Modular integrable systems, their deformations and classification

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I'll discuss the moduli space approach to finite dimensional integrable systems admitting "good" rational Lax representations, as well as their deformations to isomonodromy systems. This viewpoint was initiated by Hitchin in the higher genus case with no poles and then extended by Nitsure/Bottacin/Markman to cover the cases with poles, and in particular to incorporate existing systems in the genus zero case (rational matrices) where many of the applications lie. A key step (Biquard-B. 2004) was to show such integrable systems have complete hyperkahler metrics on their total space, and so (by hyperkahler rotation), are canonically diffeomorphic to moduli spaces of meromorphic connections. This step amounts to a generalisation of nonabelian Hodge theory to incorporate poles of any order, and this rotation is a modern intrinsic way to understand the autonomous limit of an isomonodromy system to an isospectral integrable system (dating back to Painlevé and Garnier).

Finally one can also consider the moduli spaces of monodromy/Stokes data (the wild character varieties) corresponding to such spaces: we will view these as "global" analogues of Lie groups and explain some first steps in their classification, via diagrams, analogous to Dynkin diagrams. One of the simplest class of examples of wild character varieties (related to Painlevé II) are the Flaschka–Newell surfaces:

$$x\,y\,z + x + y + z = b - b^{-1}$$

where $b \in \mathbb{C}^*$ is constant. These complex surfaces underly the simplest examples of the wild nonabelian Hodge hyperkahler manifolds, and their diagram is affine A_1.

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