entries $l^i_{jk} x^k$ $\simeq$ left-symmetric algebras

One property of left-symmetric algebras

Recall the following fundamental property of associative algebras.

Proposition

Every associative algebra $(\mathfrak{a}, *)$ carries a natural Lie algebra structure $(\mathfrak{a}, [\ ,\ ])$, namely

$$[\xi, \eta] = \xi * \eta - \eta * \xi.$$

In fact, the associativity condition can be relaxed.

Proposition

Every left-symmetric algebra $(\mathfrak{a}, *)$ carries a natural Lie algebra structure $(\mathfrak{a}, [\ ,\ ])$, namely

$$[\xi, \eta] = \xi * \eta - \eta * \xi.$$

For this reason, left-symmetric algebras are also known under the name *pre-Lie algebras.*

Proof (exercise)

The operation is bilinear and skew-symmetric, thus, the only thing we need to prove is the Jacobi identity. For arbitrary triple $\xi, \eta, \zeta \in \mathfrak{a}$ we have

$$\begin{aligned}
&[\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]] + [\zeta,[\xi,\eta]] = \\
=&[\xi, \eta * \zeta - \zeta * \eta] + [\eta, \zeta * \xi - \xi * \zeta] + [\zeta, \xi * \eta - \eta * \xi] = \\
=&\xi * (\eta * \zeta) - \xi * (\zeta * \eta) - (\eta * \zeta) * \xi + (\zeta * \eta) * \xi + \\
+&\eta * (\zeta * \xi) - \eta * (\xi * \zeta) - (\zeta * \xi) * \eta + (\xi * \zeta) * \eta + \\
+&\zeta * (\xi * \eta) - \zeta * (\eta * \xi) - (\xi * \eta) * \zeta + (\eta * \xi) * \zeta = \\
=&\mathcal{A}(\xi,\eta,\zeta) - \mathcal{A}(\xi,\zeta,\eta) + \mathcal{A}(\eta,\zeta,\xi) - \mathcal{A}(\eta,\xi,\zeta) + \\
+&\mathcal{A}(\zeta,\xi,\eta) - \mathcal{A}(\zeta,\eta,\xi)
\end{aligned}$$

We see, that the Jacobi condition is an alternated sum of associators (this holds for arbitrary algebra). Thus, the left-symmetry (as well as right-symmetry or symmetry in first and third argument) yields zero: the corresponding terms cancel out.

Poisson: Linearisation and non-degeneracy problems

Definition

Let P be a Poisson bivector and $P(p_0) = 0$ so that $P(x) = P_{\mathrm{lin}}(x) + \dots$, where P_{lin} is the linear part of P.
We will say that P is *linearisable* at p if there exists a coordinate transformation that reduces P to its linear part P_{lin}.

Linearisation problem. Given P such that $P(p_0) = 0$, find out whether P is linearisable or not?

Definition

A Lie algebra $\mathfrak{g}$ is called *non-degenerate* if *every* Poisson structure P, whose linearisation P_{lin} 'coincides' with $\mathfrak{g}$ at a singular point, is linearisable at this point.

Non-degeneracy problem. Describe all non-degenerate Lie algebras.

Nijenhuis: Linearisation and non-degeneracy problems

Definition

Let L be a Nijenhuis operator and $L(p_0) = 0$ so that $L(x) = L_{\mathrm{lin}}(x) + \dots$, where L_{lin} is the linear part of L.
We will say that L is *linearisable* at p if there exists a coordinate transformation that reduces L to its linear part L_{lin}.

Linearisation problem. Given L such that $L(p) = 0$, find out whether L is linearisable or not?

Definition

A left-symmetric algebra $\mathfrak{a}$ is called *non-degenerate* if any Nijenhuis operator L, whose linearisation 'coincides' with $\mathfrak{a}_L$ at a singular point $p \in \mathrm{M}$, is linearisable at this point.

Non-degeneracy problem. Describe all non-degenerate left-symmetric algebras.

From Poisson geometry . . .

Some well known facts:

- In dimension 2, let $\{x, y\} = f(x, y)$, with $f(0, 0) = 0$, $\mathrm{d}\, f(0, 0) \neq 0$. Then this Poisson structure is linearisable, i.e., there exist local coordinates $\tilde{x}, \tilde{y}$ such that $\{\tilde{x}, \tilde{y}\} = \tilde{y}$. In other words, the non-commutative 2-dim Lie algebra (defined by the relation $[e_1, e_2] = e_2$) is non-degenerate.
- Let P be a (smooth) Poisson structure in $\mathbb{R}^3(x, y, z)$ such that

  $$\{x, y\} = z + \dots, \quad \{y, z\} = x + \dots, \quad \{z, x\} = y + \dots$$

  where 'dots' denote higher order terms. Then there exists another coordinate system $\tilde{x}, \tilde{y}, \tilde{z}$ such that

  $$\{\tilde{x}, \tilde{y}\} = \tilde{z}, \quad \{\tilde{y}, \tilde{z}\} = \tilde{x}, \quad \{\tilde{z}, \tilde{x}\} = \tilde{y}.$$

  In other words, P is always linearisable, i.e., the Lie algebra $so(3)$ is non-degenerate (in the smooth sense).

- Let P be a (smooth) Poisson structure in $\mathbb{R}^3(x, y, z)$ such that

$$\{x, y\} = 2y + \dots, \quad \{x, z\} = -2z + \dots, \quad \{y, z\} = x + \dots$$

  where 'dots' denote higher order terms. Then there exists another coordinate system $\tilde{x}, \tilde{y}, \tilde{z}$ such that

$$\{\tilde{x}, \tilde{y}\} = 2\tilde{y}, \quad \{\tilde{x}, \tilde{z}\} = -2\tilde{z}, \quad \{\tilde{y}, \tilde{z}\} = \tilde{x}.$$

  In other words, P is linearisable, i.e., the Lie algebra $sl(2)$ is non-degenerate in the real analytic sense
  (but not in the smooth sense!).
- Every semisimple Lie algebra is non-degenerate in the real analytic sense (bit not necessarily in the smooth sense).
- Every compact Lie algebra is non-degenerate in the smooth sense.

Classification of LSAs in dimension one

Theorem (Exercise)
There are two non-isomorphic left-symmetric algebras $\mathfrak{a} = \mathrm{Span}(\eta)$ in dimension 1 defined by the relations:

1. $\eta * \eta = 0$ (trivial algebra);
2. $\eta * \eta = \eta$ (non-trivial algebra)

Indeed, a one-dimensional algebra is defined by one single relation

$$\eta * \eta = a\eta, \quad a \in \mathbb{R}.$$

If $a \neq 0$, in can be made equal to 1 by rescaling $\eta \mapsto \frac{1}{a}\eta$.

Classification theorem in dim $= 2$

Theorem

Up to isomorphism there are two continuous families and 10 exceptional two dimensional real left-symmetric algebras.
The complete list of normal forms is presented in Table 1 and Table 2.

Comments for Tables 1 and 2. For every algebra we give

- *All non-zero structure relations for a basis η_1, η_2;*
- *The operator $R = R_\eta$ of right multiplication by $\eta = x\eta_1 + y\eta_2$ (this is a Nijenhuis operator!)*
- *The operator $L = L_\eta$ of left multiplication by $\eta = x\eta_1 + y\eta_2$.*

The $\mathfrak{b}$-series corresponds to non-Abelian algebras and $\mathfrak{c}$-series to Abelian algebras.

Classification theorem: Table 1

Name	Structure relations	L	R
$\mathfrak{b}_{1,\alpha}$	$\eta_2 * \eta_1 = \eta_1,$ $\eta_2 * \eta_2 = \alpha\eta_2$	$\begin{pmatrix} y & 0 \\ 0 & \alpha y \end{pmatrix}$	$\begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$
$\mathfrak{b}_{2,\beta}, \beta \neq 1$	$\eta_1 * \eta_2 = \eta_1,$ $\eta_2 * \eta_1 = \beta\eta_1$ $\eta_2 * \eta_2 = \eta_2$	$\begin{pmatrix} \beta y & x \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} y & \beta x \\ 0 & y \end{pmatrix}$
$\mathfrak{b}_3$	$\eta_2 * \eta_1 = \eta_1,$ $\eta_2 * \eta_2 = \eta_1 + \eta_2$	$\begin{pmatrix} y & y \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} 0 & x + y \\ 0 & y \end{pmatrix}$
$\mathfrak{b}_4^+$	$\eta_1 * \eta_1 = \eta_2,$ $\eta_2 * \eta_1 = -\eta_1$ $\eta_2 * \eta_2 = -2\eta_2$	$\begin{pmatrix} -y & 0 \\ x & -2y \end{pmatrix}$	$\begin{pmatrix} 0 & -x \\ x & -2y \end{pmatrix}$
$\mathfrak{b}_4^-$	$\eta_1 * \eta_1 = -\eta_2,$ $\eta_2 * \eta_1 = -\eta_1$ $\eta_2 * \eta_2 = -2\eta_2$	$\begin{pmatrix} -y & 0 \\ -x & -2y \end{pmatrix}$	$\begin{pmatrix} 0 & -x \\ -x & -2y \end{pmatrix}$
$\mathfrak{b}_5$	$\eta_1 * \eta_2 = \eta_1,$ $\eta_2 * \eta_2 = \eta_1 + \eta_2$	$\begin{pmatrix} 0 & x + y \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} y & y \\ 0 & y \end{pmatrix}$

Classification theorem: Table 2

Name	Structure relations	$L = R$
$\mathfrak{c}_1$		$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathfrak{c}_2$	$\eta_2 * \eta_2 = \eta_2$	$\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$
$\mathfrak{c}_3$	$\eta_2 * \eta_2 = \eta_1$	$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$
$\mathfrak{c}_4$	$\eta_2 * \eta_2 = \eta_2$ $\eta_2 * \eta_1 = \eta_1$ $\eta_1 * \eta_2 = \eta_1$	$\begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$
$\mathfrak{c}_5^+$	$\eta_2 * \eta_2 = \eta_2$ $\eta_2 * \eta_1 = \eta_1$ $\eta_1 * \eta_2 = \eta_1$ $\eta_1 * \eta_1 = \eta_2$	$\begin{pmatrix} y & x \\ x & y \end{pmatrix}$
$\mathfrak{c}_5^-$	$\eta_2 * \eta_2 = \eta_2$ $\eta_2 * \eta_1 = \eta_1$ $\eta_1 * \eta_2 = \eta_1$ $\eta_1 * \eta_1 = -\eta_2$	$\begin{pmatrix} y & x \\ -x & y \end{pmatrix}$

Example of a non-degenerate LSA

Proposition

The left-symmetric algebra $\mathfrak{c}_5^+$ is non-degenerate. Equivalently, if $L = L(x, y)$ is a Nijenhuis operator of the form

$$L = \begin{pmatrix} y & x \\ x & y \end{pmatrix} + \text{higher order} \geqslant 2 \text{ terms} ,$$

Then there exists a smooth coordinate change, centred at 0 that transforms L into its linear part.

Proof. Let $f = \operatorname{tr} L$ and $g = \det L$. Then

$$\begin{aligned} f &= 2y + \dots, \\ g &= y^2 - x^2 + \dots. \end{aligned}$$

Here $\dots$ stand for terms of orders $\geqslant 2$ and $\geqslant 3$ respectively. Instead of reducing L to the required form, we will be looking for a suitable coordinate transformations to simplify f and g.

Step 1. As $f = \operatorname{tr} L = 2y + \dots$, we may set $y_{\rm new} = \frac{1}{2} f$ so that $f = \operatorname{tr} L = 2y_{\rm new}$. The first coordinate remains unchanged. Keeping the same notation x, y for the new coordinates, we now have

$$\begin{aligned} f &= 2y \\ g &= y^2 - x^2 + \dots . \end{aligned}$$

Step 2. To simplify g, we treat y as a parameter and apply the parametric Morse lemma, which says that we can introduce a new variable $x_{\rm new} = h(x, y) = x + \dots$ (y remains unchanged) in such a way that

$$g = -x_{\rm new}^2 + k(y), \quad \text{with } k(y) = y^2 + \dots$$

Keeping the same notation x, y for the new coordinates, we now have

$$\begin{aligned} f &= 2y \\ g &= -x^2 + k(y) \end{aligned}$$

where f and g are the trace and determinant of our Nijenhuis operator L.
Question. What can we say about L in this situation?

Proof of Proposition (continued...)

Recall that a Nijenhuis operator can be reconstructed from the coefficients of the characteristic polynomial by using the following fundamental formula

$$L = J^{-1}\begin{pmatrix}\sigma_1 & 1\\ \sigma_2 & 0\end{pmatrix} J, \quad \text{where } J = \left(\frac{\partial(\sigma_1,\sigma_2)}{\partial(x,y)}\right),$$

and $\sigma_1 = \operatorname{tr} L, \sigma_2 = -\det L$ are the coefficients of the characteristic polynomial of L. A simple computation gives the following formula for L

$$L = \begin{pmatrix} \frac{1}{2}k' & x + \frac{k'(4y-k')-4k}{4x} \\ x & 2y - \frac{1}{2}k' \end{pmatrix}$$

Now another miracle in Nijenhuis geometry... Look at the fraction $\frac{k'(4y-k')-4k}{4x}$. Its numerator is a function of y only. Hence this fraction is smooth at the origin if and only if $k'(4y-k')-4k \equiv 0$. It shows that $k = k(y)$ must be very special!

This relation (after differentiating by y) implies $k''(y)(2y-k') = 0$ and since $k''(y) \neq 0$, we get $k' = 2y$ and $k = y^2$, giving finally $L = \begin{pmatrix} y & x\\ x & y\end{pmatrix}$, as required.

Example of a degenerate LSA

Proposition

The left-symmetric algebra $\mathfrak{c}_4$ is degenerate.
More specifically, the following non-linear perturbation

$$\begin{pmatrix} y & x \\ 0 & y \end{pmatrix} \quad \mapsto \quad L = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} + \begin{pmatrix} yx^2 & x^3 \\ -xy^2 & -yx^2 \end{pmatrix}$$

gives a non-linearisable Nijenhuis operator L.

Proof. Consider the trace and determinant of L:

$$\operatorname{tr} L = 2y, \quad \det L = y^2 + y^2x^2$$

Obviously, the discriminant of the characteristic polynomial $\chi_L(t)$ is not identically zero, so that generically L has two different eigenvalues, while its linear part $L_{\mathrm{lin}} = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$ has one single eigenvalue y of multiplicity 2.

Hence, L and L_{lin} are essentially different and cannot be reduced to each other by a coordinate transformation.

Another classification theorem

Theorem (Smooth case)

In the smooth category

Degenerate LSA	*Non-degenerate LSA*
$\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4,$ $\mathfrak{b}_5, \mathfrak{b}_{2,\beta}$ $\mathfrak{b}_{1,\alpha}$ for $\alpha \in \Sigma_{\mathrm{sm}}$	$\mathfrak{b}_4^+, \mathfrak{b}_4^-, \mathfrak{c}_5^+, \mathfrak{c}_5^-$ $\mathfrak{b}_3, \mathfrak{b}_{1,\alpha}$ for $\alpha \notin \Sigma_{\mathrm{sm}}$

Theorem (Analytic case)

In the real analytic category

Degenerate LSA	*Non-degenerate LSA*
$\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4,$ $\mathfrak{b}_5, \mathfrak{b}_{2,\beta}$ $\mathfrak{b}_{1,\alpha}$ for $\alpha \in \Sigma_{\mathrm{an}}$	$\mathfrak{b}_4^+, \mathfrak{b}_4^-, \mathfrak{c}_5^+, \mathfrak{c}_5^-$ $\mathfrak{b}_3, \mathfrak{b}_{1,\alpha}$ for $\alpha \notin \Sigma_{\mathrm{an}}$

Comments on the sets Σ_{sm} and Σ_{an} related to $\begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$

The continuous fraction for α is a decomposition of α in the form

$$\alpha = q_0 + \cfrac{1}{q_1 + \frac{1}{q_2 + \dots}},$$

where $q_0 \in \mathbb{Z}$ and $q_i, i \geqslant 1$ are in $\mathbb{N}$. Let $[q_0, q_1, q_2, \dots]$ be a decomposition of an irrational α into the continuous fraction. If the series

$$B(x) = \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i}$$

converges, then α is a Brjuno number.

- Σ_{sm} contains $\alpha < 0$, $\alpha = \frac{1}{m}$ for $m \geqslant 2$ and s for $s \geqslant 3$.
- Σ_{an} contains $\alpha = -\frac{p}{q}$, negative irrational numbers that are not Brjuno numbers, $\alpha = \frac{1}{m}$ for $m \geqslant 2$ and s for $s \geqslant 3$.

Non-degeneracy of the diagonal algebra

Theorem (Real analytic or formal)

Let $L(x) = L_{\text{lin}}(x) + L_2(x) + L_3(x) + \dots$ *with*

$$L_{\text{lin}}(x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}.$$

Then $L(x)$ *is linearisable. In other words, the diagonal left-symmetric algebra is non-degenerate.*

Symmetries and conservation laws

Let $A = (A^i_j)$ be an operator (not necessarily Nijenhuis).

Definition

A function f is a *conservation law* for A, if the form $A^* \mathrm{d}\, f$ is closed. (Today all the constructions are local so that this condition is equivalent to the existence of a function g such that $\mathrm{d}\, g = A^* \mathrm{d}\, f$.)

Definition

An operator $B = (B^i_j)$ is called a *strong symmetry* (resp. just *symmetry*) for the operator A, if

- $AB = BA$
- (i) *strong symmetry:*

$$\langle A, B\rangle(\xi, \eta) \overset{\mathrm{def}}{=} A[\xi, B\eta] + B[A\xi, \eta] - [A\xi, B\xi] - AB[\xi, \eta] = 0,$$

  (ii) *symmetry:*

$$\langle A, B\rangle(\xi, \xi) = A[\xi, B\xi] + B[A\xi, \xi] - [A\xi, B\xi] = 0.$$

Symmetries and conservation laws in dynamical systems

Instead of an operator A, consider a vector field ξ and the corresponding dynamical system

$$\frac{\mathrm{d}\,u}{\mathrm{d}\,t} = \xi(u) \qquad \text{in more detail:} \quad \frac{\mathrm{d}\,u^i}{dt} = \xi^i(u^1, \ldots, u^n). \tag{1}$$

Definition

A *first integral* of ξ (or of the dynamical system (1)) is a function f such that $\xi(f) = \sum \xi^i \frac{\partial f}{\partial u^i} = 0$. In other words, $\mathcal{L}_\xi f = 0$.

Definition

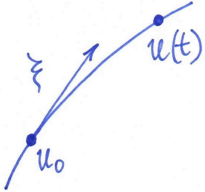
A *symmetry* (or *symmetry field*) of the system (1) is a vector field η such that $[\xi, \eta] = 0$. In other words, $\mathcal{L}_\xi \eta = 0$.

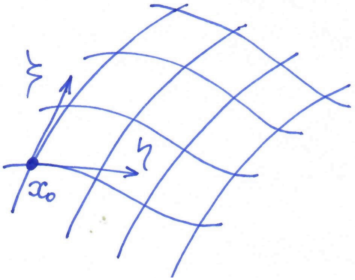
Properties.

- If η is a symmetry and f is an integral $\Rightarrow$ $f\eta$ is a symmetry and $\eta(f)$ is an integral.
- Let $\eta_1, \ldots, \eta_k$ be linearly independent symmetries.
  The linear combination $\eta = \sum f_i \eta_i$ is a symmetry if and only if $f_1, \ldots, f_k$ are first integrals.

Comments

The equation $\dfrac{\mathrm{d}\,u}{\mathrm{d}\,t} = \xi(u)$ describes an evolution of a point on the manifold.



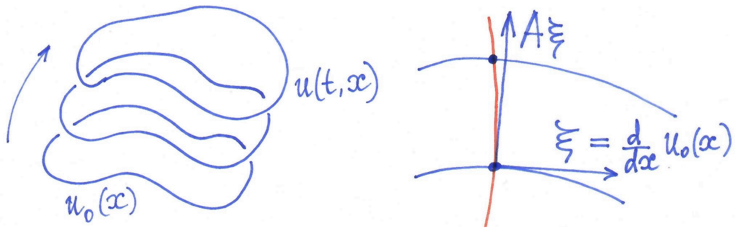


f is an integral $\Leftrightarrow$
$f(u(t)) = f(u_0)$

η is a symmetry $\Leftrightarrow$
for any point u_0
there is a surface $u(t,s)$
such that $\dfrac{\partial u}{\partial t} = \xi$, $\dfrac{\partial u}{\partial s} = \eta$

Comments

The equation $\dfrac{\partial u}{\partial t} = A(u)\dfrac{\partial u}{\partial x}$ describes an evolution of curves on the manifold.



- f is a conservation law $\Leftrightarrow \oint f(u(x,t))\mathrm{d}\,x = \oint f(u_0(x))\mathrm{d}\,x$, i.e. does not change under evolution.
- B is a symmetry $\Leftrightarrow$ for any initial curve $u_0(x)$ there is a "surface in the space of curves $u(t,s,x)$ such that

$$\frac{\partial u}{\partial t} = A(u)\frac{\partial u}{\partial x} \qquad \frac{\partial u}{\partial s} = B(u)\frac{\partial u}{\partial x}\,.$$

Symmetries and conservation laws for Nijenhuis operators

- Splitting: If $L = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix} = L_1(x) \oplus L_2(y)$ (with no common eigenvalues), then every conservation law splits as $f = f_1(x) + f_2(y)$. Similarly, every symmetry has the form
$$S = \begin{pmatrix} S_1(x) & 0 \\ 0 & S_2(y) \end{pmatrix} = S_1(x) \oplus S_2(y).$$
- If L is diagonal with distinct eigenvalues, e.g. $L(x) = \mathrm{diag}(x_1, ..., x_n)$, then its conservation laws are $f(x) = \sum f_i(x_i)$ and symmetries have the form $S = \mathrm{diag}\big(s_1(x_1), \dots, s_n(x_n)\big)$
- If L is gl-regular, then
    - every symmetry of L is automatically strong;
    - the space $\mathrm{Sym}(L)$ of the symmetries L is closed under pointwise multiplication;
    - any two $S_1, S_2 \in \mathrm{Sym}(L)$ are strong symmetries of each other;
    - if f is a conservation law of L, then f is a conservation law for all $S \in \mathrm{Sym}(L)$.
- There is an explicit description of conservation laws and symmetries for Nijenhuis Jordan blocks, and therefore for all gl-regular operators at generic points.