ALGEBRA AND ARITHMETIC (ALGAR) 2025 :
LINEAR ALGEBRAIC GROUPS AND ALGEBRAS WITH INVOLUTION

Contents

Date: July 2025.

1. Content description

The ALGAR Summer School 2025 is devoted to the topic of linear algebraic groups and algebras with involution. It intends to provide a bridge between a well-studied class of algebraic structures to the study of properties of varieties.

Two central problems in arithmetic and algebraic geometry are to decide whether some variety over a given field has rational points or whether it is even rational, that is, birational to some affine n-space for some n. This problem is particularly well studied for linear algebraic groups and their torsors.

If K is an algebraically closed field, then it turns out that every connected linear algebraic group over K is rational.

We will work over a general field K of characteristic different from 2. A typical example of a linear algebraic group is the orthogonal group $\mathsf{O}(\varphi)$ of some non-degenerate quadratic form φ over K. It has two connected components, one of them being the special orthogonal group $\mathsf{SO}(\varphi)$. The latter group is always a rational variety. The situation changes if we extend the group $\mathsf{SO}(\varphi)$ to the group of projective similitudes $\mathsf{PSim}(\varphi)$ and its connected subgroup $\mathsf{PSim}^+(\varphi)$. We assume now that $\mathsf{dim}(\varphi) = 2n$ with $n \in \mathbb{N}$. It was shown by Merkurjev [30] that $\mathsf{PSim}^+(\varphi)$ is non-rational in general when $n \geqslant 3$. This crucially relies on the fact that a rational variety is in particular rationally connected, a property first studied by Y. Manin around 1974. This property is generally weaker than rationality, and for certain linear algebraic groups like $\mathsf{PSim}^+(\varphi)$, it has a very interesting algebraic interpretation, also recognised by Merkurjev. One aim of the summer school is to shed light on these phenomena.

The groups $\mathsf{SO}(\varphi)$ and $\mathsf{PSim}^+(\varphi)$ are examples of connected semi-simple linear algebraic groups, and the latter one is adjoint, i.e. its center is trivial. Over an algebraically closed field, linear algebraic groups are classified in terms of root systems and corresponding Dynkin diagrams. An extension of this classification which works over arbitrary fields was provided by André Weil [42] in 1961. The classification makes crucial use of algebras with involution. Let V be the K-vector space on which φ is defined. Then φ induces a (so-called adjoint) involution on the endomorphism algebra $\mathsf{End}_K(V)$. More generally, one considers automorphism groups of a K-algebra with involution (A, σ), that is, where A is a central simple K-algebra and σ is an involution on A. By Weil's classification, all classical simply connected linear algebraic groups are obtained in a similar fashion.

Note that an algebraic group G over K always has K-rational points. To obtain varieties related to G for which the presence of rational points becomes an interesting question is that of a G-torsor, that is, a K-variety on which G acts in a simply transitive way. Given a G-torsor X, there will always be a finite Galois extension L/K where X has an L-rational point, and this fact can be used to classify G-torsors over K by a Galois cohomology set. The question then arises for which type of groups G and over which type of fields K a non-trivial G-torsor can arise. Serre's Conjectures I and II postulate that this does not happen over fields of small cohomological dimension. This can be seen as an extension of the fact that over an algebraically closed field, connected linear algebraic groups are rational. One of the central topics of this summer school will be to formulate, illustrate and explain these two famous conjectures.

2. Preliminary days and prequisites

Throughout the summer school, we assume fluency with basic algebraic structures covered in most bachelor's programmes, as well as some familiarity with the following concepts:

- Tensor products of modules and of algebras over commutative rings – [29]
- Projective modules – [29]
- Quadratic forms
- Quaternion algebras – [21, Chap. 1]
- Central simple algebras – [21, Sections 2.1-2.4]
- Discrete valuations – [21, Appendix A.6]

The preliminary days will consist of the following six half-day workshops, starting each with a one-hour lecture, followed by an interactive exercise session.

2.1. **Functorial approach to affine schemes.** We give a light introduction to the theory of *affine schemes* from the functorial point of view. We explain why the idea of algebraic varieties defined by polynomial equations is captured by (representable) functors on categories of commutative algebras, which leads to the concept of affine scheme. We define *morphisms* between schemes, their algebra of functions, their *open* and *closed subschemes*, the notion of *connectedness*, as well as *birational maps.*

2.2. **Quadratic forms.** We introduce the basic concepts on quadratic forms over fields. In this context, we will restrict to fields of characteristic different from 2. In particular, *isometry*, *isotropy*, *hyperbolicity*, the operations *orthogonal sum* and *tensor product*, as well as *isometry* and the *Witt decomposition theorem* are revisited. We look at the *orthogonal group* of a quadratic form as an example of a linear algebraic group. We define the group of similarity factors of quadratic forms and relate it to the group of proper projective similitudes. We also introduce the so-called *Pfister forms.*

2.3. **Central simple algebras and involutions.** We give a brief introduction to *central simple algebras* and formulate *Wedderburn's Theorem.* We discuss splitting fields and introduce the *reduced norm* and *reduced trace* of elements. Quaternion algebras provide a first interesting class of examples of central simple algebras. We look at *involutions* on a central simple algebras and show how on a matrix algebra a quadratic form gives rise to an (orthogonal) involution. We will see the different types of involutions, namely *orthogonal*, *symplectic* and *unitary* involutions, with basic examples.

2.4. **Affine group schemes.** We introduce affine group schemes, in several equivalent ways: as group objects in the category of schemes, as representable functors from commutative algebras to groups, or as the spectrum of a Hopf algebra. We give various examples of linear algebraic groups, which are affine group schemes of finite type over fields, related to isometries of bilinear forms, and automorphisms of algebras.

2.5. **Galois cohomology.** We establish the basic notions of *Galois cohomology*, a cohomology theory which is well-tailored to the theory of fields. We define and discuss *profinite groups*, in order to introduce *continuous cochains, cocycles* and *coboundaries*. Using these, one can define the cohomology groups of a Galois field extension, which exhibit some nice functorial properties. Towards the end, we will also state some well-known results, as well as touch upon the partial generalisation to a non-abelian setting.

2.6. **Clifford algebra and spin group.** We introduce the *Clifford algebra* of a quadratic form. This leads to multiple connections to the different objects already introduced. Depending on the parity of the dimension, either the full Clifford algebra or its even part is a central simple algebra and actually a tensor product of quaternion algebras. It also gives rise to the definition of a cohomological invariant, the *Clifford invariant*. Finally, it is used to define the *spin group* of a quadratic form, which is an example of a semi-simple simply connected linear algebraic group.

3. Main lectures

3.1. **The classical groups (N. Garrel).** In this session we introduce the classical groups, and their description in terms of algebras with involution, following [28] as our main reference. Given an algebra with involution (A, σ), we describe the algebraic group $\mathsf{Sim}(A, \sigma)$ of similitudes of (A, σ), which generalizes the group of similitudes $\mathsf{Sim}(V, h)$ of a bilinear or hermitian form. From it various subquotients are derived, such as the group of isometries $\mathsf{Iso}(A, \sigma)$, proper isometries $\mathsf{Iso}^+(A, \sigma)$, or the groups of projectives similitudes $\mathsf{PSim}(A, \sigma)$ and proper projective similitudes $\mathsf{PSim}^+(A, \sigma)$, among others. Those groups are called the *classical groups.*

To explain the place of these groups in the classification of algebraic groups, we introduce the classes of *reductive*, *semisimple* and *simple* algebraic groups, and show how reductive groups can be reconstructed from absolutely simple groups. In the 1950s, the Chevalley Seminar gave a classification of those absolutely simple groups in terms of types: the classical types A_n, B_n, C_n and D_n, and the exceptional types, following the earlier Cartan-Killing classification of Lie algebras from the 1890s.

In 1960, Weil [41] showed that the classical groups we introduced earlier are exactly the absolutely simple groups of classical type.

3.2. ***R*-equivalence on algebraic groups I (P. Gille).** Given a k-variety X, Y. Manin defined the R-equivalence on the set of k–points $X(k)$ as the equivalence relation generated by the following elementary relation. Denote by $\mathcal{O}$ the semi-local ring of $\mathbf{A}^1_k$ at 0 and 1. Two points x_0, $x_1 \in X(k)$ are elementary R-equivalent is there exists $x(t) \in X(\mathcal{O})$, such that $x(0) = x_0$ and $x(1) = x_1$. This extends to an equivalence relation on $X(k)$, called *R-equivalence.* We denote by $X(k)/R$ the set of R-equivalence classes. This quotient set provides us an invariant that measures somehow the defect for parametrizing rationally the k-points of X.

If G is an algebraic k–group, $G(k)/R$ carries a natural group structure. We start with generalities as functorialities in the group, in the field, and relation with the R-equivalence of a compactification of G.

We will then discuss several notion of rationality and how it applies to nice situations as split groups or special orthogonal groups. An important feature is that of R-triviality. We say that a k–group G is *R-trivial* if $G(F)/R = 1$ for each field F/k. If G is retract rational, then it is R-trivial. The converse is an open question.

3.3. **Projective similitudes and Merkurjev's rationality criteria (M. Archita).** Consider a field K and a central simple K-algebra with involution (A, σ). We will study the linear algebraic group of proper projective similitudes $\mathsf{PSim}^+(A, \sigma)$. This is a connected linear algebraic group of adjoint type, and we shall investigate whether it is rationally connected.

As a guiding example, we take the case where $A = \mathsf{End}_K V$ for a finite-dimensional K-vector space V and where $\sigma = \mathsf{ad}_\varphi$, the adjoint involution of some regular quadratic form φ. (This covers precisely the case where A is split and σ is orthogonal.)

An element $x \in A$ is called a *similitude of* (A, σ) if $x\sigma(x) \in K^\times$, and in this case we set $\mu(x) = x\sigma(x)$ and call this the corresponding *multiplier of similitude.* We denote by $\mathrm{Sim}(A, \sigma)$ the group of similitudes of (A, σ) and obtain that

$$\mu : \mathrm{Sim}(A, \sigma) \to K^\times, x \mapsto x\sigma(x)$$

is a group homomorphism.

Now $\mathsf{PSim}(A, \sigma)$ is the linear algebraic groups over K whose set of K-rational points are given by $\mathrm{Sim}(A, \sigma)/K^\times$. Its connected component of the identity is denoted by $\mathsf{PSim}^+(A, \sigma)$.

In [30, Theorem 1] Merkurjev characterised whether $\mathsf{PSim}^+(A, \sigma)$ is rationally connected in terms of the group of multipliers of similitudes

$$\mathsf{G}(\sigma) = \mu(\mathrm{Sim}(A, \sigma))$$

and certain subgroups $\mathsf{H}(\sigma) \subseteq \mathsf{G}^+(\sigma) \subseteq \mathsf{G}(\sigma)$, where $\mathsf{H}(\sigma)$ is generated by $K^{\times 2}$ and the norms from finite extensions where σ becomes hyperbolic.

Theorem (Merkurjev)**.** *The linear algebraic group $\mathsf{PSim}^+(\sigma)$ is rationally connected if and only if $\mathsf{G}^+(\sigma_{K'}) = \mathsf{H}(\sigma_{K'})$ holds for every field extension K'/K.*

Based on this theorem, we will see various cases where we can decide whether $\mathsf{PSim}^+(\sigma)$ is rationally connected. For that we will focus on the case where $A = \mathsf{End}_K V$ and $\sigma = \mathsf{ad}_\varphi$ for a quadratic form φ, and we will simply write $\mathsf{PSim}^+(\varphi)$, $\mathsf{G}(\varphi)$ etc. So we are discussing rational connectedness of $\mathsf{PSim}^+(\varphi)$. We have

$$\mathsf{G}(\varphi) = \mathsf{G}^+(\varphi) = \{a \in K^\times \mid a\varphi \simeq \varphi\} .$$

Here is a list of cases where $\mathsf{PSim}^+(\varphi)$ is rationally connected:

- $\mathsf{dim}(\varphi)$ is odd or φ is Pfister form. (In both cases $\mathsf{PSim}^+(\varphi)$ is rational.)
- φ is a tensor product of a Pfister form and a form of odd dimension. (In this case $\mathsf{PSim}^+(\varphi)$ is stably rational.)
- $\varphi = \pi \perp \rho$ where π and ρ are scaled Pfister forms of different dimension.

This covers in particular all forms φ with $\mathsf{dim}\varphi < 6$. Using his criterion, Merkurjev found the first examples of 6-dimensional forms φ for which $\mathsf{PSim}^+(\varphi)$ is not rationally connected.

3.4. **Cohomological dimension of fields (D. Izquierdo).** We will mainly follow the notes [24]. In the first part, we will introduce the Galois cohomology of fields, which is a quite technical but also very versatile tool to encode algebraic and arithmetic properties of fields. We will see for instance that it allows to encode Kummer theory for characteristic 0 fields and Artin-Schreier theory for positive characteristic fields.

We will then focus on an invariant that measures the complexity of the Galois cohomology of a field, called the *cohomological dimension* of the field. We will state some of its main properties. For instance, we will see that a perfect field has cohomological dimension 0 if and only if it is algebraically closed, and that finite fields have cohomological dimension 1. We will also see how the cohomological dimension behaves for function fields and for complete discretely valued fields. In particular, we will see that the cohomological dimension of the rational function field $k(x_1, \dots, x_n)$ over some field k is equal to n plus the cohomological dimension of k. This property explains why the cohomological dimension can be understood as a dimension for fields.

3.5. **Counterexamples for rational connectedness (M. Archita).** We consider a base field k with $\mathsf{char}(k) \neq 2$. Consider elements $a_1, a_2, a_3 \in k^\times$ such that $[k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : k] = 8$. We set $\ell = k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ and attach the following group:

$$\Lambda(\ell/k) = \left(\mathsf{N}^*_{k(\sqrt{a_1})/k} \cap \mathsf{N}^*_{k(\sqrt{a_2})/k} \cap \mathsf{N}^*_{k(\sqrt{a_3})/k}\right) /k^{\times 2} \cdot \mathsf{N}^*_{\ell/k}\,.$$

It turns out that this group is not always trivial. However, to construct and confirm such examples is not easy. The first examples of such triquadratic extensions were constructed by J.-P. Tignol in [38]. They were in characteristic 0 and the construction was based on the presence of a valuation with residue characteristic 2. Other constructions leaving more flexibility, were obtained by Sivatski in [34] and [35]. These examples where $\Lambda(\ell/k)$ is nontrivial are crucial in the construction of important examples for diverse problems in quadratic form theory, such as the existence of indecomposable division algebras of exponent 2 and degree 8.

It was observed by Ph. Gille in [15] that such constructions can also be used to produce forms φ where $\mathsf{PSim}^+(\varphi)$ is not rationally connected.

Based on this technique, we will see, following [1], that there exist quadratic forms φ of trivial discriminant over $\mathbb{C}(X, Y, Z)$ such that $\mathsf{PSim}^+(\varphi)$ is not rationally connected. Furthermore, for any $n \geqslant 3$, we construct a quadratic form φ which is a difference of two n-fold Pfister forms such that $\mathsf{PSim}^+(\varphi)$ is not rationally connected. This strengthens a result of N. Bhaskhar [4].

3.6. ***R*-equivalence on algebraic groups II (P. Gille).** In this sequel to Section 3.2, we plan to discuss the following topics.

(a) *The case of algebraic tori.* It is has been investigated by Colliot-Thélène and Sansuc [12] by means of flasque resolutions of tori and Galois cohomology. There is a nice characterization of R-trivial tori which is a first step towards Voskresenskiĭ's conjecture stating that stably k–rational tori are k–rational.

(b) *The case of special linear algebraic groups.* This is the case of $G = \mathrm{SL}_n(D)$ where D is a central simple division k–algebra. The first result is that $G(k)/R \simeq \mathrm{SK}_1(D)$ (Voskresenskiĭ); it can be expressed in terms of the K-theory of D and is independent of n. The second result is that of Wang [21, §2.10] stating that

$\mathrm{SK}_1(D) = 1$ when D has squarefree index. The third result is the vanishing of $\mathrm{SK}_1(D)$ if k is a field of cohomological dimension $\leqslant 2$ (Yanchevskiĭ [43]). We will explain also Platonov's examples of non trivial $\mathrm{SK}_1(D)$ following Wadsworth's survey on valued algebras [40]. This leads to the famous Suslin's conjecture stating that $\mathrm{SL}_n(D)$ is R-trivial if and only if D has squarefree index.

(c) *Simply connected isotropic groups.* It generalizes the $\mathrm{SL}_n(D)$ case ($n \geqslant 2$). Let G be a semisimple simply connected algebraic k–group G/k assumed absolutely k-simple (that is, its absolute Dynkin diagram is connected) and isotropic (i.e. contains $\mathbb{G}_m$). In this case the group $RG(k)$ of R–trivial elements is the Kneser-Tits subgroup $G(k)^+$ generated by unipotent elements [18, §7]. By Tits' simplicity theorem, it follows that $RG(k)$ modulo its center is a simple group. This has numerous consequences for the simplicity of the group $G(k)$ modulo center. We shall discuss specific cases as $\mathrm{Spin}(q)$, groups of type D_4, etc.

3.7. **Galois descent and classification of classical groups (N. Garrel).** In this session we shall see why the groups introduced in Session 3.1 give a complete list of classical groups, using Galois descent, as shown by Weil in [41].

The key method is Galois descent: given a base algebraic group G_0 (resp. algebra with involution (A_0, σ_0)), it allows us to describe through cohomological methods all algebraic groups G (resp. algebras with involution (A, σ)) such that G and G_0 (resp. (A, σ) and (A_0, σ_0)) become isomorphic over some Galois extension.

Each algebra with involution and each reductive group becomes split over some large enough Galois extension, so we can describe any such algebra or group by Galois descent once we know the split ones. The classification of split algebras with involution is easy, while the classification of split simple algebraic groups is heavy work, which we formulate without elaborating upon (see the work of Chevalley et al.). It turns out that split groups can be described in terms of split algebras with involution, and all classical groups are thus described in terms of algebras with involution, which yields the list that was established in Session 3.1.

3.8. **Conditions on powers of the fundamental ideal where rational connectedness occurs (M. Archita).** In this lecture we will discuss some results from [1]. We continue to consider a quadratic form φ over a field K and the question whether $\mathsf{PSim}^+(\varphi)$ is rationally connected. We have seen that this is characterized in terms of norm maps for certain field extensions. So we study how norms behave in quadratic extensions and extensions of higher degree. We see a crucial lemma about norms in biquadratic extensions, from which one can derive several positive results. In particular, we see that $\mathsf{PSim}^+(\varphi)$ is rationally connected when φ is a sum of two scaled Pfister forms of different dimension.

We then look at the powers $\mathsf{I}^n K$ of the fundamental ideal $\mathsf{I}K$ in the Witt ring of quadratic forms over K, where $n \in \mathbb{N}$. We show that, if $\mathsf{I}^{n+1}K = 0$, then for every quadratic form $\varphi \in K$, we have $\mathsf{G}(\varphi) = \mathsf{H}(\varphi) = K^{\times 2}$. This applies in particular over fields of cohomological dimension n. However, this does not yet imply that

$\mathsf{PSim}^+(\varphi)$ is rationally connected in this case, as there might exist a transcendental extension K'/K such that $\mathsf{G}(\varphi_{K'}) \neq \mathsf{H}(\varphi_{K'})$.

3.9. **Principal homogeneous spaces and Serre's Conjectures (D. Izquierdo).** We will start by introducing principal homogeneous spaces under algebraic groups. Over a field of characteristic 0, these are algebraic varieties X endowed with an algebraic action of an algebraic group G so that the induced action on geometric points is simply transitive. We will give various concrete examples.

We will then be ready to state Serre's Conjectures I and II ([33]), that aim at characterizing fields with cohomological dimension 1 or 2 in terms of rational points on principal homogeneous spaces. More precisely, we will see that Serre's Conjecture I predicts that principal homogeneous spaces under connected linear groups over fields with cohomological dimension 1 have rational points, and that Serre's Conjecture II predicts that principal homogeneous spaces under semisimple simply connected linear groups over fields with cohomological dimension 2 have rational points. Serre's Conjecture I was positively settled by Steinberg [36] and Borel-Springer [6], while Serre's Conjecture II is still widely open.

3.10. **Root data and structure of classical groups (N. Garrel).** In Session 3.7 we formulated the classification of split simple algebraic groups. The necessary theory was established in the Chevalley Seminar in the 1950s: maximal tori, Borel subgroups, Weyl group, roots and coroots, Dynkin diagram, etc. We refer to [31] for a modern exposition in the language of schemes. It was extended to non-split groups by Satake [32] and Tits [39]: in essence, each group can be decomposed in an 'anisotropic part' and a 'split part', and Satake-Tits theory explains how to understand the whole group from those two parts.

We will not prove any of those statements, even in the split case, but we will work our way through a few examples, both split and non-split, to explain in a hands-on manner how those notions emerge, how to compute them, and where the split and non-split cases agree or diverge.

3.11. ***R*-equivalence for group schemes I (P. Gille).** The recent paper [20] deals with the generalization of R-equivalence for group schemes defined over a ring B. The definition is very similar with that over a field. We consider the localized ring $B[t]_\Sigma$ where Σ is the multiplicative subset of polynomials P satisfying $P(0)\,P(1) \in B^\times$. Given a B–scheme X, two points x_0, $x_1 \in X(B)$ are elementary R-equivalent is there exists $x(t) \in X(B[t]_\Sigma)$, such that $x(0) = x_0$ and $x(1) = x_1$. We denote then by $X(B)/R$ the set of R-equivalence classes. It turns out that several generalities from the field case behave well in this framework. Functoriality issues become more subtle in this framework and depend of the ring.

To be more precise, the best case is when B is (semi)-local domain with infinite residue fields, we make this assumption from now on and denote by F the fraction field of B. For example, if G is an affine smooth connected group B–scheme such that G_K is retract rational, we will show that $G(B)/R = 1$ [20, prop. 2.20].

3.12. **R-equivalence for group schemes II (P. Gille).** The leading idea is to reexamine in the ring case the topics presented in the field case in Section 3.6 and one new.

(a) *The case of tori.* The theory generalizes well over rings by means of the flasque resolutions in this setting [13]. The interest is the functorality. For example if B is local henselian of residue field k, both base change maps $T(B)/R \to T(k)/R$ and $T(A)/R \to T(F)/R$ are isomomorphisms.

(b) *The case of special linear groups schemes.* This is the case of $G = \mathrm{SL}_n(\mathcal{D})$ where $\mathcal{D}$ is an Azumaya B–algebra. It turns out that we still have $G(B)/R \simeq \mathrm{SK}_1(\mathcal{D})$ as well with Wang's result. We have also nice behaviour in the henselian case.

(c) *Simply connected isotropic group schemes.* Let G be a semisimple simply connected algebraic B–group G assumed absolutely B-simple (that is, its absolute Dynkin diagram is connected) and isotropic (i.e. contains $\mathbb{G}_{m,B}$). Under additional conditions (rank $\geqslant 2$, B contains a field), the group $RG(B)$ of R–trivial elements is again the Kneser-Tits subgroup $G(B)^+$ generated by unipotent elements [20, Thm. 6.5]. In many examples, it permits to show that $G(B)$ is generated by unipotent elements.

(d) Specialization of R-equivalence Assume that B is an henselian DVR and G is a reductive B–group scheme. We have maps $G(B)/R \to G(F)/R$ and $G(B)/R \to G(k)/R$. We are interested to have a third map $G(F)/R \to G(k)/R$ called the specialization map satisfying a natural compatibility. This exists [17] and is used in the arithmetic of algebraic groups over semi-global fields [11]. There are partial results in dimension 2 [11, §A.3] and for regular local rings containing a field [20, §8.5].

3.13. **Known results about Serre's Conjecture II (D. Izquierdo).** We will show some of the most important results about Serre's Conjecture II. Some of them allow to prove the whole conjecture for given fields, while others prove the conjecture for specific types of algebraic groups over arbitrary fields.

In terms of fields, Serre's Conjecture II is for instance known over complete discretely valued fields (Bruhat-Tits [7], Gille-I.-Lucchini Arteche [?GIL]), totally imaginary number fields (Kneser [26] [27], Harder [22] [23], Chernousov [8]), function fields in two variables over $\mathbb{C}$ (de Jong, He, Starr [14]) and finite extensions of $\mathbb{C}((x,y))$ (Colliot-Thélène-Gille-Parimala [9]). In terms of groups, the conjecture is known for groups of classical types (Merkurjev-Suslin [37], Gille [16], Bayer-Fluckiger-Parimala [2], Berhuy-Frings-Tignol [3]), as well as some exceptional types (G_2, F_4). It remains open in general for types E_6, E_7, E_8 and trialitary D_4. At the end, I will discuss a recent result together with Lucchini Arteche [25] that allows to reduce the conjecture to the case of fields of characteristic 0.

4. Special talks

4.1. **Springer's theorem for quadratic lattices over Dedekind domains (Jing Liu).** A classical Theorem due to T.A. Springer states that a quadratic form over a field is already isotropic if it becomes isotropic over a base change of odd degree. In this talk, I will report on Springer-type results for representation and isometry of quadratic lattices over Dedekind domains. As key ingredient in the proof, certain norm principles will be mentioned. This is based on my PhD thesis and in part on joint work with Yong Hu and Fei Xu.

4.2. **The Artin-Springer Theorem for algebras with involution (Abhigyan Writwik Medhi).** A theorem for quadratic forms due to Artin and Springer states that, if a quadratic form over a field F becomes isotropic over an odd-degree extension of F, then it must be isotropic over the base field F. In this talk, I will recall from [28, Chapter 1] how the theory of central simple F-algebras with involution/quadratic pair can be thought of as a natural generalization of the theory of quadratic spaces. This motivates us to find an analogous result to Artin-Springer's theorem for quadratic forms, in the context of algebras with involution/quadratic pair. I will talk about the known results in this direction, focusing on the different types of involutions separately. (See also [5].) If time permits, I will end the talk by mentioning my work which relates the isotropy of a quadratic pair over a generic splitting field and isotropy over an odd degree extension of the base field.

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