Integrable Systems: from Poisson to Nijenhuis
Exercises

Alexey Bolsinov
Loughborough University, UK

EXERCISE 1. Using the second (alternative) definition of the Nijenhuis torsion, prove the following important property (used for constructing many of integrable hierarchies). Let L be Nijenhuis and α_0 a closed differential 1-form such that $\alpha_1 = L^*\alpha_0$ is closed also. Then $\alpha_2 = L^*\alpha_1$, $\alpha_3 = L^*\alpha_2$, etc. are all closed.

EXERCISE 2. Prove that if the eigenvalues of 2×2 operator L are constant, then L is Nijenhuis.

EXERCISE 3. Prove the following necessary and sufficient condition in dimension 2:

L is Nijenhuis if and only if $d \det L = (\operatorname{adj} L)^* \, d \operatorname{tr} L$. In more detail:

$$\Big((\det L)_x, (\det L)_y\Big) = \Big((\operatorname{tr} L)_x, (\operatorname{tr} L)_y\Big) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{where } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

EXERCISE 4. Use this criterion to check that $L = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$ with $u = u(x, y)$, $v = v(x, y)$ and $v \neq 0$, is Nijenhuis if and only if the function $u + iv$ is holomorphic in complex variable $z = x + iy$.

EXERCISE 5. Construct an example of a Nijenhuis operator $L(u)$ whose components $L^i_j(u)$ are homogeneous polynomials in u of degree k.

Let $x_1, \dots, x_n$ be local coordinates.

EXERCISE 6. Prove that $L = \begin{pmatrix} x_1 & 1 & & \\ x_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ x_n & 0 & \dots & 0 \end{pmatrix}$ is Nijenhuis. Check that $f = x_1$ is a conservation law of L and construct the corresponding hierarchy $f_0 = f, f_1, f_2, \dots$, where $d f_k = (L^k)^* \, d f$.

EXERCISE 7. Prove that the *Toeplitz matrix* $L = \begin{pmatrix} x_1 & 0 & & \\ x_2 & x_1 & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ x_n & \dots & x_2 & x_1 \end{pmatrix}$ defines a Nijenhuis operator. Check that x_i is a conservation law of L.

Exercise 8. Consider the left-symmetric algebra associated with the Nijenhuis operator $L = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$. Prove that this algebra is non-degenerate.

Exercise 9. Consider the left-symmetric algebra associated with the Nijenhuis operator $L = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$. Prove that this algebra is degenerate.

Exercise 10. Let $L = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$. Prove that the conservation laws of L are of the form $\sum_i f_i(x_i)$.

Exercise 11. Let $L = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$. Prove that the symmetries of L are of the form $M = \begin{pmatrix} h_1(x_1) & & & \\ & h_2(x_2) & & \\ & & \ddots & \\ & & & h_n(x_n) \end{pmatrix}$.

Poisson 2026 Summer School, Antwerp, 3-7 August
Email address: `A.Bolsinov@lboro.ac.uk`