Hamiltonian group actions and fixed points

Mantra: act globally, compute locally

Poisson 2026
Summer School
3-7 August 2026

Tara S. Holm
Cornell University

Outline - Lecture 4

- Few words on symplectic reduction
- Symplectic embeddings

Symplectic reduction

We have the moment map

$$\begin{array}{rcl} \Phi : M & \to & \mathfrak{t}^* \\ p & \mapsto & \left(\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \varphi^{\xi}(p) \end{array} \right). \end{array}$$

It can be used to prove localization results because φ^{ξ} behaves like a **Morse function**, with critical set M^T (for most ξ).

The moment map is also an **equivariant** map: $T \circlearrowright \Phi^{-1}(\mu)$ for every $\mu \in \mathfrak{t}$. If μ is a regular value, $\Phi^{-1}(\mu)$ is a manifold.

$$\begin{aligned} \dim(\Phi^{-1}(\mu)) &= \dim(M) - \dim(T) \\ &= 2n - d \end{aligned}$$

$$\begin{aligned} \dim(\Phi^{-1}(\mu)/T) &= \dim(M) - 2 \cdot \dim(T) \\ &= 2n - 2d \end{aligned}$$

Symplectic reduction

We have the moment map

$$\begin{aligned} \Phi : M &\rightarrow \mathfrak{t}^* \\ p &\mapsto \left(\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \phi^{\xi}(p) \end{array} \right). \end{aligned}$$

It can be used to prove localization results because ϕ^{ξ} behaves like a **Morse function**, with critical set M^T (for most ξ).

The moment map is also an **equivariant** map: $T \circlearrowright \Phi^{-1}(\mu)$ for every $\mu \in \mathfrak{t}$. If μ is a regular value, $\Phi^{-1}(\mu)$ is a manifold.

<u>Theorem</u> [Marsden-Weinstein]:

If $T \circlearrowright M$ is a compact Hamiltonian T-manifold, and μ is a regular value of Φ, then

$$M//T(\mu) = \Phi^{-1}(\mu)/T$$

is symplectic, with at worst orbifold singularities.

Example of symplectic reduction

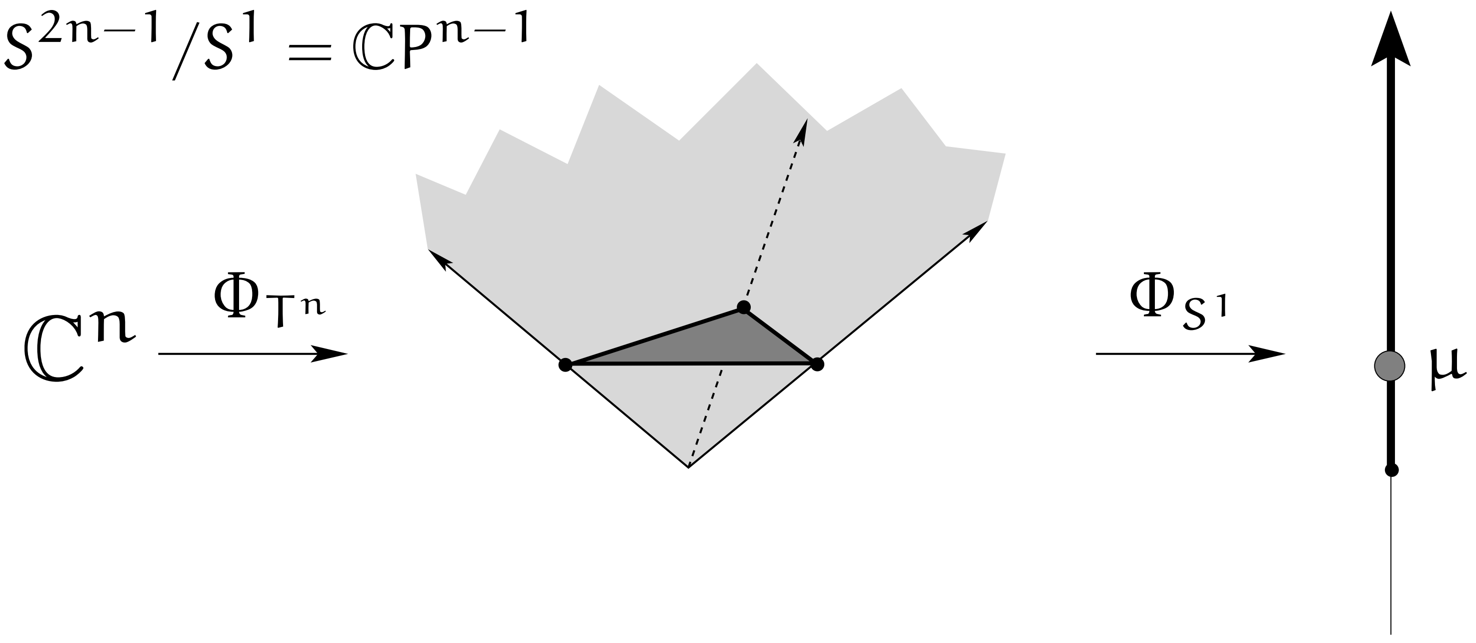
$S^1 \circlearrowright \mathbb{C}^n$

$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n)$

$$\begin{aligned} \Phi : \mathbb{C}^n &\to \mathbb{R} \\ (z_1, \ldots, z_n) &\mapsto \sum |z_i|^2 \end{aligned}$$

$$\Phi^{-1}(\mu) = \left\{ (z_1, \ldots, z_n) \;\middle|\; \sum |z_i|^2 = \mu \right\} = S^{2n-1}$$

$$\Phi^{-1}(\mu)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}$$



Example of symplectic reduction

$$S^1 \circlearrowright \mathbb{C}^n$$
$$t \cdot (z_1, \dots, z_n) = (t \cdot z_1, \dots, t \cdot z_n)$$

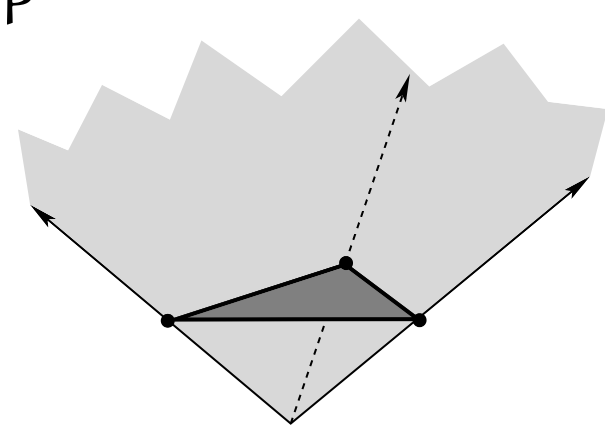
A circle action

$$\begin{aligned} \Phi : \mathbb{C}^n &\to \mathbb{R} \\ (z_1, \dots, z_n) &\mapsto \sum |z_i|^2 \end{aligned}$$

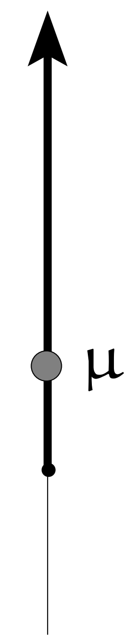
$$\Phi^{-1}(\mu) = \left\{ (z_1, \dots, z_n) \,\middle|\, \sum |z_i|^2 = \mu \right\} = S^{2n-1}$$

$$\Phi^{-1}(\mu)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}$$

$$\mathbb{C}^n \xrightarrow{\Phi_{T^n}}$$



$$\xrightarrow{\Phi_{S^1}}$$



Example of symplectic reduction

$S^1 \circlearrowright \mathbb{C}^n$

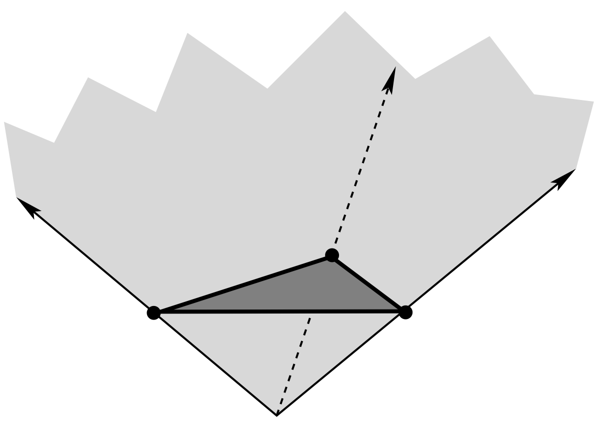
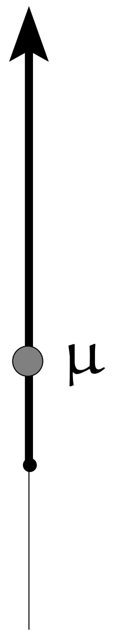
$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n)$

Moment map

$$\boxed{\begin{array}{rcl} \Phi : \mathbb{C}^n & \to & \mathbb{R} \\ (z_1, \ldots, z_n) & \mapsto & \sum |z_i|^2 \end{array}}$$

$$\Phi^{-1}(\mu) = \left\{ (z_1, \ldots, z_n) \,\middle|\, \sum |z_i|^2 = \mu \right\} = S^{2n-1}$$

$$\Phi^{-1}(\mu)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}$$

$\mathbb{C}^n \xrightarrow{\Phi_{T^n}}$  $\xrightarrow{\Phi_{S^1}}$ 

Example of symplectic reduction

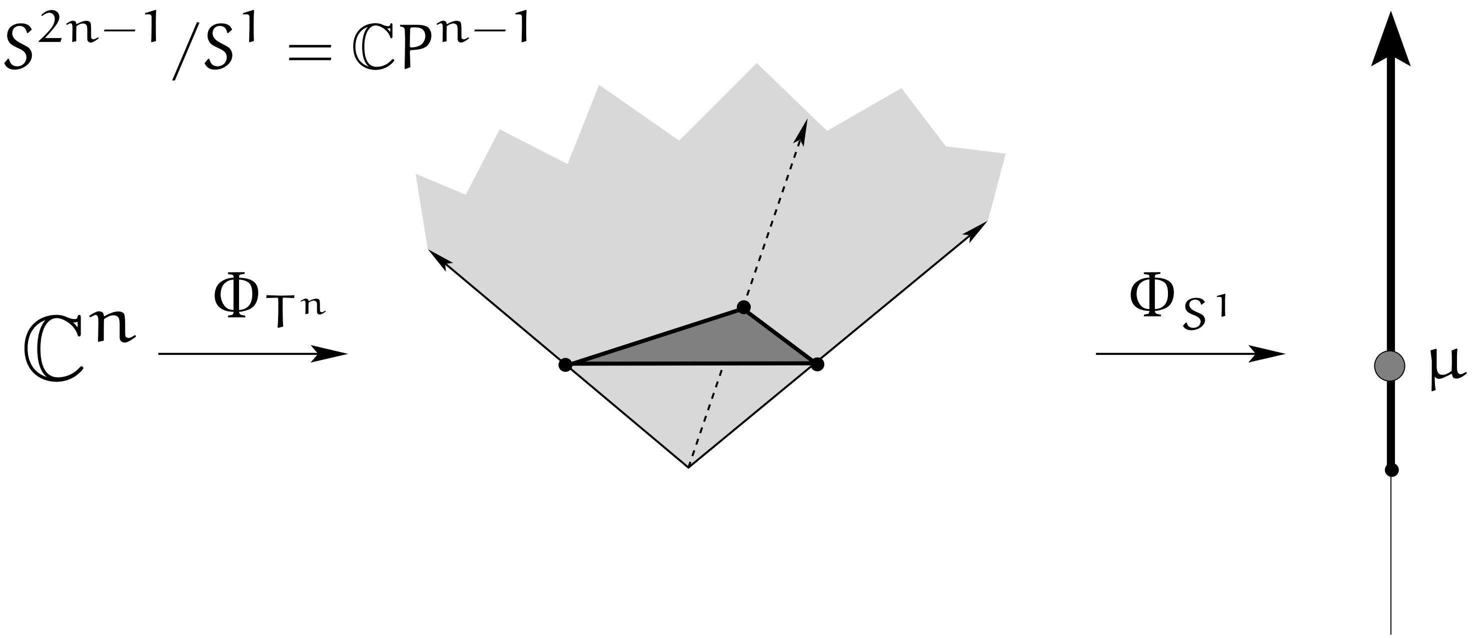
$S^1 \circlearrowright \mathbb{C}^n$

$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n)$

$$\begin{aligned} \Phi : \mathbb{C}^n &\to \mathbb{R} \\ (z_1, \ldots, z_n) &\mapsto \sum |z_i|^2 \end{aligned}$$

$$\boxed{\Phi^{-1}(\mu) = \left\{ (z_1, \ldots, z_n) \;\middle|\; \sum |z_i|^2 = \mu \right\} = S^{2n-1}}$$ Level set

$\Phi^{-1}(\mu)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}$



Example of symplectic reduction

$S^1 \circlearrowright \mathbb{C}^n$

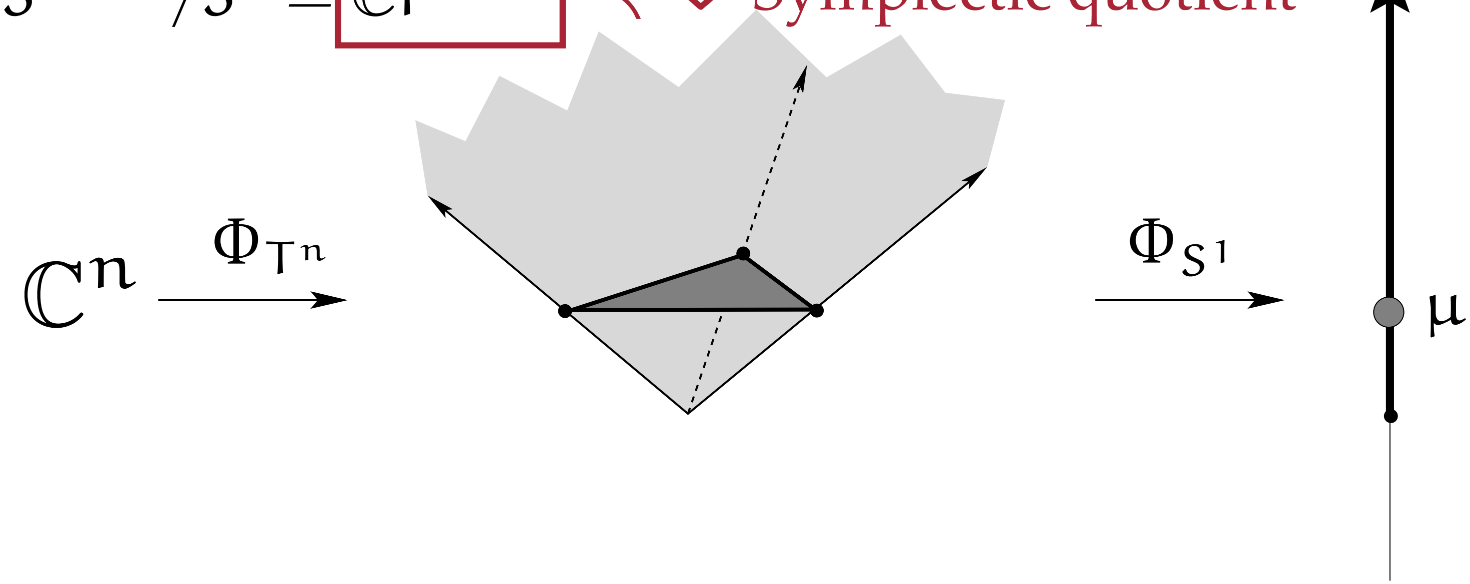
$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n)$

$$\begin{aligned} \Phi : \mathbb{C}^n &\to \mathbb{R} \\ (z_1, \ldots, z_n) &\mapsto \sum |z_i|^2 \end{aligned}$$

$$\Phi^{-1}(\mu) = \left\{ (z_1, \ldots, z_n) \;\middle|\; \sum |z_i|^2 = \mu \right\} = S^{2n-1}$$

$$\Phi^{-1}(\mu)/S^1 = S^{2n-1}/S^1 = \boxed{\mathbb{C}P^{n-1}}$$

Symplectic quotient



More examples

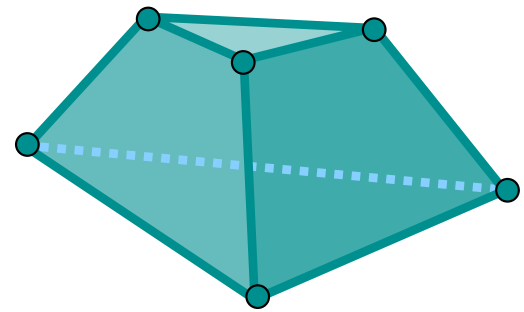
Delzant's Theorem:

$$\left\{\begin{array}{c}\text{compact toric}\\\text{symplectic manifolds}\end{array}\right\} \overset{1:1}{\leftrightsquigarrow} \left\{\begin{array}{c}\text{simple rational}\\\text{smooth convex polytopes}\end{array}\right\}$$

$$M \rightsquigarrow \Phi(M)$$

$$\Phi^{-1}(\alpha)/K \circlearrowleft T^N/K$$

$$\cong \mathbb{C}^N//T^d \leftrightsquigarrow$$



$\subseteq \mathbb{R}^n$

$$T^N \circlearrowright \mathbb{C}^N \xrightarrow{\Phi} (\mathbb{R}^N)^* \to \mathfrak{k}^* \ni \alpha$$

$\vee$

$K = T^d$

&

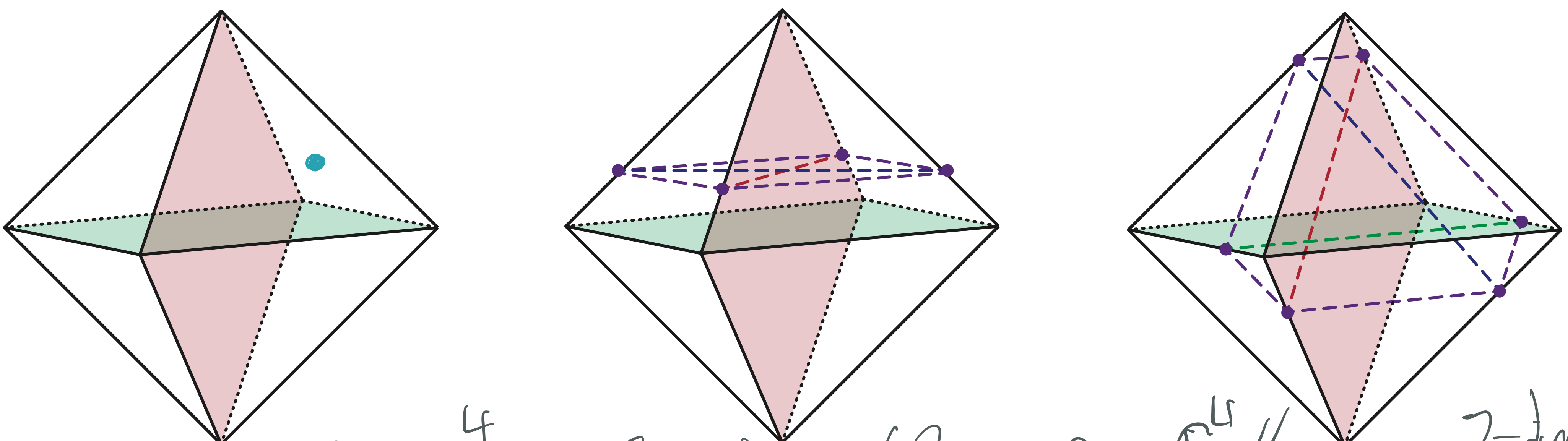
N facets

combin. of Δ

determine $K \leq T^N, \alpha$

More examples

$$\left.\begin{array}{l} T^n \circlearrowright \mathcal{F}\ell ags(\mathbb{C}^n) \\ T^n \circlearrowright \mathcal{G}r(k,\mathbb{C}^n) \\ T \circlearrowright \mathcal{O}_\lambda \end{array}\right\} \mathcal{O}_\lambda/\!/T \text{ is a weight variety.}$$



$T^3 \circlearrowright Gr_2\mathbb{C}^4$: 8-dim'l

$Gr_2\mathbb{C}^4/\!/T^3 = $ 2-dim $= S^2$

https://www.desmos.com/3d/lv7bzaxjez

Yet another example

$$\begin{aligned}
\mathcal{P}ol_3(a_1,\ldots,a_n) &= \frac{\left\{(\overrightarrow{v_1},\ldots,\overrightarrow{v_n}) \;\middle|\; \overrightarrow{v_i}\in\mathbb{R}^3, |v_i| = a_i, \sum \overrightarrow{v_i} = \overrightarrow{0}\right\}}{SO(3)} \\
&= S^2_{a_1}\times\cdots\times S^2_{a_n} /\!\!/ SO(3)
\end{aligned}$$

$$\begin{tikzcd}
 & (\mathbb{C}^2)^n \arrow[dl, "/\!\!/ U(2)"'] \arrow[dr, "/\!\!/ T^n"] & \\
\mathcal{G}r(2,\mathbb{C}^n) \arrow[dr, "/\!\!/ T^{n-1}"'] & & S^2\times\cdots\times S^2 \arrow[dl, "/\!\!/ SO(3)"] \\
 & \mathcal{P}ol_3(a_1,\ldots,a_n) &
\end{tikzcd}$$

More examples

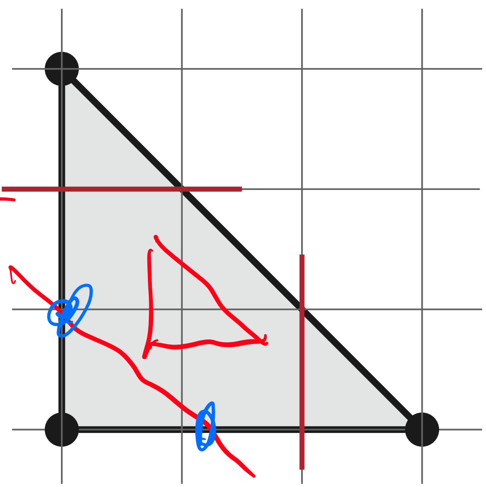
Delzant's Theorem:

$$\left\{ \begin{array}{c} \text{compact toric} \\ \text{symplectic manifolds} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{simple rational} \\ \text{smooth convex polytopes} \end{array} \right\}$$

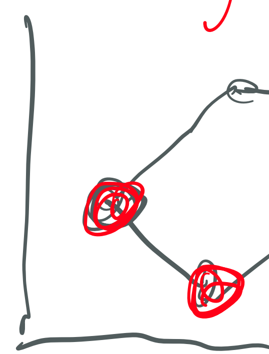
$\mathcal{P} = \mathcal{P}ol_3(1,1,1,1,1)$

$= \mathbb{CP}^2_3 \# 4\overline{\mathbb{CP}^2_1}$

Traynor trick



Locus is an $S^2 \subseteq \mathcal{P}$



$\mathcal{P}$ does not admit a toric action!

In fact, no Ham. S^1-action

Symplectic embeddings

Question.

When is there a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$?

Symplectic embeddings

Question.

When is there a symplectic embedding $(M_1, \omega_1) \xhookrightarrow{s} (M_2, \omega_2)$?

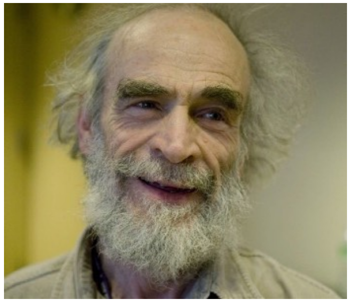
Partial answers

Theorem (Gromov).

There is a symplectic embedding

$$B^{2n}(R) \xhookrightarrow{s} B^2(r) \times \mathbb{R}^{2n-2}$$

if and only if $R \leq r$.



Symplectic embeddings

Question.

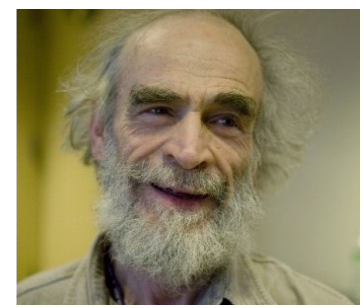
When is there a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$?

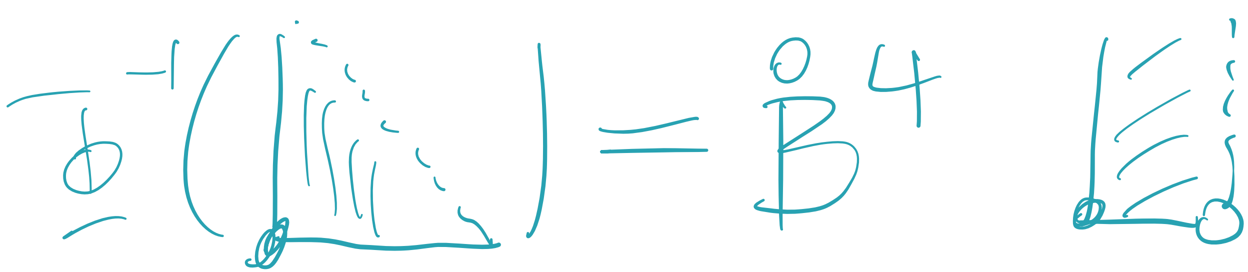
Partial answers

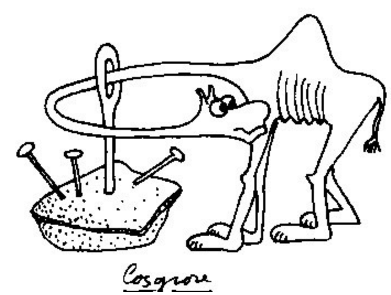
Theorem (Gromov).
There is a symplectic embedding

$$B^{2n}(R) \overset{s}{\hookrightarrow} B^2(r) \times \mathbb{R}^{2n-2}$$

if and only if $R \leq r$.





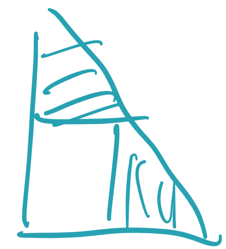


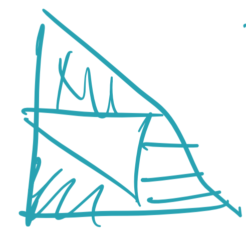
Symplectic embeddings

Question.

When is there a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$?

Partial answers





Theorem (Biran).

There is a symplectic packing

$$\mathrm{Vol}\Big(\coprod_{\ell} B^4(\delta)\Big) \overset{s}{\hookrightarrow} \Big(B^4(\Delta)\Big)$$

filling the fraction p_ℓ of $B^4(\Delta)$, for

ℓ	1	2	3	4	5	6	7	8	$\geqslant 9$
p_ℓ	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

packing stability

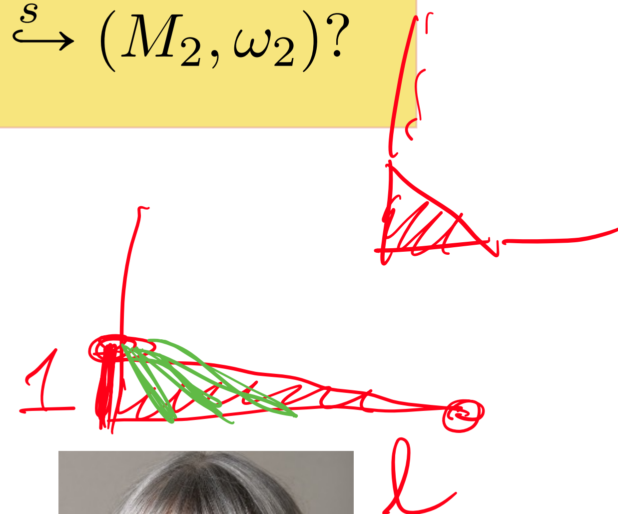
Symplectic embeddings

$\mathbb{C}^2 \to \mathbb{R}^2$

Question.

When is there a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$?

Partial answers



Theorem (McDuff).

There is a symplectic packing

$$\coprod_{\ell} B^4(1) \overset{s}{\hookrightarrow} B^4(\Delta)$$

if and only if

$$\mathscr{E}(1, \ell) \overset{s}{\hookrightarrow} B^4(\Delta)$$

Symplectic embeddings

Question.

When is there a symplectic embedding $(M_1, \omega_1) \xhookrightarrow{s} (M_2, \omega_2)$?

Partial answers

$$\mathscr{E}(a,b) = \left\{ (z_1, z_2) \;\middle|\; \pi\frac{|z_1|^2}{a} + \pi\frac{|z_2|^2}{b} < 1 \right\}$$

Theorem (McDuff. Hutchings.).
There is a symplectic embedding

$$\mathscr{E}(a,b) \xhookrightarrow{s} \mathscr{E}(c,d)$$

if and only if $\mathscr{N}(a,b)_k \leqslant \mathscr{N}(c,d)_k \ \forall\ k$.

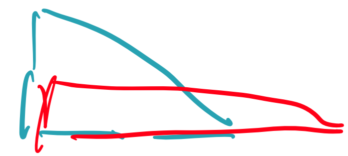
$$\begin{pmatrix} 0 & a & 2a & 3a & \cdots \\ b & a+b & 2a+b & 3a+b & \cdots \\ 2b & a+2b & 2a+2b & 3a+2b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Symplectic embeddings

Question.

When is there a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$?

Partial answers



Theorem (McDuff. Hutchings.).
There is a symplectic embedding

$$\mathscr{E}(a,b) \overset{s}{\hookrightarrow} \mathscr{E}(c,d)$$

if and only if $\mathscr{N}(a,b)_k \leqslant \mathscr{N}(c,d)_k \ \forall\ k$.

$$\mathscr{N}(1,4) = (0,1,2,3,4,4,5,5,6,6,7,7,8,8,8,9,9,9,\dots)$$

$$\mathscr{N}(2,2) = (0,2,2,4,4,4,6,6,6,6,8,8,8,8,8,10,10,10,\dots)$$

Symplectic embeddings

Question.

When is there a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$?

Partial answers

Theorem (McDuff. Hutchings.).
There is a symplectic embedding

$$\mathscr{E}(a, b) \overset{s}{\hookrightarrow} \mathscr{E}(c, d)$$

if and only if $\mathscr{N}(a,b)_k \leqslant \mathscr{N}(c,d)_k \ \forall\ k$.

$$\left.\begin{array}{l}\mathscr{N}(1,4) = (0,1,2,3,4,4,5,5,6,6,7,7,8,8,8,9,9,9,\dots) \\ \mathscr{N}(2,2) = (0,2,2,4,4,4,6,6,6,6,8,8,8,8,8,10,10,10,\dots)\end{array}\right\} \implies \mathscr{E}(1,4) \overset{s}{\hookrightarrow} \mathscr{E}(2,2)$$

Symplectic embeddings

89/34

Infinite staircases

34/13

Theorem (McDuff–Schlenk).

The function $c_{B^4}(a) = \inf \left\{ \lambda \;\middle|\; \mathscr{E}(1,a) \overset{s}{\hookrightarrow} B^4(\lambda) = \mathscr{E}(\lambda,\lambda) \right\}$

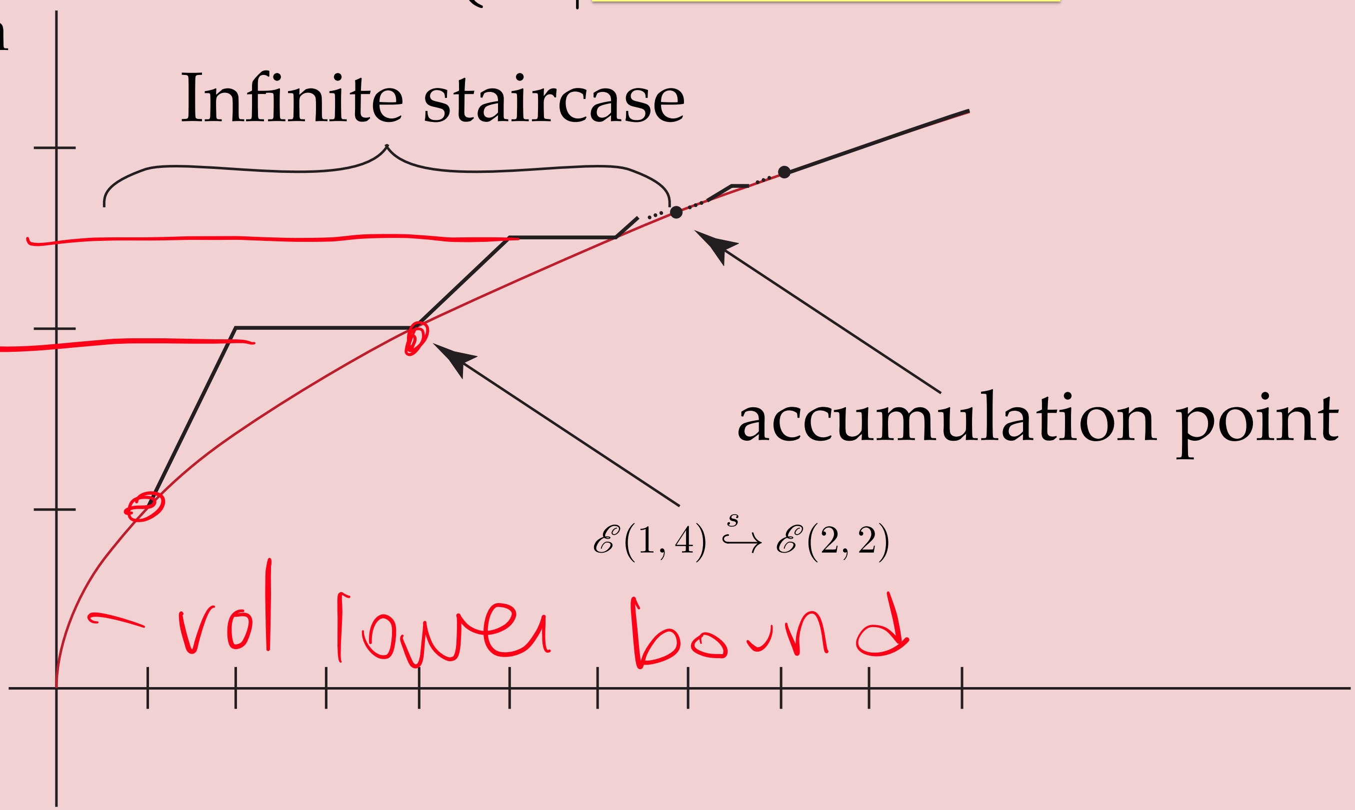
has graph

13/5

5/2

2/1

1/1



Symplectic embeddings

Infinite staircases

- Fibonacci staircase (McDuff–Schlenk)  
- Pell staircase (Frenkel–Müller) 
- Ellipsoid staircase (Cristofaro-Gardiner–Kleinman)  
- More generally, study $\mathscr{E}(1,a) \overset{s}{\hookrightarrow} X$

Convex toric domains

Definition.

A convex toric domain is the inverse image of a convex subset of $\mathbb{R}^2$ under the map

$$\Phi : \mathbb{C}^2 \to \mathbb{R}^2$$

that sends $\Phi(z, w) = \pi(|z|^2, |w|^2)$.

Convex toric domains

Definition.

A convex toric domain is the inverse image of a convex subset of $\mathbb{R}^2$ under the map

$$\Phi : \mathbb{C}^2 \to \mathbb{R}^2$$

that sends $\Phi(z, w) = \pi(|z|^2, |w|^2)$.

Theorem (Cristofaro-Gardiner – H. – Mandini – Pires).

Let Ω be a Delzant polygon, X_Ω the corresponding convex toric domain, and M_Ω the corresponding closed toric symplectic manifold. Then

$$\mathscr{E}(c, d) \overset{s}{\hookrightarrow} X_\Omega \Longleftrightarrow \mathscr{E}(c, d) \overset{s}{\hookrightarrow} M_\Omega.$$

Ellipsoid embedding functions

Properties of $c_X(a)$. (Cristofaro-Gardiner – H. – Mandini – Pires).

For a convex toric domain X of finite type, the ellipsoid embedding function is:

- continuous;
- non-decreasing;
- sublinear: $c_X(t \cdot a) \leq t \cdot c_X(a)\ \forall t \geq 1$;
- equal to the vol curve for $a >> 0$;
- piecewise linear when not vol or accum; and
- may equal the embedding function of a compact X.




Infinite staircases: an obstruction

Theorem (Cristofaro-Gardiner – H. – Mandini – Pires).

Let X be a convex toric domain with finite blowup vector. If $c_X(a)$ has an infinite staircase, then it accumulates at a solution a_0 of

$$a^2 - \left(\frac{\mathrm{per}^2}{\mathrm{vol}} - 2\right) a + 1 = 0.$$

Furthermore, at a_0,

$$c_X(a_0) = \sqrt{\frac{a_0}{\mathrm{vol}}}\ .$$

Infinite staircases: an obstruction

Theorem (Cristofaro-Gardiner – H. – Mandini – Pires).

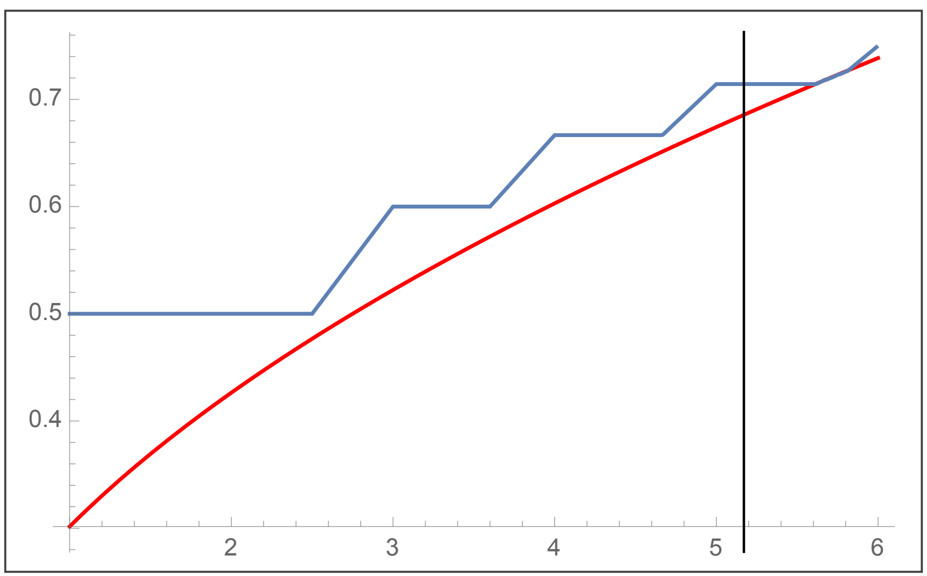
Let X be a convex toric domain with finite blowup vector. If $c_X(a)$ has an infinite staircase, then it accumulates at a solution a_0 of

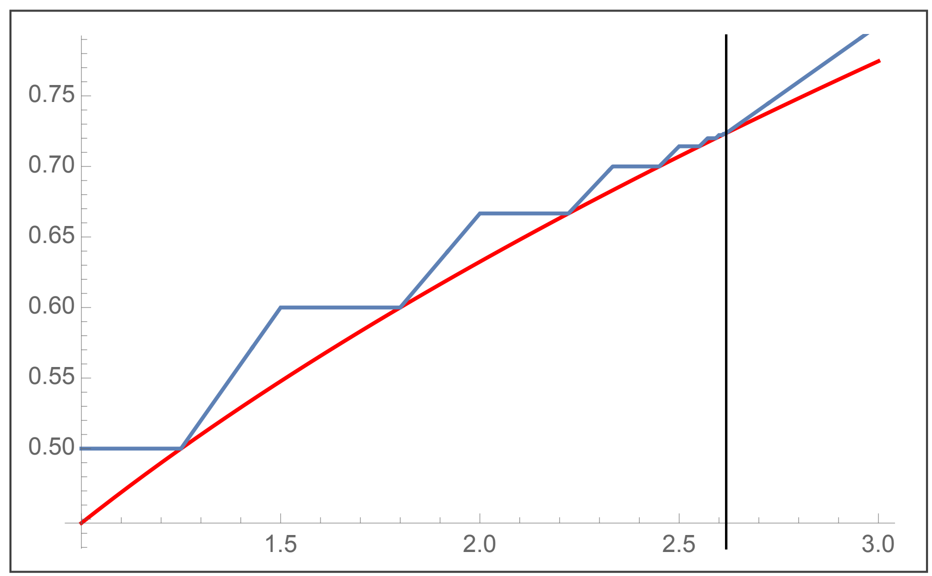
$$a^2 - \left(\frac{\mathrm{per}^2}{\mathrm{vol}} - 2\right) a + 1 = 0.$$

Furthermore, at a_0,

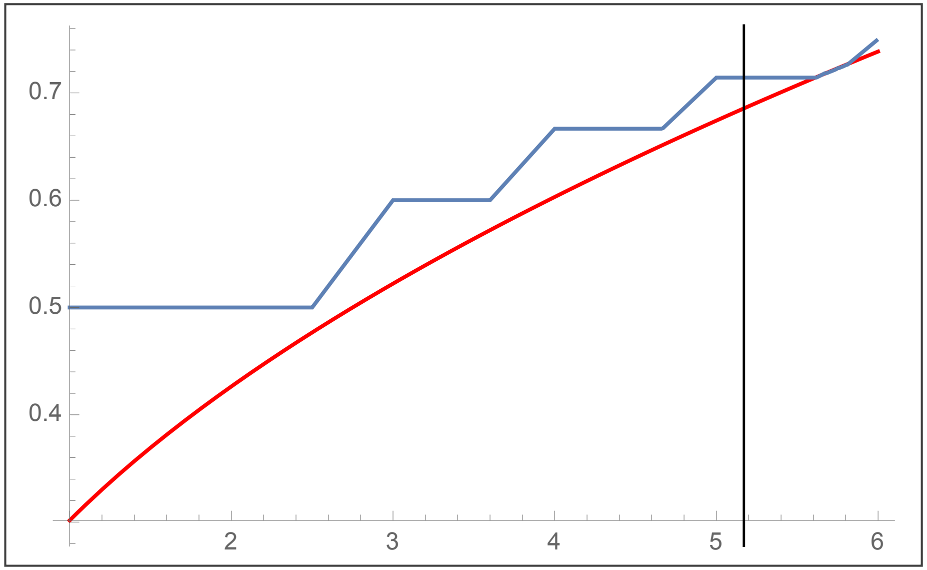
$$c_X(a_0) = \sqrt{\frac{a_0}{\mathrm{vol}}}\ .$$

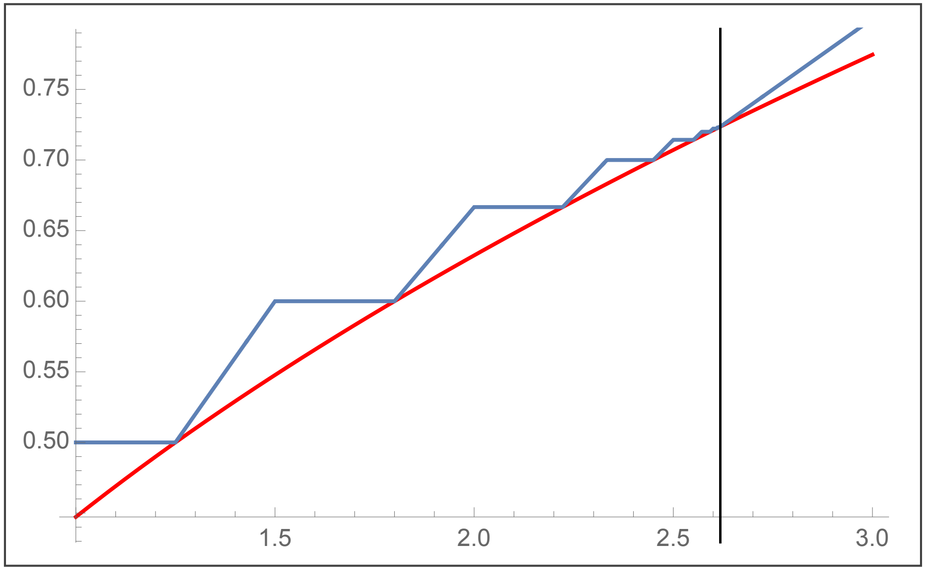


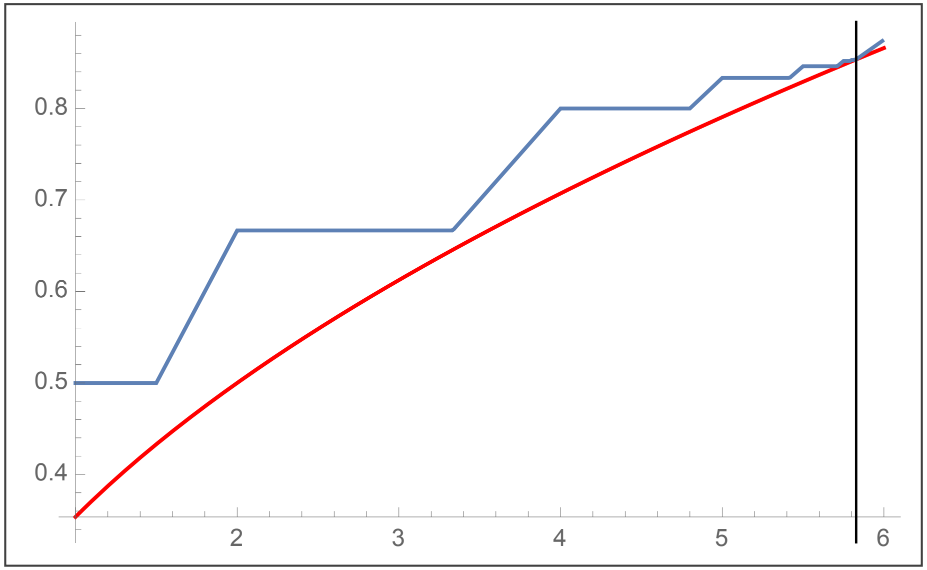



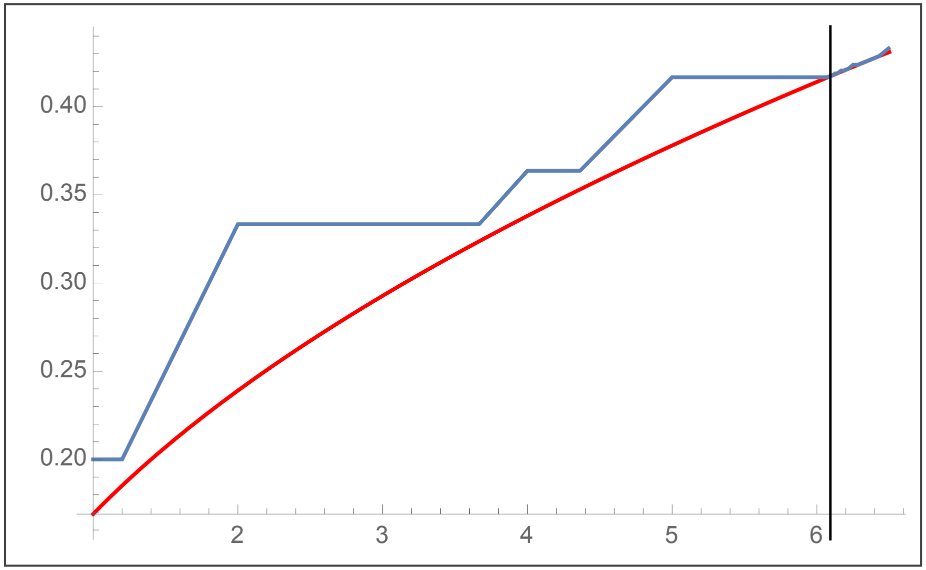


Infinite staircases: an obstruction









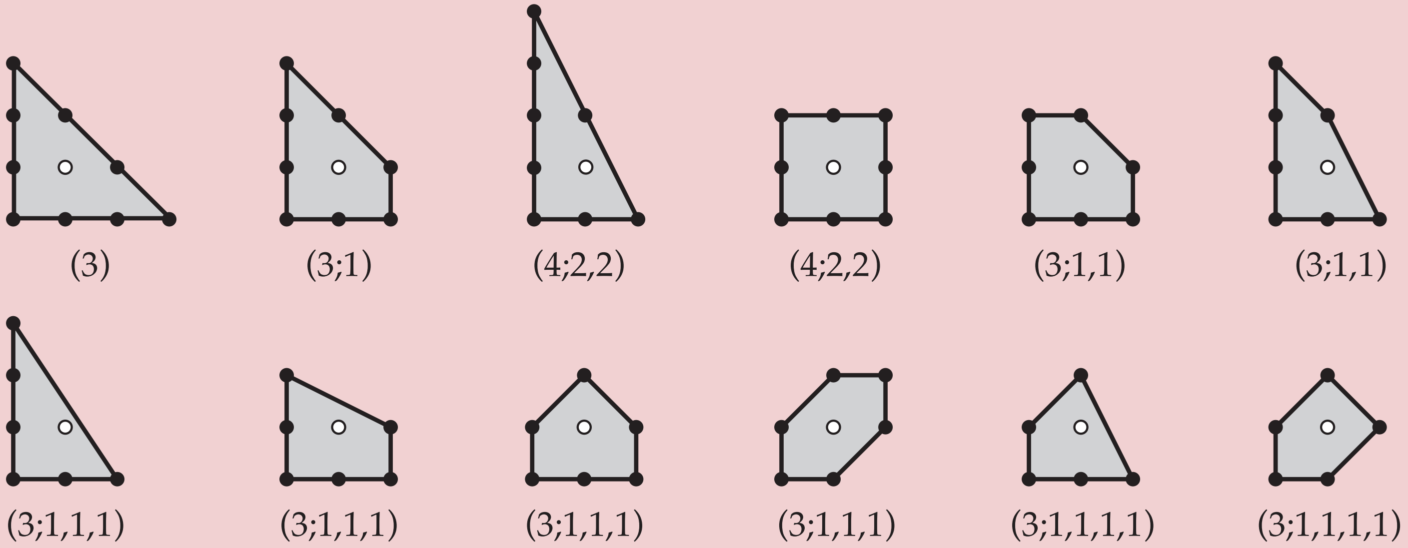
Symplectic embeddings

Infinite staircases

Theorem (Cristofaro-Gardiner – H. – Mandini – Pires).
There is an infinite staircase in the ellipsoid embedding function

$$c_X(a) = \left\{ \lambda \;\middle|\; \mathscr{E}(1,a) \overset{s}{\hookrightarrow} \lambda \cdot X \right\}$$

for the following toric domains:



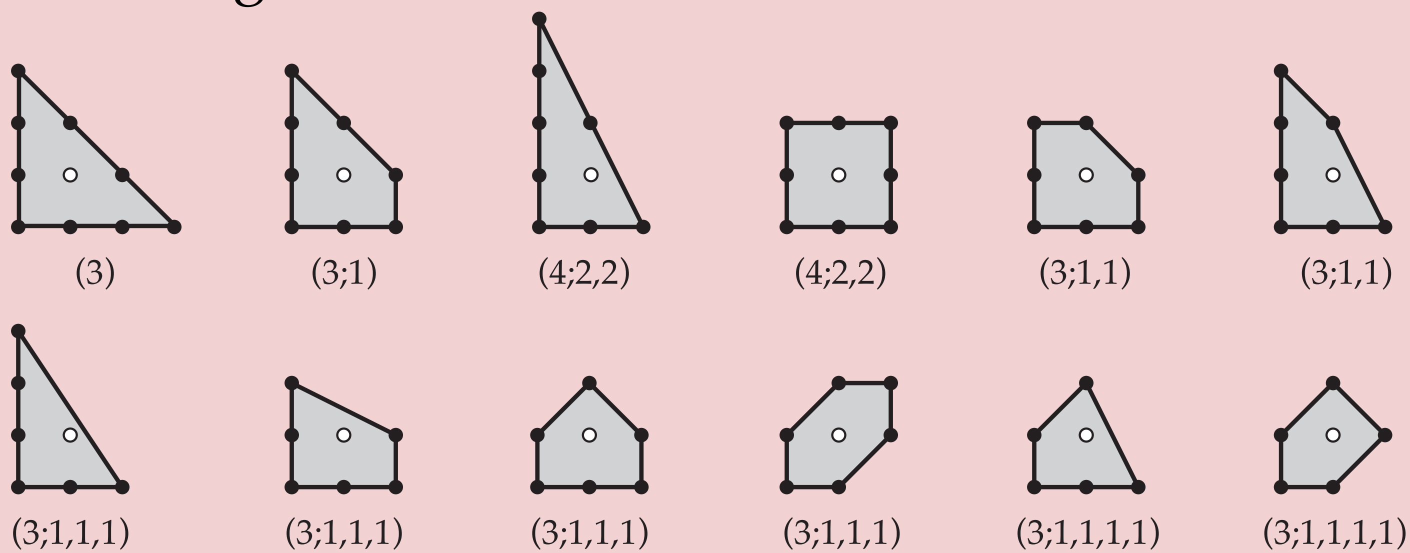
Symplectic embeddings

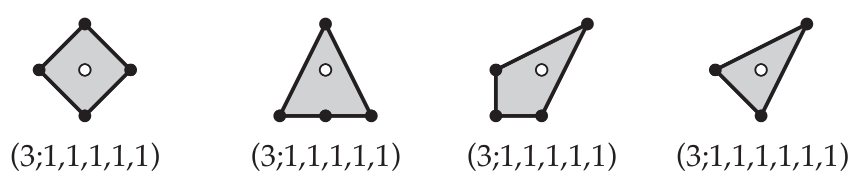
Infinite staircases

Theorem (Cristofaro-Gardiner – H. – Mandini – Pires).
There is an infinite staircase in the ellipsoid embedding function

$$c_X(a) = \left\{ \lambda \;\middle|\; \mathscr{E}(1,a) \overset{s}{\hookrightarrow} \lambda \cdot X \right\}$$

for the following toric domains:





Symplectic embeddings

Infinite staircases

This site is supported by donations to [The OEIS Foundation](#).

THE ON–LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

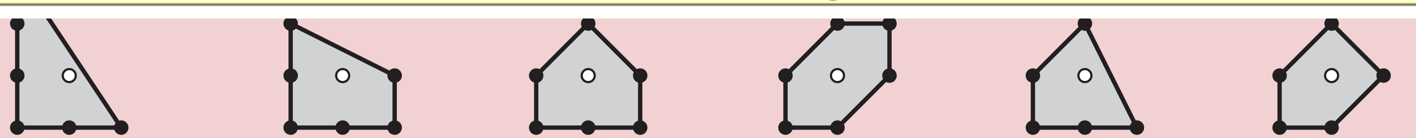
2,5,8,11,13,15,30 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

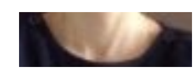
Search: **seq:2,5,8,11,13,15,30**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data



(3;1,1,1) (3;1,1,1) (3;1,1,1) (3;1,1,1) (3;1,1,1,1) (3;1,1,1,1)



Symplectic embeddings

Infinite staircases

This site is supported by donations to The OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

A007826 Numbered stops on the Market-Frankford rapid transit (SEPTA) railway line in Philadelphia, PA USA. +20 6

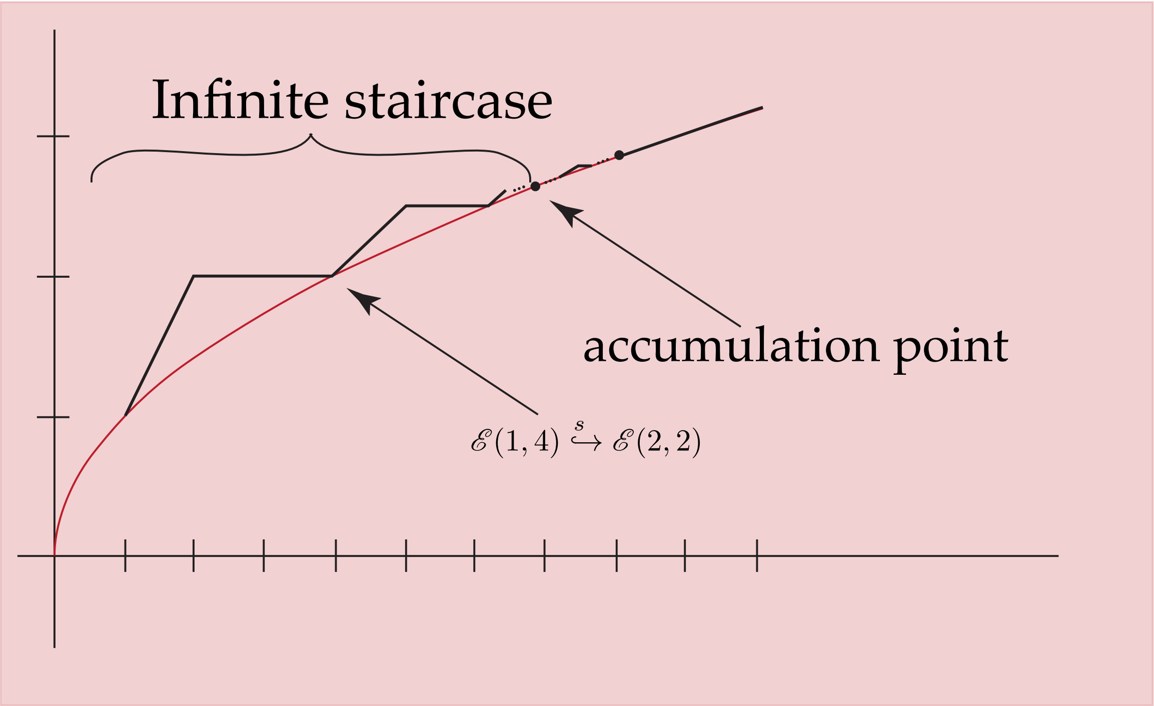
2, 5, 8, 11, 13, 15, 30, 34, 40, 46, 52, 56, 60, 63, 69 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS Formally abbreviated as The Blue Line (and known informally as 'The El'), the Market-Frankford Line extends East to West from slightly to the east of 2nd Street through the city line to the western suburbs at 63rd Street and then on to 69th Street Transportation Center, lined up almost entirely with the major dividing thoroughfare Market Street. It is actually a subway at the eastern end of this portion and through to beyond the 40th Street stop (a(1)-a(9) represent subway stops), passing under the Schuylkill River (along with trolley lines 10, 11, 13, 34 and 36) closer to 30th than to 15th Street. The only non-numbered stop on this end is suburban Milbourne between 63rd and 69th. The 'Frankford end' runs in a somewhat northeasterly direction and has all stops only with non-number names (and is entirely above ground). The semi-express A and B versions of the train both skip certain stops at peak travel times, and the only regular trains are

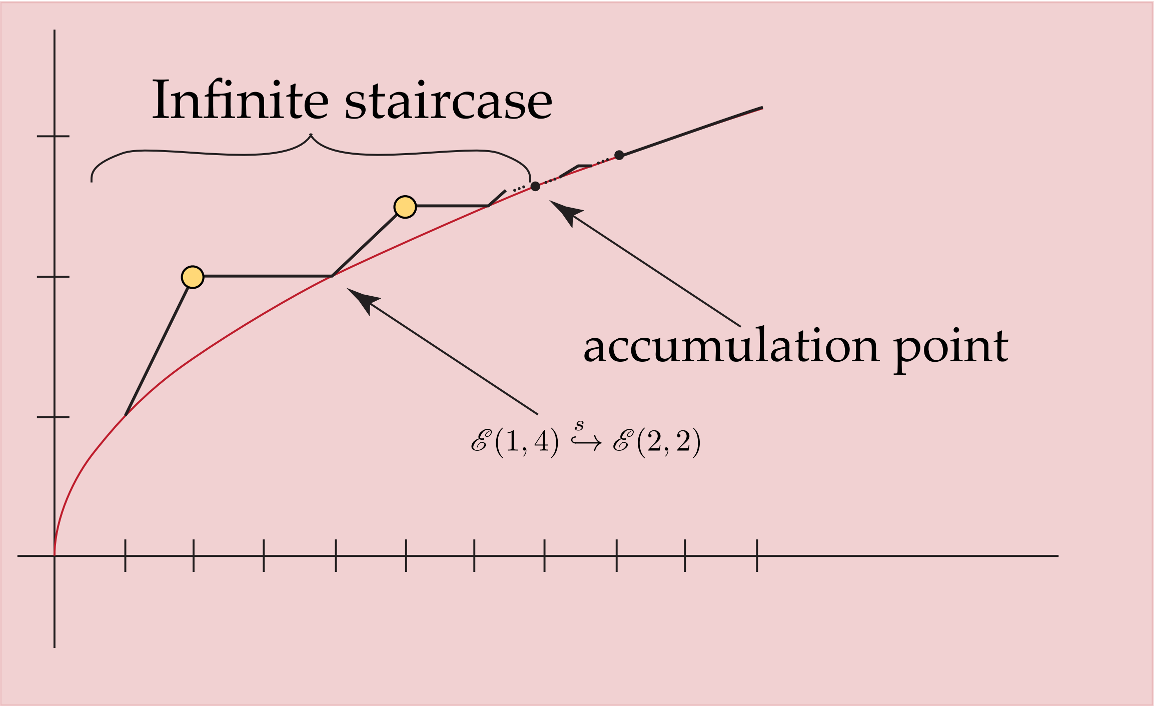
Infinite staircases: the proof

- Convex toric domain $\rightsquigarrow$ $(b; b_1, \ldots, b_n)$ the blowup form
- For any convex toric domain X, $c_X(a)$ is continuous, non-decreasing, and piecewise linear when not equal to the volume lower bound.



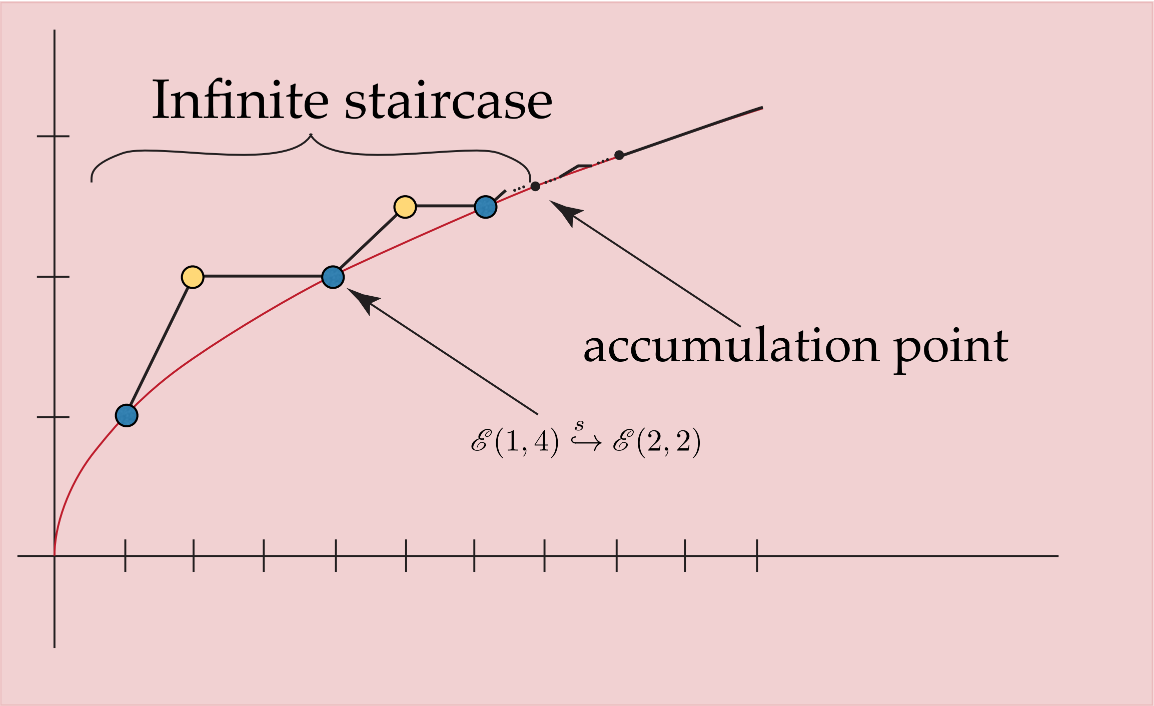
Infinite staircases: the proof

- Convex toric domain $\rightsquigarrow$ $(b; b_1, \ldots, b_n)$ the blowup form
- For any convex toric domain X, $c_X(a)$ is continuous, non-decreasing, and piecewise linear when not equal to the volume lower bound.

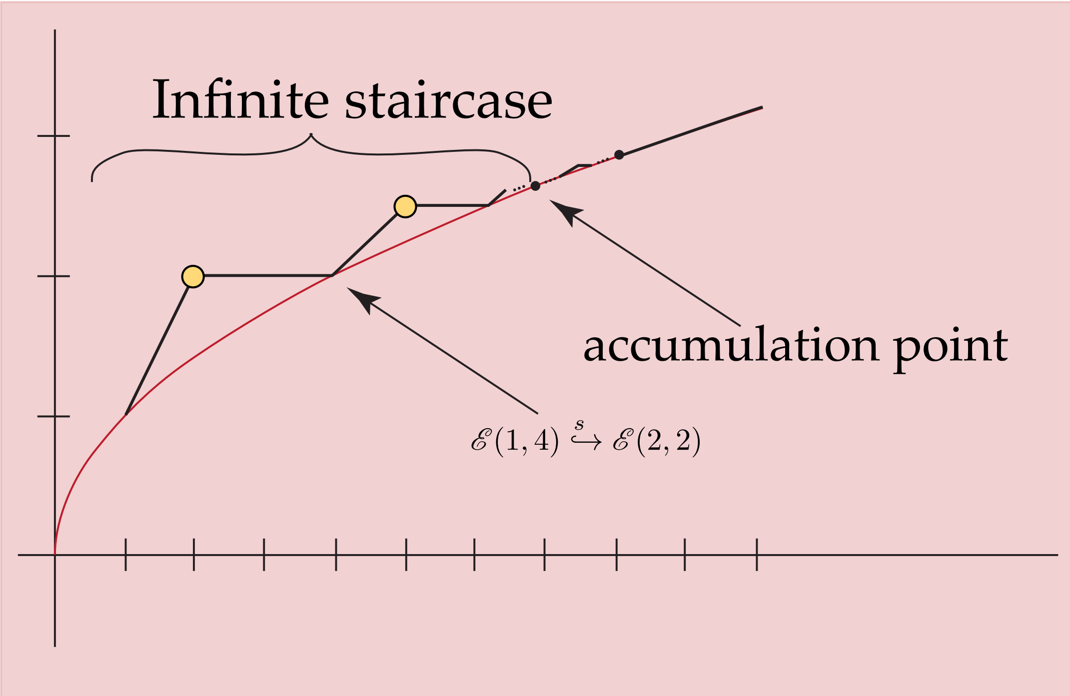


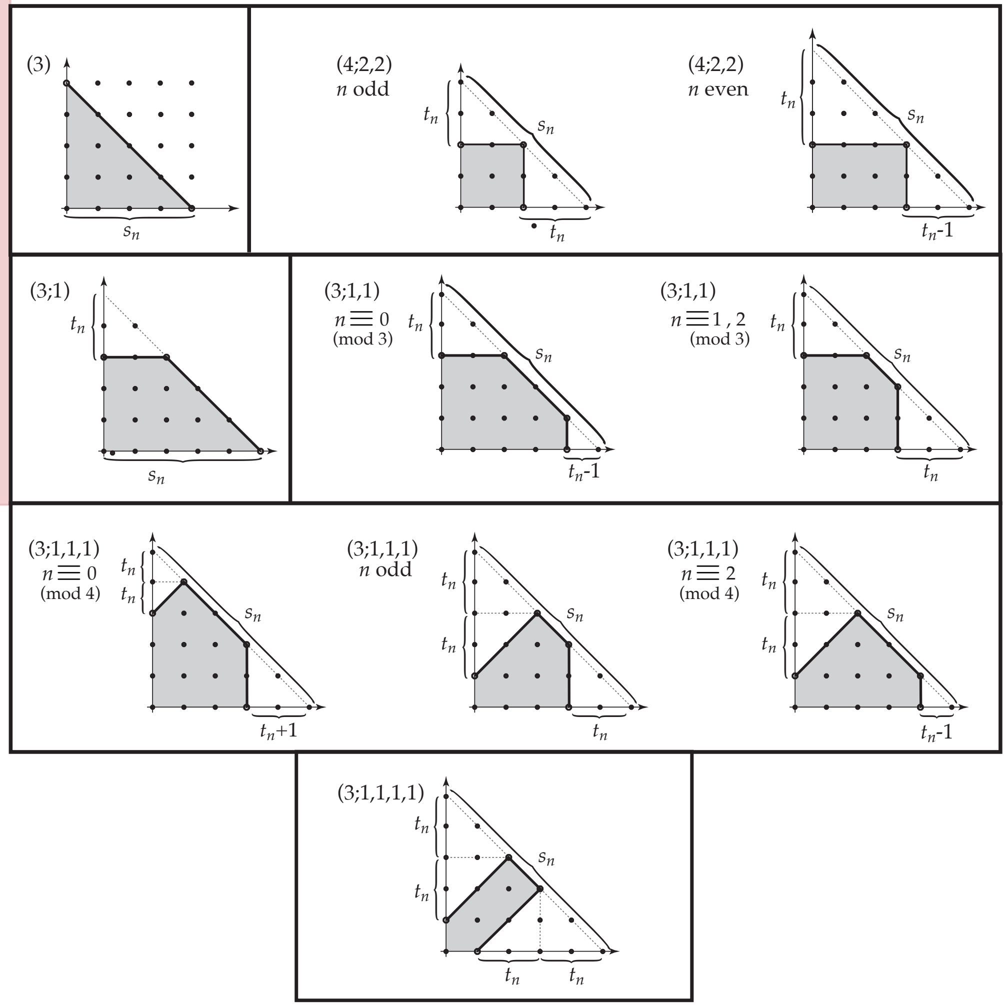
Infinite staircases: the proof

- Convex toric domain $\rightsquigarrow$ $(b; b_1, \ldots, b_n)$ the blowup form
- For any convex toric domain X, $c_X(a)$ is continuous, non-decreasing, and piecewise linear when not equal to the volume lower bound.



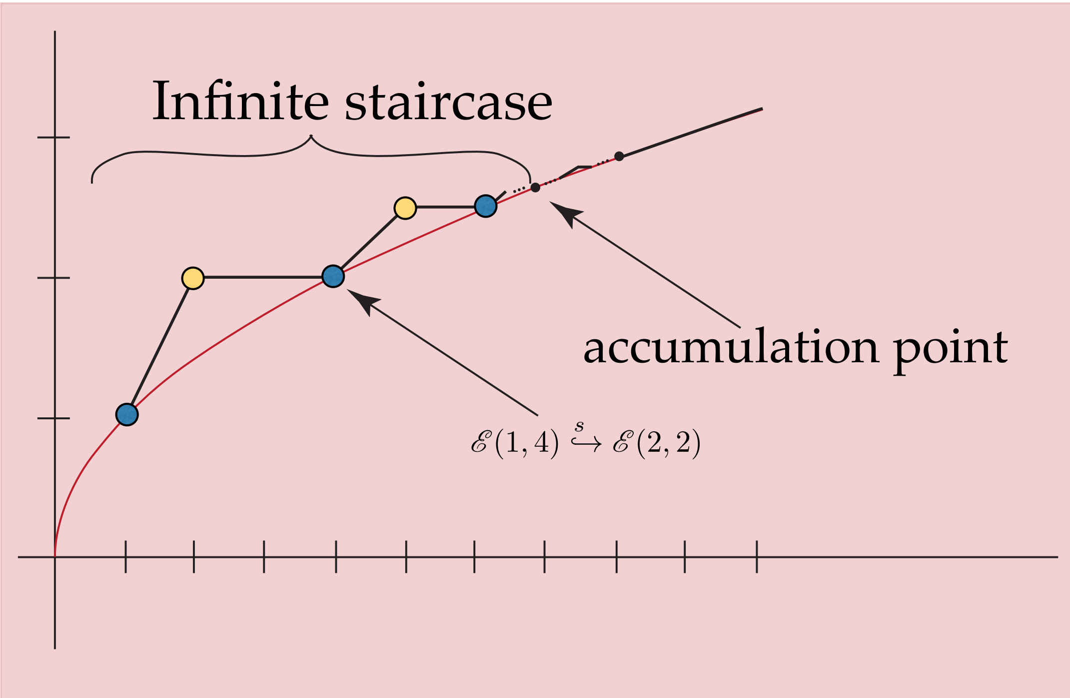
Infinite staircases: outer corners

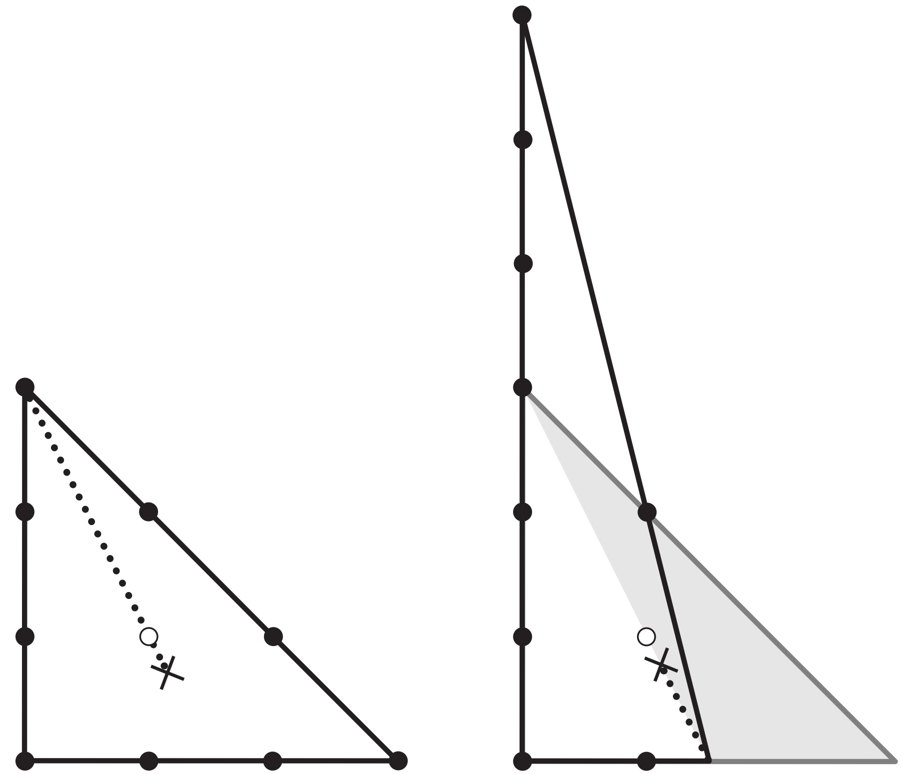




ECH capacities by counting lattice points

Infinite staircases: inner corners





Finding embeddings via ATFs