Integrable Systems: from Poisson to Nijenhuis
Lecture 4

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Plan

- Integrable geodesic flows with integral quadratic in momenta
- Quadratic integrals and Killing tensors
- Stäckel construction
- Nijenhuis operators and related objects
- Frobenius algebras
- Mixing Nijenhuis and Frobenius
- New general construction of integrable systems with quadratic integrals
- Inverse theorem

Integrable geodesic flows with integral quadratic in momenta

- (M, g) (pseudo-)Riemannian manifold
- Geodesic flow of g:
  Hamiltonian system on T^*M with
  $$H = \tfrac{1}{2} h^{ij}(x) p_i p_j = h(p, p), \quad h = \tfrac{1}{2} g^{-1}$$
  $x = (x^1, \dots, x^n)$ and $p = (p_1, \dots, p_n)$ are canonical coordinates on the cotangent bundle T^*M
- Liouville Integrability:
  n independent Poisson commuting integrals
  $F_1 = H, F_2 \dots, F_n : T^*M \to \mathbb{R}$

Example: C. G. J. Jacobi, Lectures on Dynamics, 1842-43

Geodesic flow on the ellipsoid $\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{z^2}{c} = 1$

Euclidean metric in elliptic coordinates (λ, μ, ν) in $\mathbb{R}^3$:

$$\mathrm{d}\, s^2_{\mathbb{R}^3} = \frac{(\mu-\lambda)(\nu-\lambda)\ \mathrm{d}\,\lambda^2}{4(a-\lambda)(b-\lambda)(c-\lambda)} + \frac{(\nu-\mu)(\lambda-\mu)\ \mathrm{d}\,\mu^2}{4(a-\mu)(b-\mu)(c-\mu)} + \frac{(\mu-\nu)(\lambda-\nu)\ \mathrm{d}\,\nu^2}{4(a-\nu)(b-\nu)(c-\nu)}$$

The ellipsoid is given by $\lambda = 0$. Hence

$$\mathrm{d}\, s^2_{\mathrm{ell}} = (\mu-\nu)\left(\frac{\mu\ \mathrm{d}\,\mu^2}{4(a-\mu)(b-\mu)(c-\mu)} - \frac{\nu\ \mathrm{d}\,\nu^2}{4(a-\nu)(b-\nu)(c-\nu)}\right)$$

$$H = \frac{f(\mu)\, p_\mu^2 - f(\nu)\, p_\nu^2}{\mu-\nu}, \quad f(t) = \frac{4(a-t)(b-t)(c-t)}{t}.$$

$$F = \frac{\mu\, f(\nu)\, p_\nu^2 - \nu\, f(\mu)\, p_\mu^2}{\mu-\nu}.$$

Notice that this integral F is quadratic in momenta, and this works for any $f(\cdot)$, moreover we can take two different functions $f_1(\mu)$ and $f_2(\nu)$.

Quadratic integrals and Killing tensors

We are interested in Hamiltonian systems admitting n Poisson commuting integrals of the form

$$\boxed{F_s = h_s(p,p) = \sum h_s^{ij}(x) p_i p_j = h(K_s^* p, p), \quad h = h_1} \tag{1}$$

where K_s is an operator, known as Killing $(1,1)$-tensor of the metric g. Here $g = \frac{1}{2} h^{-1}$, $F_1 = h(p,p)$ and $K_1 = \mathrm{Id}$.

Equivalently, we want to describe metrics g admitting (or, together with) n Killing $(1,1)$-tensors

$$K_1 = \mathrm{Id},\ K_2,\ \dots,\ K_n$$

such that the corresponding quadratic functions $F_s = g^{-1}(K_s^* p, p)$ Poisson commute (and are independent).

Terminology: Killing $(1,1)$-tensor $K = (K^i_j)$ means that
$F = g^{-1}(K^* p, p)$
is a first integral of the geodesic flow. In a more classical setting, one usually consider $K_{ij} = g_{i\alpha} K^\alpha_j$. Then the Killing condition is:
$\nabla_i K_{jk} + \nabla_j K_{ki} + \nabla_k K_{ij} = 0$.

Stäckel construction, 1891

Consider the following matrix formula

$$\begin{pmatrix} s_{11}(x_1) & s_{12}(x_1) & \dots & s_{1n}(x_1) \\ s_{21}(x_2) & s_{22}(x_2) & \dots & s_{2n}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1}(x_n) & s_{n2}(x_n) & \dots & s_{nn}(x_n) \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} f_1(x_1)p_1^2 \\ f_2(x_2)p_2^2 \\ \vdots \\ f_n(x_n)p_n^2 \end{pmatrix}$$

or shortly $SF = P$, where S is known as Stäckel matrix.

Theorem

If $F_1, \dots, F_n$ satisfy this relation, i.e. $F = S^{-1}P$, then they Poisson commute.

Explanation: The common level surface $\mathcal{X}_c = \{F_\alpha(x,p) = c_\alpha\}$ of the integrals is given by the equations of the form

$$f_i(x_i)p_i^2 = \sum_\alpha s_{i\alpha}(x_i)c_\alpha, \quad i = 1, \dots, n, \quad c_\alpha \in \mathbb{R}.$$

The variables are separated, implying immediately that $\mathcal{X}_c$ is Lagrangian for all c. This is equivalent to the fact that $F_1, \dots, F_n$ Poisson commute.

Properties of Stäckel integrals

$$\begin{pmatrix} s_{11}(x_1) & s_{12}(x_1) & \dots & s_{1n}(x_1) \\ s_{21}(x_2) & s_{22}(x_2) & \dots & s_{2n}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1}(x_n) & s_{n2}(x_n) & \dots & s_{nn}(x_n) \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} f_1(x_1)p_1^2 \\ f_2(x_2)p_2^2 \\ \vdots \\ f_n(x_n)p_n^2 \end{pmatrix}$$

Properties of $F_1, \dots, F_n$:

- Poisson commute
- quadratic in momenta
- diagonal in coordinates $x_1, \dots, x_n$
- K_i algebraically commute, $K_i K_j = K_j K_i$

Ellipsoid again

Example

For $H = \frac{f(\mu)\,p_\mu^2 - f(\nu)\,p_\nu^2}{\mu-\nu}$ and $F = \frac{\mu\,f(\nu)\,p_\nu^2 - \nu\,f(\mu)\,p_\mu^2}{\mu-\nu}$, the Stäckel matrix is $S = \begin{pmatrix} \mu & 1 \\ \nu & 1 \end{pmatrix}$ so that we have

$$\begin{pmatrix} \mu & 1 \\ \nu & 1 \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} = \begin{pmatrix} f(\mu)p_\mu^1 \\ f(\nu)p_\nu^2 \end{pmatrix}, \qquad f(t) = \frac{4(a-t)(b-t)(c-t)}{t}.$$

For the n-dimensional ellipsoid, everything is basically the same:
S is the Vandermonde matrix:

$$S = \begin{pmatrix} \lambda_1^{n-1} & \dots & \lambda_1 & 1 \\ \lambda_2^{n-1} & \dots & \lambda_2 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_n^{n-1} & \dots & \lambda_n & 1 \end{pmatrix}$$

and $f(t) = \frac{\prod_{i=1}^{n+1}(a_i - t)}{t}$.

Some historical remarks

L. Eisenhart in 1934 asked the question under which conditions the existence of Poisson commuting integrals of the form (1) would imply that the integrals and the metric come from the Stäckel construction, and gave first results in this direction.

The question was answered in full generality by E. Kalnins and W. Miller in 1980. They proved that linearly and functionally independent Poisson commuting integrals (1) necessarily come from the Stäckel construction if one assumes that at every point $p \in \mathrm{M}$ there exists a basis in $T_p\mathrm{M}^n$ in which the operators K_s are all diagonal.

Question. But what about non-diagonal case? For pseudo-Riemannian metrics, diagonalisability is not a natural assumption.

Examples of Nijenhuis operators

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$

Diagonal

$$\begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_n & \dots & x_2 & x_1 \end{pmatrix}$$

Toeplitz

$$\begin{pmatrix} x_1 & 1 & & \\ x_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ x_n & 0 & \dots & 0 \end{pmatrix}$$

Companion form

$$\begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & 0 & \ddots & \\ x_n & \dots & x_2 & x_1 \end{pmatrix}$$

???

Frobenius algebras

$(\mathfrak{a}, \star)$ vector space with multiplication $\star$

- commutative
- associative
- with unity
- admits a non-degenerate invariant symmetric bilinear form b

$$b\,(\xi \star \eta, \zeta) = b\,(\eta, \xi \star \zeta).$$

Each Frobenius algebra of dimension n admits a faithful representation $\mathrm{R} : \mathfrak{a} \to \mathrm{gl}(n, \mathbb{R})$:

$$\xi \mapsto \mathrm{R}_\xi = \text{operator of (right) multiplication by } \xi$$

Thus, we can think of $\mathfrak{a}$ as an n-dimensional commutative subalgebra in $\mathrm{gl}(n, \mathbb{R})$. This subalgebra $\mathfrak{a} \subset \mathrm{gl}(n, \mathbb{R})$ satisfies two important properties:

- $\mathfrak{a}$ is a maximal commutative subalgebra
- there exists an inner product g on $\mathbb{R}^n$ w.r.t. which all $A \in \mathfrak{a}$ are self-adjoint

Dual bases: We say that two bases $K_1, \dots, K_n \in \mathfrak{a}$ and $M^1, \dots, M^n \in \mathfrak{a}$ are *dual* w.r.t. the Frobenius form b if $b(M^i, K_j) = \delta^i_j$.

Examples of Frobenius algebras

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \quad \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_n & \dots & x_2 & x_1 \end{pmatrix} \quad \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & 0 & \ddots & \\ x_n & \dots & x_2 & x_1 \end{pmatrix}$$

Not an accident!

Theorem

Let $\mathfrak{a}$ be a Frobenius algebra, $e_1, \dots, e_n$ a basis of $\mathfrak{a}$ and $A_i = \mathrm{R}_{e_i}$ be the corresponding multiplication operators. Then

$$A = \sum_i x_i A_i$$

is a Nijenhuis operator on $\mathbb{R}^n(x_1, \dots, x_n)$.

Important example. Let L be a gl-regular operator ($n \times n$ constant matrix). Then $\mathfrak{a} = \mathrm{Span}(\mathrm{Id}, L, \dots, L^{n-1})$ is an n-dimensional Frobenius algebra.

Combining Frobenius and Nijenhuis

Let $M_1, \dots, M_n$ be linearly independent Nijenhuis operators such that $\mathrm{Span}_{\mathbb{R}}(M_1(x), \dots, M_n(x)) \subset \mathrm{gl}(T_x\mathrm{M})$ is a Frobenius subalgebra.
In addition, assume that $\langle M_i, M_j \rangle = 0$.

Examples:

- $M_k = L^{k-1}$, $k = 1, \dots, n$, where L is a gl-regular Nijenhuis operator;
- $M_k = \mathrm{R}_{e_i} = \mathrm{const}$, where $e_1, \dots, e_n$ is a basis of a Frobenius algebra.

Symmetry algebra of $M_1, \dots, M_n$:

$$\mathsf{Sym} = \mathsf{Sym}(M_1, \dots, M_n) = \{\text{all common symmetries of } M_i\text{'s}\}.$$

Fundamental facts about Sym

- Sym is an infinite-dimensional commutative associative algebra under usual matrix multiplication
- elements of Sym are all Nijenhuis
- Sym admits many common conservation laws, moreover there exists $\alpha = \mathrm{d}\, f$ such that $M_1^*\alpha, \dots, M_n^*\alpha$ are closed and linearly independent (at least in the real analytic case)
- if $M_1', \dots, M_n' \in \mathsf{Sym}$ are linearly independent pointwise, then $\mathsf{Sym}(M_1', \dots, M_n') = \mathsf{Sym}(M_1, \dots, M_n)$

Example: diagonal case

Let $M_1, \dots, M_n$ be diagonal Nijenhuis operators. Then

$$\mathrm{Sym} = \left\{ \begin{pmatrix} f_1(x_1) & & & \\ & f_2(x_2) & & \\ & & \ddots & \\ & & & f_n(x_n) \end{pmatrix}, \ f_i \in C^\infty \right\}$$

Common conservation laws are

$$\alpha = \mathrm{d}\, g, \qquad g = g_1(x_1) + g_2(x_2) + \dots + g_n(x_n)$$

Notice that

$$\mathrm{Sym} = \mathrm{Sym}(D_1, \dots, D_n),$$

where $D_1, \dots, D_n$ are constant diagonal matrices. Moreover, $\mathrm{Sym} = \mathrm{Sym}(D)$, where D is a single constant diagonal matrix with simple spectrum.

A symmetry algebra Sym is called flat if it admits a basis that consists of constant operators (in a certain coordinate system). Flat symmetry algebras are in one-to-one correspondence with Frobenius algebras.

General construction and duality

We study Poisson commuting functions $F_s : T^*\mathrm{M} \to \mathbb{R}$, $s = 1, \dots, n$, quadratic in momenta.

$$\boxed{F_s = h_s(p,p) = \sum h_s^{ij}(x) p_i p_j = h(K_s^* p, p), \quad h = h_1}$$

They can be naturally identified with symmetric bilinear forms h_s on $T^*\mathrm{M}^n$:

$$F_s(x,p) = \sum_{i,j} h_s^{ij}(x) p_i p_j = h_s(p,p)$$

If h_1 is non-degenerate, then we can interpret it as a contravariant (pseudo-)Riemann metric and the functions can be equivalently written in the form $F_s = h_1(K_s^* p, p) = g^{-1}(K_s^* p, p)$, where the operators

$$K_s = h_s h_1^{-1}. \tag{2}$$

are Killing $(1,1)$-tensors of the metric h_1. The next theorem provides a natural method for constructing such functions from a symmetry algebra Sym.

Stäckel-type theorem in the non-diagonal case

Theorem

Let Sym *be a symmetry algebra, $M^1, \dots, M^n$ be a basis of* Sym, *and α be a common conservation law of M^i's such that $M^{1*}\alpha, \dots, M^{n*}\alpha$ are linearly independent pointwise. Consider the canonical coordinate system $u^1, \dots, u^n, p_1, \dots, p_n$ on $T^*\mathrm{M}$ such that $\mathrm{d}\, u^i = M^{i*}\alpha$, $i = 1, \dots, n$ and set, as before, $M^i \cdot M^j = a^{ij}_s M^s$, $a^{ij}_s = a^{ij}_s(u)$. Then*

1. *The quadratic in momenta functions*

$$F_s(u, p) = a^{ij}_s(u) p_i p_j, \quad s = 1, \dots, n,$$

   pairwise Poisson commute with respect to the canonical Poisson structure on $T^*\mathrm{M}$;

2. *The functions $F_s(u, p)$ satisfy the condition*
   $\det\left(\frac{\partial F}{\partial p}\right) \neq 0$ *for a generic $p \in T^*_x\mathrm{M}$;*

3. *If $h = \left(a^{ij}_1\right)$ is non-degenerate, then $F_s = h(K^*_s p, p)$ and the corresponding Killing tensors K_s given are dual to M^i with respect to $\left(a^{ij}_1\right)$ treated as a Frobenius form, that is, $M^i = h^{is} K_s$.*

Inverse theorem

Theorem

Consider a collection of quadratic in momenta functions

$$F_s(x,p) = \sum_{i,l} h_s^{ij}(x) p_i p_j = h(K_s^* p, p), \quad s = 1, \dots, n, \quad K_1 = \mathrm{Id}.$$

Suppose that they satisfy the following conditions:

(i) *Functions F_s pairwise commute with respect to the canonical Poisson bracket on $T^*\mathrm{M}$;*

(ii) $\det\left(\frac{\partial F}{\partial p}\right) \neq 0$ *for generic p;*

(iii) $K_i \cdot K_j = K_j \cdot K_i$.

Then the functions F_s are obtained via the procedure described in Theorem 1 for an appropriate choice of a symmetry algebra Sym, *basis* $M_1, \dots, M_n \in \mathrm{Sym}$, *and a common conservation law α.*

Moreover... Let $K_i \cdot K_j = \sum a_{ij}^s K_s$. Assume that $b_{ij} = a_{ij}^1$ is non-degenerate. Then we can think of $b = \left(b_{ij}\right)$ as a Frobenius form on $\mathrm{Span}(K_1, \dots, K_n)$. Then $\mathrm{Sym} = \mathrm{Sym}(M^1, \dots, M^n)$, where $M^i = b^{ij} K_j$ are dual to K_j. Each M^j is Nijenhuis!

References (BKM = AB, A. Konyaev, V. Matveev)

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