ALGEBRA AND ARITHMETIC (ALGAR) 2023
LOCAL-GLOBAL PRINCIPLES FOR QUADRATIC FORMS

Contents

1. Content description 2
2. Introductory days 2
3. Main talks 2
3.1. Quadratic forms and valued fields (N. Daans) 2
3.2. Hasse principles over number fields (R. Parimala) 3
3.3. A first glimpse into Berkovich analytic spaces. Part I (V. Mehmeti) 3
3.4. Quadratic forms, Galois cohomology and Milnor's conjecture (D. Izquierdo) 4
3.5. Invariants for quadratic forms in Galois cohomology (R. Parimala) 4
3.6. A first glimpse into Berkovich analytic spaces. Part II (V. Mehmeti) 4
3.7. Defining valuation rings with local-global principles (N. Daans) 5
3.8. Brauer-Hasse-Noether exact sequences and the arithmetic of Pfister forms (D. Izquierdo) 5
3.9. Field patching and local-global principles (R. Parimala) 6
3.10. Patching over Berkovich analytic curves (V. Mehmeti) 6
3.11. Tate-Shafarevich groups and Poitou-Tate duality (D. Izquierdo) 6
3.12. Local–global principles for quadratic forms defined over function fields of curves (V. Mehmeti) 7
3.13. Bounded symbol lengths for function fields over p-adic fields and number fields. (R. Parimala) 7
3.14. Arithmetic of homogeneous spaces (D. Izquierdo) 8
3.15. The Pythagoras number of function fields (N. Daans) 8
4. Special talks 8
4.1. R-equivalence on adjoint groups (Archita Mondal, IIT Bombay) 8
4.2. Universal quadratic forms over p-adic and number fields (Yong Hu, Southern University of Science and Technology, Shenzen) 9
4.3. A local-global principle for rational function fields (Marco Zaninelli, University of Antwerp) 9
References 9

Date: August 14, 2023.

1. Content description

Quadratic forms are algebraic objects with particularly nice algebraic and geometric properties. Local-global principles for quadratic forms are a classical topic of algebra and number theory. The Hasse-Minkowski Theorem formulates such a local-global principle for the case of a number field or a function field of a curve over a finite field. It can for example be used to show that any sum of squares in a number field is a sum of four squares.

While quadratic forms are interesting to study over general fields, it is only for very special fields that isotropy can be determined by a local-global principle. Such a local-global principle is usually expressed in terms of completions (or henselisations) of a field with respect to valuations. In the last two decades, a new technique called field patching has led to the discovery of a series of new cases of fields where certain types of quadratic forms satisfy a local-global principle. This includes in particular function fields of curves over complete discretely valued fields, such as the fields $\mathbb{Q}_p$ of p-adic numbers, where p is prime number.

When a local-global principle is present, it can also shed light on other features of a field. This applies in particular to the study of field invariants such as the u-invariant or the Pythagoras number. Recent breakthroughs establishing upper bounds on such invariants for particular fields have been achieved in this way. Also the study of Hilbert's 10th Problem has seen recent progress based on certain local-global principles.

The aim of the summer school is to introduce the attendees to this active research area, with an emphasis on the applicability of local-global principles. This will include providing the context for different scenarios of application, such as the study of the u-invariant and the Pythagoras number of fields as well as of Hilbert's 10th Problem. Also the nuances between different types of local-global principles, for example whether formulated in terms of discrete or of more general valuations, as well as the notorious case of characteristic 2, will be highlighted. Finally, examples of failure of local-global principles will be examined.

2. Introductory days

The first two days, Thursday 17 and Friday 18 August, will give an introduction in the form of exercise workshops. We will solve exercises training the basic concepts of quadratic form theory over fields, valuations, completions and basic invariants of quadratic forms are trained. In particular, we we will look into quadratic forms and their classification over the fields of p-adic numbers $\mathbb{Q}_p$.

3. Main talks

3.1. **Quadratic forms and valued fields (N. Daans).** In this talk, we introduce quadratic forms over fields, and briefly discuss some basic properties. We will then zoom in on the behaviour of quadratic forms over fields carrying a valuation, especially a henselian valuation, like the field of p-adic numbers $\mathbb{Q}_p$ or the field of formal Laurent series $\mathbb{R}((T))$. We will see that quadratic forms over a henselian valued field can often be well understood via the valuation's value group and the quadratic form theory of its residue field, a notion often referred to as *Springer's Theorem*. In the exposition we will try wherever feasible to work over fields of arbitrary characteristic. This includes in particular fields of characteristic 2, where subtleties may arise which are not visible in fields of other characteristics.

For an accessible introduction to quadratic form theory over complete discretely valued fields of residue characteristic different from 2, see e.g. [31, Chapter VI]. Our main references for quadratic forms over general henselian valued fields (not necessarily discrete and possibly with residue characteristic 2) are [3, 13, 34].

3.2. **Hasse principles over number fields (R. Parimala).** We recall the classical theorem of Hasse and Minkowski giving a local-global principle for isotropy of quadratic forms over number fields. We explain how the proof uses certain basic results from class field theory. We then look at the case of function fields of curves over complete discretely valued fields, which we call *semi-global fields*. We state a Hasse principle for quadratic forms over such fields with respect to the divisorial discrete valuations on the field. These results lead to a proof that the u-invariant of function fields of curves over p-adic fields is 8, which is a theorem of [41] and [32]. In the next lecture we shall explain how the patching techniques of Harbater-Hartmann-Krashen are used in the proof of the Hasse principle for quadratic forms over semi-global fields [12].

3.3. **A first glimpse into Berkovich analytic spaces. Part I (V. Mehmeti).** *Berkovich spaces* are analytic varieties defined over complete rank-one valued fields. In this lecture I will give an introduction to these spaces, with a particular focus of *Berkovich analytic curves* and their structure.

Let (k, v_k) be a complete valued field, where v_k is of rank one, e.g. $k = \mathbb{Q}_p$ for a prime number p. The topology induced on k by v_k is *totally disconnected*, which makes it difficult to define analytic structures (such as analytic functions or manifolds) over k. One of the possible ways to do this is through Berkovich's theory, founded in the late 80's and vastly developed since, with many applications, namely in arithmetic geometry.

The k-analytic spaces obtained through this approach satisfy some properties analogous to those of complex analytic varieties: locally compact, locally path-connected, analytic functions are locally formal power series, etc. One-dimensional Berkovich spaces, i.e. Berkovich curves, are particularly well-behaved: they have the structure of a *real graph*.

We will start with some generalities on Berkovich's theory, with the purpose of understanding the local structure of these analytic spaces; this will eventually lead us to their definition. In the case of curves, we will see some details on their set-wise, topological, and analytic structure. We will also present an important classification of their points, which can surprisingly be interpreted both topologically and algebraically.

Examples will be presented throughout this lecture, with a particular focus at the end on the affine and projective analytic line.

3.4. **Quadratic forms, Galois cohomology and Milnor's conjecture (D. Izquierdo).** In this first lecture, we aim at classifying quadratic forms over fields of characteristic different from 2. We will start by recalling the constructions of the Witt ring and of the first invariants of quadratic forms (dimension, discriminant, Clifford invariant). We will then introduce the Galois cohomology of a field and, thanks to the Milnor conjecture, we will see how it allows to provide a complete classification of quadratic forms over fields.

References: [31] (Chap. II, III, IV, V, X.6), [43], [25].

3.5. **Invariants for quadratic forms in Galois cohomology (R. Parimala).** We define the Galois cohomology of a field with coefficients in a discrete Galois module. We describe the Galois cohomology groups of degree 1 and degree 2 with values in $\mathbb{Z}/2\mathbb{Z}$. We recall the Milnor maps from the nth power of the fundamental ideal of the Witt group to the degree-n Galois cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. We describe the maps in degree one and two as the discriminant and the Clifford invariant maps. We also define symbols in degree n and use the theorem of Orlov-Vishik-Voevodsky resolving Milnor's conjecture to get the nth Galois cohomology groups being generated by symbols. We define symbol lengths in degree n of a field. We also define the cohomological dimension of a field. We then prove a sufficient criterion for the finiteness of the u-invariant of a field in terms of finite cohomological dimension and bounded symbol lengths.

3.6. **A first glimpse into Berkovich analytic spaces. Part II (V. Mehmeti).** In this lecture I will explain how one can establish a connection between usual algebraic curves defined over a complete valued field k and Berkovich k-analytic curves. More precisely, given a nice algebraic curve C over k, one can construct its *Berkovich analytification* C^{an}, which is a Berkovich analytic curve.

The above construction gives rise to the so called *analytification functor*, which sends an algebraic object to an analytic one. We will see how this functor is constructed, and how there are principles of comparison between the properties of C^{an} and C, known as GAGA theorems. This is particularly strong in the case of curves.

As we are working with analytic objects, it is natural to define meromorphic functions on them. Seeing as it plays an important role later on, we will briefly introduce the sheaf of meromorphic functions on Berkovich curves. In the case of proper curves, this sheaf also satisfies a GAGA principle.

As a natural conclusion to this lecture, we will see the connection between Berkovich curves and rank one valuations. More precisely, let F denote the function field of the curve C. Then the points of C^{an} correspond to rank one valuations on the field F.

We will see a detailed description of this phenomenon in the case of the analytic projective line $\mathbb{P}_k^{1,\mathrm{an}}$, as well as some properties of the valuations on $k(T)$ determined by the points of $\mathbb{P}_k^{1,\mathrm{an}}$, such as their value groups and residue fields.

References for first two lectures: [5], [9], [36].

3.7. **Defining valuation rings with local-global principles (N. Daans).** When a field K carries a valuation v, it is often motivated to ask whether the valuation ring (respectively its maximal ideal) is *existentially definable* as a subset of K. Said more explicitly, we ask whether there exist some natural number k and a polynomial $f(X, Y_1, ..., Y_k)$ over K such that, for every $x \in K$, the polynomial $f(x, Y_1, ..., Y_k)$ has a zero in K^n if and only if $v(x) \geq 0$ (respectively $v(x) > 0$). If the valuation v is henselian, then the existential definability of the valuation ring and maximal ideal has been extensively studied, and can generally be obtained under mild conditions on the value group and residue field, see e.g. [2]. On the other hand, if the valuation is not henselian, then no general approach is known. We will start the talk by introducing and motivating the problem of existentially defining valuation rings. We will then discuss how local-global principles can be used to existentially define non-henselian valuation rings. In particular, we will take a look at how to define valuation rings in number fields and global function fields using the Hasse-Minkowski theorem - a technique which goes back to Robinson [44] - as well as give a sketch of more recent work on definability of valuations in function fields, which partially relies on novel local-global principles as discussed in talks 3.9 and 3.10 [12, 35]. This work can be found in my PhD thesis [8, Chapter 7] and will appear in a publication coauthored with Becher and Dittmann.

3.8. **Brauer-Hasse-Noether exact sequences and the arithmetic of Pfister forms (D. Izquierdo).** In this lecture, we will see how to compute over various fields some of the cohomology groups that control the classification of quadratic forms. After a general discussion on the possible vanishing of cohomology groups and its impact on quadratic forms, we will focus on several arithmetically and geometrically interesting fields. We will first study the classical case of number fields by introducing the Brauer-Hasse-Noether exact sequence and by understanding what it concretely means. We will then see how Brauer-Hasse-Noether exact sequences can also be obtained in other less classical settings, such as arithmetic function fields and Laurent series fields.

References: [39] (Chap. VII.1 and VIII.8), [14] (Chap. 8 and 14), [29], [30], [27].

3.9. **Field patching and local-global principles (R. Parimala).** Let K be a complete discretely valued field and X/K a smooth projective geometrically integral curve over K. Let $F = K(X)$ be the function field of the curve. Harbater, Hartmann and Krashen describe a finite family of overfields of F with respect to which they determine an obstruction set to the Hasse principle for torsors under connected linear algebraic groups. We explain this setting and give a proof of the Hasse principle for quadratic forms over F with respect to divisorial discrete valuations using the patching results.

3.10. **Patching over Berkovich analytic curves (V. Mehmeti).** From a historical point of view, one encounters the notion of *patching* for the first time in complex analysis in the nineteenth century. Since then, developments of patching techniques in different frameworks have been shown useful for various purposes such as Galois theory, inverse Galois theory and differential algebra.

In 2009, Harbater, Hartmann and Krashen introduced a version of the technique, called *field patching*, adapted to an algebraic context, and applied it for the first time to the study of local–global principles for quadratic forms defined over function fields of curves. Field patching has since been used successfully for proving many results related to the existence of rational points and zero-cycles of degree one on homogeneous varieties. Abstractly, it can be stated as follows.

The patching property. Let F, F_0, F_1, F_2 be fields such that $F \subseteq F_i \subseteq F_0$ for $i = 1, 2$. Given a linear algebraic group G/F:

$$\forall g \in G(F_0), \exists g_i \in G(F_i), i = 1, 2, \text{ such that } g = g_1 g_2 \in G(F_0).$$

In this lecture, we will see how this method can be extended to Berkovich analytic curves encountered in the first two lectures. One can isolate abstract hypotheses under which the patching property holds, and it turns out that said hypotheses are naturally satisfied in the setting of Berkovich analytic curves. A difference with the setup of Harbater, Hartmann and Krashen is that patching becomes quite geometric in nature here. In fact, loosely speaking, later on we will interpret it as a patching of meromorphic functions on the Berkovich curve. We will also spend some time working out a couple of examples.

References: [17], [18], [19], [20], [35], [42], [33].

3.11. **Tate-Shafarevich groups and Poitou-Tate duality (D. Izquierdo).** In this third lecture, we will see that over various fields having a suitable arithmetic behaviour, the Galois cohomology satisfies nice duality theorems. We will start by the classical Tate duality over p-adic fields and the Poitou-Tate duality over number fields, and we will then see how such dualities can be generalized to semi-global fields or Laurent series fields. We will constantly illustrate such dualities by applying them to various concrete questions related to quadratic forms (for instance to study square classes in a field or more generally to Grunwald-Wang-like theorems).

References: [39] (Chap. VII.2 and VIII.6), [14] (Chap. 10, 17 and 18), [15], [16], [10], [26].

3.12. **Local–global principles for quadratic forms defined over function fields of curves (V. Mehmeti).** In this lecture we will see an application of patching techniques on Berkovich curves (seen in Lecture 3.10) to *local–global principles* for quadratic forms. We recall:

Local–global principle. Let F be a field, and $(F_i)_{i\in I}$ a family of overfields of F (i.e. $F \subsetneq F_i$ for all $i \in I$). Given a quadratic form q over F,

$$q \text{ is isotropic over } F \iff q \text{ is isotropic over } F_i \ \forall i \in I.$$

In this lecture, F will be the function field of a curve C defined over a complete valued field of arbitrary characteristic (the characteristic 2 case is covered thanks to a remark of N. Daans). The first local–global principle we will prove is geometric in nature, in the sense that one can construct F and the family of overfields $(F_i)_{i\in I}$, directly from the Berkovich curve C^{an}, the analytification of C. Some general properties of the overfields will be seen, showing that this local–global principle has good algebraic attributes as well. We will present some explicit examples of these fields F_i.

In Lecture II, a connection between rank one valuations and points of C^{an} was presented. Building up on that, we will prove a second local–global principle, which is of more classical nature: the overfields will now be completions of F with respect to those rank one valuations. We stress that the completions of F appearing here are not necessarily discrete.

If time allows it, we will discuss the relationship with local–global principles where the family $(F_i)_{i\in I}$ consists of *only* discrete completions of F.

As a non-trivial consequence of these local–global principles, results on the *u-invariant* of F can be obtained. A particular case of this is that $u(\mathbb{Q}_p(T)) = 8$ when $p \neq 2$, proven in [12] and [20] through local–global principles, and more generally in [32] via different methods.

The results here generalize those of Harbater, Hartmann and Krashen's [20].

References: [12], [20], [32], [35], [37].

3.13. **Bounded symbol lengths for function fields over p-adic fields and number fields. (R. Parimala).** We state theorems on the finiteness of symbol lengths for function fields of p-adic curves. We sketch a proof of a theorem of Saltman that every element in the 2-torsion part of the Brauer group of the function field of a p-adic curve, where $p \neq 2$, is represented by a biquaternion algebra. This leads to the observation that the symbol length in H^2 is 2. One has also symbol length 1 for H^3, and together this leads to the finiteness of the u-invariant. We shall also describe some results concerning function fields of curves over number fields and discuss some open questions in this direction.

3.14. **Arithmetic of homogeneous spaces (D. Izquierdo).** In this final lecture, we will introduce homogeneous spaces, which constitute a vast and interesting generalization of equations of the form $q = a$ with q a quadratic form. We will see how their rational points over number fields behave by introducing the Brauer-Manin obstruction and by discussing Sansuc's and Borovoi's theorems. We will finally see that such tools can be generalized to semi-global fields and to Laurent series fields.

References: [45], [6], [11], [15], [16], [10], [26], [28].

3.15. **The Pythagoras number of function fields (N. Daans).** The Pythagoras number of a field K is the smallest natural number n such that every sum of squares of elements of K is a sum of n squares of elements of K, or ∞, if such a natural number does not exist. Let us denote the Pythagoras number of K by $p(K)$. Any non-zero natural number is the Pythagoras number of some field [24]. While computing the precise value of the Pythagoras number of a given field can in general be a very subtle problem, we can for a large class of naturally occurring fields bound the Pythagoras number: when K is an extension of $\mathbb{Q}$ or $\mathbb{R}$ of finite transcendence degree, then its Pythagoras number is known to grow at most exponentially as a function of the transcendence degree. On the other hand, very little is known about the behaviour of the Pythagoras number under general field extensions, in particular how quickly and freely the Pythagoras number can grow. For example, we do not know whether, when $p(K)$ is finite, then also $p(K(X))$ is finite, where $K(X)$ is a rational function field over K. In this talk, I will introduce the Pythagoras number of fields and share basic examples and observations. I will give a rough idea of why we can give bounds for the Pythagoras number for extensions of $\mathbb{Q}$ or $\mathbb{R}$ only depending on the transcendence degree. The main focus will be on recent work together with Becher, Grimm, Manzano-Flores and Zaninelli [4], where it is shown that, if $p(K(X)) = 2$, then $p(K(X,Y)) \leq 8$. Crucial ingredients for this last result are Mehmeti's local-global principle [35] as discussed in talk 3.10, and a theorem of L. Bröcker which relates the property that $p(K(X)) = 2$ for a field K to the existence of a certain henselian valuation on K [7].

4. Special talks

We will have two special talks during the summer school by two of our participants.

4.1. **R-equivalence on adjoint groups (Archita Mondal, IIT Bombay).** Let E be a field and X be an irreducible algebraic variety over E. Let $X(E)$ denote the group of E rational points of X. Y. Manin introduced the notion of *rational equivalence* on $X(E)$, and more generally on a variety defined over a field.

The set of equivalence classes for this relation is denoted by $X(E)/R$ and has a natural group structure. Later J.-L. Colliot-Thélène and J.J. Sansuc studied *R-equivalence* in the category of linear algebraic groups.

Now let G be a connected linear algebraic group. It is a birational invariant of G, and the triviality of $G(E)/R$ is closely related to the rationality of the underlying group variety given by G over E.

Let F be a field of characteristic different from 2 and with virtual cohomological dimension 2 and let G be a semi-simple adjoint classical group defined over F. We are interested in the triviality of $G(F)/R$. This is a joint work with Prof. Preeti Raman.

Reference: [1]

4.2. **Universal quadratic forms over p-adic and number fields (Yong Hu, Southern University of Science and Technology, Shenzen).** Let k and n be positive integers and $f \in \mathbb{Z}[x_1, \ldots, x_n]$ an integral quadratic form. We say that f is *k-universal* if for every integral quadratic form in k variables $g \in \mathbb{Z}[y_1, ..., y_k]$, there exist linear forms $l_1, \ldots, l_n \in \mathbb{Z}[y_1, ..., y_k]$ such that $f(l_1, ..., l_n) = g$. In this talk, I will report some recent progress on the classification of k-universal quadratic forms over p-adic fields and the local-global principle for k-universality over number fields. This is based on joint works with Zilong He and Fei Xu.

References: [21],[22],[23]

4.3. **A local-global principle for rational function fields (Marco Zaninelli, University of Antwerp).** Let K be a field of characteristic different from 2. Under certain assumptions on K, it is possible to obtain a local-global principle for Pfister forms over the rational function field $K(X)$ by using classic tools from quadratic form theory, such as Milnor's Exact Sequence and Springer's Theorem for complete non-dyadic valued fields. In particular, we assume the presence of a local-global principle for Pfister forms over F for any finite field extension F/K; this holds for example when K is a global field and when $K = K_0(\!(t)\!)$ for a global field K_0, in which cases classic local-global principles for sums of squares are retrieved via a simple argument.

References

[1] M. and Preeti Archita R., *Rational equivalence on adjoint groups of type D_n over fields of virtual cohomological dimension 2*, Trans. Amer. Math. Soc. **375** (2022), no. 10, 7373–7384, DOI 10.1090/tran/8726.

[2] S. Anscombe and A. Fehm, *Characterizing diophantine henselian valuation rings and valuation ideals*, Proc. London Math. Soc. **115** (2017), 293-322.

[3] R. Aravire and B. Jacob, *Versions of Springer's theorem for quadratic forms in characteristic 2*, American Journal of Mathematics **118** (1996), 235 - 261.

[4] K. Becher, N. Daans, D. Grimm, G. Manzano-Flores, and M. Zaninelli, *The Pythagoras number of a Rational Function Field in two variables*, 2023. Preprint available as `arXiv: 2302.11423`.

[5] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.
[6] M. Borovoi, *The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer*, Journal für die reine und angewandte Mathematik (Crelle) **473** (1996).
[7] L. Bröcker, *Characterization of Fans and Hereditarily Pythagorean Fields*, Mathematische Zeitschrift **151** (1976), 149-164.
[8] N. Daans, *Existential first-order definitions and quadratic forms*, Universiteit Antwerpen, 2022.
[9] A. Ducros, *La structure des courbes analytiques*, `https://webusers.imj-prg.fr/~antoine.ducros/trirss.pdf`.
[10] J.-L. Colliot-Thélène and D. Harari, *Dualité et principe local-global pour les tores sur une courbe au-dessus de* $\mathbf{C}((t))$, Proc. Lond. Math. Soc. **110(6)** (2015).
[11] J.-L. Colliot-Thélène, R. Parimala, and V. Suresh, *Lois de réciprocité supérieures et points rationnels*, Trans. Amer. Math. Soc **368(6)** (2016).
[12] J.-L. Colliot-Thélène, R. Parimala, and V. Suresh, *Patching and local-global principles for homogeneous spaces over function fields of p-adic curves*, Comment. Math. Helv. **87** (2012), no. 4, 1011–1033.
[13] M.A. Elomary and J.-P. Tignol, *Springer's theorem for tame quadratic forms over Henselian fields*, Mathematische Zeitschrift **269** (2011), 309-323.
[14] D. Harari, *Galois Cohomology and Class Field Theory*, Universitext, Springer, 2020.
[15] D. Harari, C. Scheiderer, and T. Szamuely, *Weak approximation for tori over p-adic function fields*, Internat. Math. Res. Notices **2015** (2015).
[16] D. Harari and T. Szamuely, *Local-global questions for tori over p-adic function fields*, J. Algebraic Geom. **25(3)** (2016).
[17] D. Harbater, *Patching and Galois theory*, Galois groups and fundamental groups, Math. Sci. Res. Inst. Publ., vol. 41, Cambridge Univ. Press, Cambridge, 2003, pp. 313–424.
[18] ———, *Patching in algebra*, Travaux mathématiques. Vol. XXIII, Trav. Math., vol. 23, Fac. Sci. Technol. Commun. Univ. Luxemb., Luxembourg, 2013, pp. 37–86.
[19] D. Harbater and J. Hartmann, *Patching over fields*, Israel J. Math. **176** (2010), 61–107.
[20] D. Harbater, J. Hartmann, and D. Krashen, *Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), no. 2, 231–263.
[21] Z. He and Y. Hu, *On k-universal quadratic lattices over unramified dyadic local fields*, J. Pure Appl. Algebra **227** (2023), no. 7, Paper No. 107334, 32, DOI 10.1016/j.jpaa.2023.107334.
[22] ———, *On n-universal quadratic forms over dyadic local fields* (2023), available at `https://doi.org/10.48550/arXiv.2204.01997`.
[23] Z. He, Y. Hu, and F. Xu, *On indefinite k-universal integral quadratic forms over number fields*, Math. Z. **304** (2023), no. 1, Paper No. 20, 26, DOI 10.1007/s00209-023-03280-z.
[24] D. Hoffmann, *Pythagoras Numbers of Fields*, Journal of the American Mathematical Society **12** (1999), no. 3, 839-848.
[25] D. Izquierdo, *Autour de la conjecture de Milnor*, Gazette SMF **166** (2020).
[26] ———, *Dualité et principe local-global pour les anneaux locaux henséliens de dimension 2 (with an appedix by J. Riou)*, Algebraic Geometry **6(2)** (2019).
[27] ———, *Vanishing theorems and Brauer-Hasse-Noether exact sequences for the cohomology of higher-dimensional fields*, Trans. AMS **372** (2019).
[28] D. Izquierdo and G. Lucchini Arteche, *Local-global principles for homogeneous spaces over some two-dimensional geometric global fields*, Journal für die reine und angewandte Mathematik (Crelle) **2021(781)** (2021).

[29] U. Jannsen, *Principe de Hasse cohomologique*, in Séminaire de Théorie des Nombres, Paris, 1989-90, Progr. Math. **102** (1992).
[30] ———, *Hasse principles for higher-dimensional fields*, Ann. Math. (2) **183(1)** (2016).
[31] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, 2005.
[32] D. B. Leep, *The u-invariant of p-adic function fields*, J. Reine Angew. Math. **679** (2013), 65–73.
[33] Q. Liu, *Tout groupe fini est un groupe de Galois sur* $\mathbf{Q}_p(T)$*, d'après Harbater*, Recent developments in the inverse Galois problem (Seattle, WA, 1993), Contemp. Math., vol. 186, Amer. Math. Soc., Providence, RI, 1995, pp. 261–265 (French, with English summary).
[34] P. Mammone, R. Moresi, and A. R. Wadsworth, *u-invariants of fields of characteristic 2.*, Mathematische Zeitschrift **208** (1991), no. 3, 335-348.
[35] V. Mehmeti, *Patching over Berkovich Curves and Quadratic Forms*, Compositio Math. **155** (2019), no. 12, 2399–2438.
[36] ———, *Patching on Berkovich Spaces and the Local–Global Principle*, PhD Thesis in Mathematics, University of Caen Normandy (2019).
[37] ———, *An analytic viewpoint on the Hasse Principle* (2022), `https://arxiv.org/abs/2203.16234`.
[38] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/70), 318–344.
[39] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields*, Grundlehren der mathematischen Wissenschaften, vol. 323, Springer, 2008.
[40] D. and Vishik Orlov A. and Voevodsky, *An exact sequence for* $K_*^M/2$ *with applications to quadratic forms*, Ann. of Math. (2) **165** (2007), no. 1, 1–13, DOI 10.4007/annals.2007.165.1.
[41] R. Parimala and V. Suresh, *The u-invariant of the function fields of p-adic curves*, Ann. of Math. (2) **172** (2010), no. 2, 1391–1405.
[42] J. Poineau, *Raccord sur les espaces de Berkovich*, Algebra Number Theory **4** (2010), no. 3, 297–334 (French, with English and French summaries).
[43] A. Quéguiner-Mathieu, *Galois cohomology, quadratic forms and Milnor K-theory*, Course notes, `https://www.math.univ-paris13.fr/~queguin/publi.htm` (2011).
[44] J. Robinson, *Definability and decision problems in arithmetic*, Journal of Symbolic Logic **14** (1949), no. 2, 98-114.
[45] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, Journal für die reine und angewandte Mathematik (Crelle) **327** (1981).
[46] V. Suresh, *Third Galois cohomology group of function fields of curves over number fields*, Algebra Number Theory **14** (2020), no. 3, 701–729.