# Integrable deformation of cluster map

Wookyung Kim

Supervised by:
Jan Grabowski (Lancaster University)
Andrew Hone (University of Kent)

Lancaster University

Conference FDIS 2023

# Cluster algebra (coefficient free case)

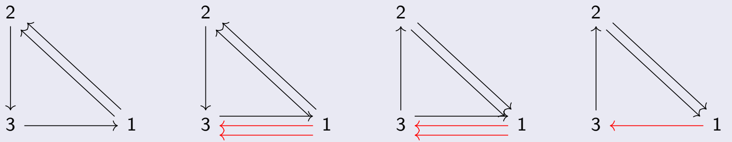
- Cluster algebras are a class of commutative algebras, which were introduced by Fomin and Zelevinsky
- Initial data formed by
  1. $N$-tuple of distinguished generators $\mathbf{x} = (x_1, x_2, \ldots x_N)$ in a field of rational functions $\mathcal{F}$. Generators $x_i$ are called **cluster variables**
  2. A **quiver**, $Q$, which is a directed graph with $N$ vertices. (In cluster algebras, we are considering the quiver, which does not possess 2-cycles or loops)

  The pair $(\mathbf{x}, Q)$ is called an **initial seed**.
- A cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is a subalgebra of the field $\mathcal{F}$ whose generators obtained from $\mathbf{x}$ by the special iterative process called **mutation**, $\mu_k$
- The mutation consists of two parts: **Quiver mutation** and **Cluster mutation**

Quiver mutation at node $k$ consists of three steps

1. For each subquiver $i \xrightarrow{p} k \xrightarrow{q} j$, insert an edge $i \xrightarrow{pq} j$
2. Reverse all arrows which are connected to $k$, $j \xrightarrow{q} k \xrightarrow{p} i$
3. Remove any 2-cycles which are formed by inserting arrows.

**Example (Quiver mutation at node 2)**



# Cluster mutations

- A quiver $Q$ can be identified with an $N \times N$ skew-symmetric matrix, $B$, known as **exchange matrix**. The exchange matrix also transforms into a new exchange matrix under the mutation:

$$(B')_{ij} = (\mu_k(B))_{ij} = \begin{cases} -b_{ij} & \text{if } \ i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise} \end{cases}$$

- **Cluster mutations**: Given an initial cluster which is n-tuple $\mathbf{x} = (x_1, \ldots x_N)$. The cluster mutation in the direction $k$ is $\mu_k(\mathbf{x}) = (x_1, \ldots, x_{k-1}, x_k', \ldots, x_N)$ where the new generator $x_k'$ is defined by the expression

$$\mu_k(x_k) = \frac{\prod_{i=1}^N x_i^{[b_{ik}]_+} + \prod_{i=1}^N x_i^{[-b_{ik}]_+}}{x_k}, \quad [\pm b_{ik}]_+ = \max(0, \pm b_{ik})$$

This expression is known as a (coefficient free) **exchange relation**.

# Periodicity and Laurent phenomenon

## Definition (Mutation periodic)

*The quiver Q with N nodes is* **mutation periodic** *with period m if*

$$\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1}(Q) = \rho^m(Q)$$

*for $N \geq m$, where $\rho : (1, 2, \ldots, N) \to (N, 1, 2, \ldots, N-1)$ is the cyclic permutation.*

The quiver corresponding to type $A_2$ is mutation periodic with period 1 as $\mu_1(Q) = \rho(Q)$. Define $\varphi = \rho^{-1}\mu_1$, then $\varphi(Q) = Q$. We refer the map $\varphi$ as **cluster map**.

## Theorem (Laurent phenomenon)

*Every cluster variable generated by the exchange relation is a Laurent polynomial in the initial variables.*

- The cluster map $\varphi$ can be considered as a discrete dynamical system. The trajectory of the system is represented by the iteration of the map
- Gekhtman, Shapiro and Vainshtein introduced a log-canonical Poisson bracket, $\{x_i, x_j\} = P_{ij}x_ix_j$, where matrix $P$ (Poisson tensor) satisfies $PB = \lambda D$
- It is compatible with cluster algebra in the sense that $\mu_k : ((x_1, \ldots, x_N), B) \to \left((\tilde{x}_1, \ldots, \tilde{x}_N), \tilde{B}\right)$ yields $\{\tilde{x}_i, \tilde{x}_j\} = \tilde{P}_{ij}\tilde{x}_i\tilde{x}_j$
- The $\varphi$ is a poisson map $\left( i.e.\ \varphi^*(\{x_i, x_j\}) = \{\varphi(x_i), \varphi(x_j)\} \iff P = \varphi(P)\right)$ *as* $\varphi(B) = B$

## Definition

*Let $P$ be Poisson tensor of rank $2r$. Then $\varphi$ is* **integrable** *if there exists*

1. *$N - 2r$ Casimir functions $\mathcal{C}_k$, i.e. $\varphi^*(\mathcal{C}_k) = \mathcal{C}_k$ satisfying $\{\mathcal{C}_k, f(\mathbf{x})\} = 0$ for all function $f(\mathbf{x})$*
2. *$r$ first integrals $h_j$, $j = 1, \ldots, r$ ($\varphi^*(h_j) = h_j$) such that $\left\{h_i, h_j\right\} = 0$*

# Presymplectic form

- In cluster algebras, there exists a skew-symmetric bilinear form, written in the log-canonical form

$$\omega = \sum_{i<j} \frac{b_{ij}}{x_i x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j = \sum_{i<j} b_{ij} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j$$

  where $b_{ij}$ are entries of the exchange matrix $B$ associated to the quiver $Q$

- It is compatible with cluster algebra i.e. $\mu_k : ((x_1, \dots, x_N), B) \to \left((\tilde{x}_1, \dots, \tilde{x}_N), \tilde{B}\right)$ yields

$$\tilde{\omega} = \sum_{i<j} \frac{\tilde{b}_{ij}}{\tilde{x}_i \tilde{x}_j} \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j$$

- The cluster map is symplectic $\varphi : (x_1, x_2, \dots, x_N) \to (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N)$ satisfies $\varphi^* \omega = \omega \circ \varphi(x_1, \dots, x_N) = \omega$ as $\varphi(B) = B$.

# Deformation of cluster mutation which preserves presymplectic form

- Restate the exchange relation as

$$x_k' = \mu_k(x_k) = \frac{f_k(M_k^+, M_k^-)}{x_k}, \quad \text{where } M_k^{\pm} = \prod_i x_i^{[\pm b_{ik}]_+}$$

Remark that if $f_k(M_k^+, M_k^-) = M_k^+ + M_k^-$, the relation above returns to the original one.

- What is the condition for the function $f_k$ in order for the map $\varphi = \rho^{-m}\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1}$ to be a symplectic map?

$$\varphi^*\omega = \sum_{i<j}^{N} \frac{b_{ij}}{\tilde{x}_i\tilde{x}_j} \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j = \sum_{i<j}^{N} \frac{b_{ij}}{x_i x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j = \omega$$

### Theorem (Hone, Kouloukas)

*Consider the cluster algebra with initial seed* $(\mathbf{x}, Q)$*. Let* $\varphi = \rho^{-m}\mu_{i_m}\mu_{i_{m-1}}\cdots\mu_{i_2}\mu_{i_1}$ *such that* $\varphi(Q) = Q$*. Then* $\varphi$ *is a symplectic map if and only if the smooth function* $f_k : \mathbb{F}\times\mathbb{F}\to\mathbb{F}$ *satisfies*

$$f_k(M_k^+, M_k^-) = M_k^+ g_k\left(\frac{M_k^-}{M_k^+}\right)$$

*for some differentiable function* $g_k : \mathbb{F}\to\mathbb{F}$

Setting $g_k(x) = x + 1 \implies f_k(M_k^+, M_k^-) = M_k^+ + M_k^-$.

Original cluster map is symplectic

# Deformation of symplectic map of type $A_6$

The quiver corresponds to Dynkin type of $A_6$,


- It is mutation periodic with period 6 i.e.

$$\varphi := \mu_6\mu_5\mu_4\mu_3\mu_2\mu_1(B) = B$$

- With setting $g_k(x) = bx + a$ for $a, b \in \mathbb{R}$, the function $f(M_k^+, M_k^-) = bM_k^- + aM_k^+$, the map $\varphi$ acting on the initial cluster variables $(x_1, x_2, x_3, x_4, x_5, x_6)$ generates the following:

$$\begin{aligned}
\mu_1 &: x_1x_1^{'} = b_1 + a_1x_2 \\
\mu_n &: x_nx_n^{'} = b_n + a_nx_{n-1}^{'}x_{n+1} \quad 2 \leq n \leq 5 \\
\mu_6 &: x_6x_6^{'} = b_6 + a_6x_5^{'}
\end{aligned}$$

- Consider the log-canonical Poisson bracket, $\{x_i, x_j\} = (B^{-1})_{ij} x_i x_j$.
- Let $L_i = (\varphi^*)^i(x_1)$. If $b_r = 1,\ a_s = 1$ (for $1 \leq r \leq 6$, $2 \leq s \leq 5$), then $\varphi$ is integrable as there exist 3 first integrals

$$I_1 = \sum_{j=0}^{8} L_j, \quad I_2 = \prod_{j=0}^{8} L_j$$

$$I_3 = \sum_{j=0}^{8} L_j L_{j+1}(L_{j+2} + L_{j+4} + L_{j+6}) + \sum_{j=0}^{2} L_j L_{j+3} L_{j+6},$$

which are in involution with respect to the Poisson bracket $\{\cdot, \cdot\}$.

- Question: Is the deformed integrable map $\varphi$ still a cluster map?
- No because it produces $\frac{a_1^2 x_2 x_3 + (x_1+x_3)a_1 + x_1 x_2}{x_2(a_1 x_2 + 1)} \notin \mathbb{Z}[\mathbf{x}^{\pm}]$
- **Laurentification** refers to a transformation that lifts the birational map to a new coordinate system where the map possesses the Laurent property.
- Note that this is not a unique procedure as several methods can Laurentify the birational map, e.g. recursive factorization and projectivization.
- The transformation can be guessed from the singularity confinement pattern, observed from **singularity analysis**. The pattern of singularities closely correlates with the transformation.

# Singularity pattern

- Let $\mathbf{x}_n = (x_{1,n}, \ldots, x_{6,n})$ be variables induced by the iteration of the map $\varphi$.
- The $\varphi : \mathbf{x}_n \to \mathbf{x}_{n+1}$ gives the variable, $x_{1,n+1} = \frac{1+a_1 x_{2,n}}{x_{1,n}}$. It has singularity at $x_{1,n} = 0$.
- Setting the initial values $x_{1,0} = \epsilon,\ x_{j,0} = u_j$ for $j \in \{1, \ldots, 6\} \backslash \{1\}$, where $\epsilon \ll 1$ and $u_j$ are regular values.
- $x_{1,0} = \epsilon(\sim 0^1) \underset{\varphi}{\to} x_{1,1} \sim \epsilon^{-1}(\sim \infty^1) \underset{\varphi}{\to} x_{1,2} = \frac{u_4 a_1 - 1}{u_3}(\sim R)$

| | | | | |
|---|---|---|---|---|
| $x_{1,n}$ | $0^1$ | $\infty^1$ | R | R |
| $x_{2,n}$ | R | $\infty^1$ | R | R |
| $x_{3,n}$ | R | $\infty^1$ | R | R |
| $x_{4,n}$ | R | $\infty^1$ | R | R |
| $x_{5,n}$ | R | $\infty^1$ | R | R |
| $x_{6,n}$ | R | $\infty^1$ | $0^1$ | R |

$\longrightarrow$

$$x_{1,n} = \frac{* \cdot \tau_{n+1}}{* \cdot \tau_n} \quad x_{2,n} = \frac{*}{* \cdot \tau_n}$$

$$x_{3,n} = \frac{*}{* \cdot \tau_n} \quad x_{4,n} = \frac{*}{* \cdot \tau_n}$$

$$x_{5,n} = \frac{*}{* \cdot \tau_n} \quad x_{6,n} = \frac{* \cdot \tau_{n-1}}{* \cdot \tau_n}$$

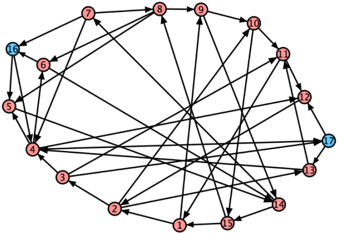
By singularity analysis, one could observe the pattern which lead us to define the map $\pi : (x_1, \ldots, x_{2N}) \to (q_{2,0}, q_{1,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, p_{1,0}, p_{2,0}, a_1, a_6)$

$$x_{1,n} = \frac{\sigma_n \tau_{n+1}}{\sigma_{n+1}\tau_n} \quad x_{2,n} = \frac{p_{1,n}}{\sigma_{n+2}\tau_n} \quad x_{3,n} = \frac{p_{2,n}}{\sigma_{n+3}\tau_n} \quad x_{4,n} = \frac{q_{2,n}}{\sigma_{n+4}\tau_n}$$

$$x_{5,n} = \frac{q_{1,n}}{\sigma_{n+5}\tau_n} \quad x_{6,n} = \frac{\sigma_{n+7}\tau_{n-1}}{\sigma_{n+6}\tau_n}$$

By setting the $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14}, \tilde{x}_{15}, \tilde{x}_{16}, \tilde{x}_{17}) =$ $(q_{2,0}, q_{1,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, p_{1,0}, p_{2,0}, a_1, a_6)$

$$\tilde{\omega} = \pi^* \omega = \sum_{i<j} \frac{\tilde{b}_{ij}}{\tilde{x}_i \tilde{x}_j} \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j \longrightarrow$$



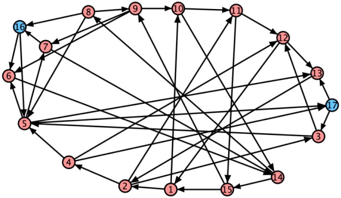
Deformed quiver $Q_6$

# Deformation of type $A_6$ (Laurentification)

- The relations, which is given by $\varphi \circ \pi$, is equivalent to the exchange relations induced by $\mu_3\mu_2\mu_{115}\mu_{14}\mu_6(\tilde{\mathbf{x}}_0)$

$$\mu_3\mu_2\mu_1\mu_{15}\mu_{14}\mu_6(\tilde{\mathbf{x}}_0, \tilde{Q}_{A_6}) = (\tilde{\mathbf{x}}_1, \rho\tilde{Q}_{A_6})$$

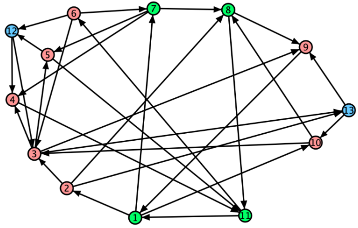
$$\begin{aligned}
\tau_{n+2}\sigma_n &= \sigma_{n+2}\tau_n + a_1 p_n \\
p_{n+1}p_n &= \sigma_{n+3}\sigma_{n+2}\tau_n\tau_{n+1} + r_n\sigma_{n+1}\tau_{n+2} \\
r_{n+1}r_n &= \sigma_{n+4}\sigma_{n+3}\tau_n\tau_{n+1} + q_n p_{n+1} \\
q_{n+1}q_n &= \sigma_{n+5}\sigma_{n+4}\tau_n\tau_{n+1} + w_n r_{n+1} \\
w_{n+1}w_n &= \sigma_{n+6}\sigma_{n+5}\tau_n\tau_{n+1} + \sigma_{n+7}q_{n+1}\tau_{n-1} \\
\sigma_{n+8}\tau_{n-1} &= \sigma_{n+6}\tau_{n+1} + a_6 w_{n+1}
\end{aligned} \quad \longleftrightarrow \quad \varphi \circ \pi$$




**Theorem**

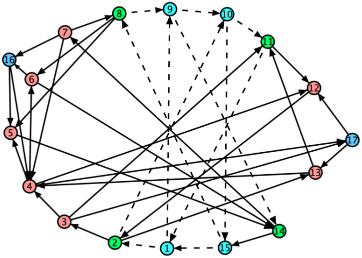
*The sequence mutations in a cluster algebra of type $A_6$ with including two frozen variables $a_1, a_6$ generates the sequence of tau functions, $(\sigma_n)$,$(p_n)$,$(r_n)$,$(w_n)$,$(q_n)$,$(\tau_n)$, which are elements of the Laurent polynomial ring*
$\mathbb{Z}_{>0}\left[a_1, a_6, \sigma_0^{\pm}, \sigma_1^{\pm}, \sigma_2^{\pm}, \sigma_3^{\pm}, \sigma_4^{\pm}, \sigma_5^{\pm}, \sigma_6^{\pm}, \sigma_7^{\pm}, \tau_{-1}^{\pm}, \tau_0^{\pm}, \tau_1^{\pm}, p_0^{\pm}, r_0^{\pm}, w_0^{\pm}, q_0^{\pm}\right]$.

# What is the difference between type $A_4$ and type $A_6$?




Deformed quiver $\tilde{Q}_{A_4}$

$\longrightarrow$




Deformed quiver $\tilde{Q}_{A_6}$

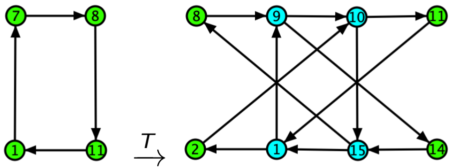



Figure 1: Local expansion on the four cycle

# Alternative form of $\tilde{Q}_{A_6}$

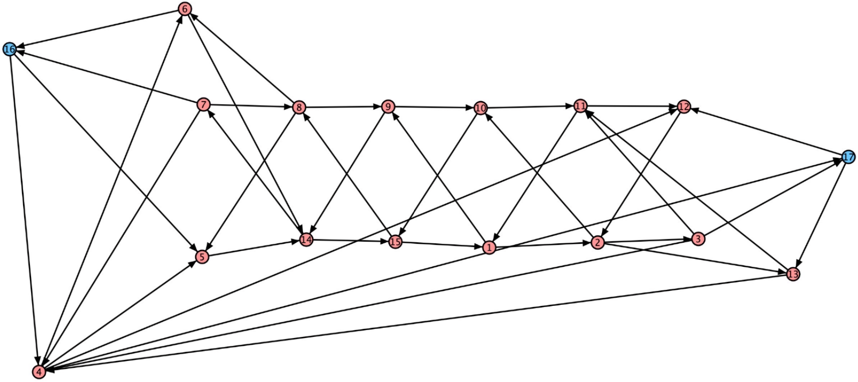



Figure 2: Alternative form of $\tilde{Q}_{A_6}$

# Local expansion of the deformed quiver

- Each node in the deformed quiver $Q_4$ corresponds to tau-functions $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) = (q_{1,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, p_{1,0}, a_1, a_4)$
- By the local expansion above, the deformed quiver $Q_6$ has vertices corresponding to $(q_{2,0}, q_{1,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, p_{1,0}, p_{2,0}, a_1, a_6)$
- The extension from the deformed quiver of type $A_4$ to $A_6$, suggests that the map $\pi : (x_1, \ldots, x_{2N}) \to (q_{N-1,0}, \ldots, q_{1,0}, \tau_{-1}, \tau_0, \tau_1, \sigma_0, \ldots, \sigma_{2N}, p_{1,0}, \ldots, p_{N-1,0}, a_1, a_{2N})$

$$x_1 = \frac{\sigma_0\tau_1}{\sigma_1\tau_0} \quad x_2 = \frac{p_0}{\sigma_2\tau_0} \quad x_3 = \frac{p_1}{\sigma_3\tau_0}, \ldots, x_N = \frac{p_{N-2}}{\sigma_N\tau_0},$$
$$x_{N+1} = \frac{q_{N-2}}{\sigma_{N+1}\tau_0}, \ldots, x_{2N-1} = \frac{q_0}{\sigma_{2N-1}\tau_0} \quad x_{2N} = \frac{\sigma_{2N+1}\tau_{-1}}{\sigma_{2N}\tau_0}$$

# Deformed quiver $Q_{2N}$ (type $A_{2N}$)

Let $\tilde{\mu} = \mu_{N-1}\mu_{N-2}\cdots\mu_2\mu_1\mu_{4N+3}\mu_{4N+2}\cdots\mu_{3N+6}\mu_{3N+5}$. Then the sequence of mutations $\mu_N\tilde{\mu}\mu_{N+3}$ is equivalent to cyclic permutation of $2N+5$ nodes, which corresponds to tau-functions $\tau$ and $\sigma$.

$$\mu_N\tilde{\mu}\mu_{N+3}(Q) = \rho(Q)$$

Subsequent mutation $\mu_{N+1}\tilde{\mu}\mu_{N+4}$ will give similar result. Applying the $2N+3$ of these blocks of the mutations,

$$\hat{\mu}(Q) = Q$$

where $\hat{\mu} = \mu_{3N+4}\tilde{\mu}\mu_{N+2}\mu_{3N+3}\tilde{\mu}\mu_{N+1}\mu_{3N+2}\tilde{\mu}\mu_N\mu_{3N+1}\tilde{\mu}\mu_{3N+4}\cdots\mu_N\tilde{\mu}\mu_{N+3}$

The birational map (cluster map )$\varphi = \mu_{2N} \cdots \mu_1$ associated to type $A_{2N}$, which is equivalent to

$$\begin{aligned}
&\mu_1 : x_1 x_1^{'} = 1 + a_1 x_2 \\
&\mu_n : x_n x_n^{'} = 1 + x_{n-1}^{'} x_{n+1} \quad 2 \leq n \leq 2N-1 \\
&\mu_{2N} : x_{2N} x_{2N-1}^{'} = 1 + a_{2N} x_5^{'},
\end{aligned}$$

can be Laurentified to the cluster map $\varphi\pi$ on the space of tau-functions, that is, the generated variables are in the Laurent polynomial ring $\mathbb{Z}_{>0}\left[a_1, a_{2N}, \sigma^{\pm}, p^{\pm}, q^{\pm}\right]$

## Future direction

- Deformation of cluster map of type $A_{2N+1}$.
- Generalizing deformation of the cluster map of type $B$ and $D$.(Up to so far we have successfully Laurentified the deformations of type $B_2$, $B_3$ and $D_4$)
- Potential relations between deformed integrable cluster map and other integrable map (We have found that deformed particular type B3 and D4 maps are closely related to a particular solution of Somos-7 recurrences)
- Investigating symmetries in deformed exchange matrix type $A_{2N}$

Thank you for your attention!