ALGEBRA AND ARITHMETIC (ALGAR) 2025 : LINEAR ALGEBRAIC GROUPS AND ALGEBRAS WITH INVOLUTION

1. Content description

The ALGAR Summer School 2025 is devoted to the topic of linear algebraic groups and algebras with involution. It intends to provide a bridge between a well-studied class of algebraic structures to the study of properties of varieties.

Two central problems in arithmetic and algebraic geometry are to decide whether some variety over a given field has rational points or whether it is even rational, that is, birational to some affine n-space for some n. This problem is particularly well studied for linear algebraic groups and their torsors.

If K is an algebraically closed field, then it turns out that every connected linear algebraic group over K is rational.

We will work over a general field K of characteristic different from 2. A typical example of a linear algebraic group is the orthogonal group $\mathsf{O}(\varphi)$ of some non-degenerate quadratic form φ over K. It has two connected components, one of them being the special orthogonal group $\mathsf{SO}(\varphi)$. The latter group is always a rational variety. The situation changes if we extend the group $\mathsf{SO}(\varphi)$ to the group of projective similitudes $\mathsf{PSim}(\varphi)$. We assume now that $\mathsf{dim}(\varphi) = 2n$ with $n \in \mathbb{N}$. It was shown by Merkurjev [3] that $\mathsf{PSim}(\varphi)$ is non-rational in general when $n \geq 3$. This crucially relies on the fact that a rational variety is in particular rationally connected, a property first studied by Y. Manin around 1974. This property is generally weaker than rationality, and for certain linear algebraic groups like $\mathsf{PSim}(\varphi)$, it has a very interesting algebraic interpretation, also recognised by Merkurjev. One aim of the summer school is to shed light on these phenomena.

The groups $\mathsf{SO}(\varphi)$ and $\mathsf{PSim}(\varphi)$ are examples of semi-simple linear algebraic groups that are simply connected, respectively adjoint. A classification of such groups was provided by André Weil [4] in 1961 in terms of root systems and corresponding Dynkin diagrams. The classification makes crucial use of algebras with involution. Consider that the quadratic form φ is given on a K-vector space V. Then φ induces a (so-called adjoint) involution on the endomorphism algebra $\mathsf{End}_K(V)$. More generally, one considers automorphism groups of a K-algebra with involution (A, σ), that is, where A is a central simple K-algebra and σ is an involution on A. By Weil's classification, all simply connected linear algebraic groups are obtained in this way.

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Note that an algebraic group G over K always has K-rational points. To obtain varieties related to G for which the presence of rational points becomes an interesting question is that of a G-torsor, that is, a K-variety on which G acts in a simply transitive way. Given a G-torsor X, there will always be a finite Galois extension L/K where X has an L-rational point, and this fact can be used to classify G-torsors over K by a Galois cohomology set. The question then arises for which type of groups G and over which type of fields K a non-trivial G-torsor can arise. Serre's Conjectures I and II postulate that this does not happen over fields of small cohomological dimension. This can be seen as an extension of the fact that over an algebraically closed field, connected linear algebraic groups are rational. One of the central topics of this summer school will be to formulate, illustrate and explain these two famous conjectures.

2. Prerequisites

Throughout the summer school, we assume fluency with basic algebraic structures covered in most bachelor's programmes, and some familiarity with the following concepts:

- Tensor products of modules and of algebras over commutative rings [2]
- Projective modules [2]
- Quadratic forms
- Quaternion algebras – [1, Chap. 1]
- Central simple algebras – [1, Sections 2.1-2.4]
- Discrete valuations – [1, Appendix A.6]

[**Note:** a more detailed overview of the contents of the preliminary days and of the main week will be added here soon.]

References

[1] Philippe Gille and Tamás Szamuely, *Central simple algebras and Galois cohomology*, Second, Cambridge Studies in Advanced Mathematics, vol. 165, Cambridge University Press, Cambridge, 2017.

[2] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[3] A. S. Merkurjev, *R-equivalence and rationality problem for semisimple adjoint classical algebraic groups*, Publications Mathématiques de l'IH/'ES 84.1 (1996) 189-213.

[4] A. Weil, *Algebras with involutions and the classical groups*, J. Ind. Math. Soc. 24 (1961) 589–623.