ALGEBRA AND ARITHMETIC (ALGAR) 2026 :
SEMIGLOBAL FIELDS AND THEIR VALUATIONS

Contents

Date: July 2026.

1. Content description

The ALGAR Summer School 2026 is devoted to semiglobal fields and their valuations. A *semiglobal field* is a function field of a curve defined over a complete discretely valued base field. The typical example for the base field is the field of p-adic numbers $\mathbb{Q}_p$ for a prime number p. Function fields over such fields appear naturally in many ways in number theory and arithmetic geometry. During the last two decades, a series of breakthrough results in the study of quadratic forms and linear algebraic groups over such fields were achieved, leading in particular to unexpected local-global principles. In this context, the term *semiglobal field* was coined recently (probably in [22]), alluding to the presence of mixed phenomena from classical *local* and *global fields*. These local-global results are typically expressed in terms of valuations. In the first place one encounters the discrete valuations given by prime divisors of some *arithmetic model* of the field: rather than seeing the field only as the function field of a curve, one widens the perspective by viewing that curve embedded into an *arithmetic surface*, in which the irreducible curves (codimension-one subvarieties) correspond to certain discrete valuations of the field.

In the summer school we aim to understand the different kinds of valuations that appear on a semiglobal field and how they can be viewed as points in a space. We will further see how this understanding in combination with methods from valuation theory (not only for discrete valuations) helps in the study of quadratic forms over those fields and the determination of field invariants such as the *Pythagoras number* or the *u-invariant*. When looking at local conditions, the understanding of valued fields which are *henselian* is of crucial importance.

The summer school will consist of 12-14 lectures during the main week, which are complemented by exercise sessions and some research talks. It will be preceded by three preliminary days, in which the basic concepts from valuation theory, quadratic form theory and basic algebraic geometry shall be introduced and studied through exercises.

2. Preliminary days and prequisites

Throughout the main week of the summer school, we will assume some familiarity with valuation theory and with basic facts about quadratic forms and function fields in one variable. They will be trained during the preliminary days through six half-day workshops, each combining a lecture of at most 60 minutes with an interactive exercise session.

2.1. **Valuations.** We will assume basic familarity with discrete valuations (value group $\mathbb{Z}$), and with complete discretely valued fields such as $\mathbb{Q}_p$, and formulate (but somehow assume for $\mathbb{Q}_p$) Hensel's lemma for complete local rings. We introduce valuations with a general ordered abelian value group, in additive notation. We define the rank of a valuation and relate rank-1 valuations to absolute values. We talk about extensions of valuations and the Fundamental Inequality for finite field extensions.

2.2. **Useful concepts from algebraic geometry.** A finitely generated field extension F/K is the field of fractions of a finitely generated integral K-algebra, which corresponds to an affine variety. This algebra is not unique, and for example from the point of view of studying all K-trivial valuations on F, it is desirable to be flexible in the choice of the algebra. The concept of a projective variety provides a good framework to manage these choices. In the case where K is the field of fractions of a discrete valuation ring T, one can similarly consider F as the field of fractions of a finitely generated T-algebra, which leads to the analogous framework of projective schemes over T. We will explore this by way of examples.

In this way, one can relate valuations on F to points on a geometric *model of* F (over K, or T). However, even in the case where F/K has transcendence degree 1, one cannot always recover uniquely the valuation from the corresponding point on a given projective curve model of F/K. The issue occurs for singular points on the curve. Hence, one aims to refine the projective curve so that it has no singular point. This can be obtained using for example *blowing-ups*, as we will explore in examples. In the case of projective schemes over T, something similar can be done, but the story is a little bit more complicated...

2.3. **Corsenings, refinements, completion and henselisation.** We saw in Session 2.1 that a rank-1 valuation of a field K induces an absolute value turning K into a metric space, hence one can speak of Cauchy sequences and convergent sequences in K. One can consequently construct, from an arbitrary rank-1 valued field (K, v), its *completion*, which is the smallest extension of the valued field (K, v) which is complete. We will discuss its most important properties, following [14, Sect. 1.3 & 2.4]. One of these properties is that complete rank-1 valued fields are *henselian*, that is, their valuation rings satisfy (the conclusion of) Hensel's Lemma.

For many algebraic applications, it turns out that the henselian property is more important than completeness as such. To any valued field (K, v), one can associate its *henselisation*, which is the smallest extension of (K, v) that is henselian. The henselisation exists for valuations of arbitrary rank, and other than the completion, it is always an algebraic extension. We will discuss basic properties of the henselisation, and compare it with the completion, following [14, Sect. 4.1 & 5.2].

Finally, we address how different valuations on a field may be related via the concepts of *coarsening*, *refinement* and *composition*, following [14, Sect. 2.3].

2.4. **Function fields in one variable.** We recall how a function field in one variable F/K arises from an irreducible curve over K. We then see how points of the curve give rise to discrete valuations on F which are trivial on K. We outline how smoothness of the curve can be seen in terms of these discrete valuations. We describe base change from both points of view.

We consider valuations on a rational function field in one variable $K(t)$. Given a valuation v on K, we define its *Gauss extension with respect to t*. More generally, we look at valuations on F that restrict to a Gauss extension of a valuation from K to $K(t)$ for some $t \in F$ transcendental over K and their residue fields.

2.5. **Local theory of quadratic forms.** We introduce basic concepts from the study of quadratic forms over fields of characteristic different from 2. In particular, *isometry*, *isotropy*, *hyperbolicity*, the operations *orthogonal sum* and *tensor product*, as well as *isometry* and the *Witt's decomposition theorem* are recalled. We introduce *Pfister forms*. We then describe a theorem due to W.H. Durfee [13] and T.A. Springer [33] concerning quadratic forms over a henselian valued field.

3. Main lectures

3.1. **Local-global principles for quadratic forms (D. Grimm).** The Hasse-Minkowski Theorem states that, over a number field K, a quadratic form is *isotropic* (i.e. it has a nontrivial zero) if and only if it is isotropic over each completion of K with respect to some ultrametric. Those ultrametrics are given by embeddings into the real numbers (if such embeddings exists) and by discrete valuations. In general, the fields over which a similar local-global principle for isotropy of quadratic forms holds are rare, but an important somehow similar scenario has been discovered in [12], namely that of function fields in one variable over a complete (or henselian) rank-1 valued base field (such as the field of p-adic numbers $\mathbb{Q}_p$ for a prime number p, or $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$). Due to the partial analogy to the case of global number fields, such fields were baptized *semiglobal fields*. In this lecture, the local-global principle for semiglobal fields from [12] shall be explained and illustrated by some consequences. In difference to the classical situation over number fields, the local-global principle does not apply for quadratic forms of dimension 2: the *Local Square Theorem* can fail, that is, an element can be locally a square at each completion without being itself a square.

In general, controlling the set of all valuations on a semiglobal field is difficult. In the case where the base field valuation is discrete, this failure can be expressed in terms of the topology of a certain graph associated to what is called an *arithmetic model* of the function field over the base field valuation ring.

3.2. **Smooth algebraic surfaces over a field (Q. Liu).** To give some geometric intuition for the arithmetic surfaces that we will introduce in Session 3.5, we will first review some basic concepts and results for smooth projective surfaces over a field. This includes the notion of divisors (formal finite sums of irreducible curves contained in the surface) and the theory of intersection between the divisors. As an application we prove Bézout's theorem about the intersection number of two curves in a projective plane.

Surfaces with the same function field are said to be *birational.* A basic tool in this context is given by the blowing-up morphisms. They will allow us to describe completely the way to pass from one surface to a birational one.

Up to blowing-ups, any surface S can be given a *fibration* over a curve, that is, a non-constant map $\pi\colon S \to B$ to a curve B. Such surfaces are closer to arithmetic surfaces. We will study the variation of the fibers $\pi^{-1}(b)$ when b runs through B. As a general fact, we will see that all but finitely many fibers are smooth curves, but some of them can become singular (degeneration).

A reference for this material is [17, Chap. V].

3.3. **The Tate algebra over a valued field (K. Hübner).** We consider a nonarchimedean (i.e. complete rank-1 valued) field K whose valuation ring we denote by K°. In K we study disks $D(a, r)$ with center $a \in K$ and radius $r \in |K^\times|$. The *unit disc* has center 0 and radius 1 and identifies with K°. In fact, it makes sense to define disks of radius $r \in \Gamma$ for some totally ordered abelian group Γ containing $|K^\times|$.

Next, we equip the polynomial ring $K[T]$ with the Gauß norm studied in section 2.4. After completion, we obtain the *Tate algebra* $K\langle T\rangle$, which can be interpreted as the ring of power series converging on the unit disk in K. Varying the radius and center of the Gauß valuation, we can describe the ring of convergent power series for any disk in K.

We finish this lecture by studying intersections of series of nested disks $D_1 \supseteq D_2 \supseteq D_3 \supseteq \ldots$. It may happen that the intersection of such a series is empty even though the radii of the disks do not tend to 0. We call the valued field K *spherically complete* if this phenomenon does not occur. References for this topic are [3, Part I, §2] and [4, Part B, §5].

3.4. **Computing field invariants of semiglobal fields (N. Daans).** Consider a function field in one variable F/K over a complete rank-1 valued field K. To study questions about quadratic forms (or other nice classes of polynomial equations) over F, local-global principles like those discussed in Session 3.1 can aid to reduce the problem to the study of an analogous problem over the completions of F with respect to certain valuations on F (see Session 3.12). In particular, certain invariants of F which are determined by the behaviour of quadratic forms, can be computed from the values of these invariants over completions of F - provided one knows how to compute the latter. Here, the local theory developed in Session 2.5 can help.

In this talk, we discuss one such invariant for semiglobal fields: the *u-invariant.* The talk presents some of the work from [9] and [10] on quadratic form invariants for semiglobal fields. For an introduction to quadratic form invariants for fields, see for example [32, Chap. 3, 7 & 8].

3.5. **Algebraic curves over a discretely valued field - I (Q. Liu).** For a fibered surface $\pi\colon S \to B$ over a curve, when we look at the fibers $\pi^{-1}(b)$ around a given point $b_0 \in B$, we can replace B with the spectrum of the discrete valuation ring $\mathcal{O}_{B,b_0}$. More generally, if we consider an arbitrary discrete valuation ring $\mathcal{O}_K$ instead of $\mathcal{O}_{B,b_0}$, we get an arithmetic surface, i.e., an integral scheme X of dimension 2 endowed with a projective and surjective morphism to $\mathrm{Spec}(\mathcal{O}_K)$. Then most all of the tools for smooth projective surfaces are also available for X.

We will then review these tools for arithmetic surface. However, assuming X regular, the intersection theory is more constraint, as the base is affine.

We will also describe the discrete valuations on the function field $K(X)$. Under some conditions, they correspond exactly to those given by codimension 1 points of various blown-ups of X.

3.6. **The adic unit disk (K. Hübner).** We continue with the setup from Session 3.3: (K, K°) is a nonarchimedean field, which we assume to be algebraically closed for simplicity. When searching for a nonarchimedean analogue of complex manifolds, one encounters the problem that K^n is totally disconnected, which makes it hard to glue open subsets of K^n to a global object. Several frameworks of nonarchimedean spaces have been developed in order to address this problem. We are going to stay within one of those: the theory of *adic spaces*.

We will not discuss adic spaces in generality but rather focus on understanding the adic spectrum of the Tate algebra $K\langle T\rangle$. It constitutes an example of an *affinoid adic space*, which are the building blocks of adic spaces in the same way as affine schemes are for schemes. The most general affinoid adic spaces we will encounter are adic spectra of *affinoid K-algebras* $K\langle T_1, \dots, T_n\rangle/I$ for an ideal I. We call these *affinoid rigid spaces*.

By definition, the points of the adic spectrum $\mathrm{Spa}(K\langle T\rangle)$ are given by continuous valuations of $K\langle T\rangle$ that are less or equal to 1 (in multiplicative notation) on $K^\circ\langle T\rangle$. We call this space the *(adic) unit disk* $\mathbb{D}$. It will turn out that almost all points of $\mathbb{D}$ are given by Gauß valuations. Referneces for the adic unit disk can be found in [5, §5] and [6].

3.7. **Complete and henselian valued fields (N. Daans).** Recall that a rank-1 valued field K is called *complete* if every Cauchy sequence in K has a limit. In such fields *Hensel's Lemma* holds, from which many algebraic properties of the valued field may be deduced.

Although complete rank-1 valued fields form a very natural class from an analytic-geometric perspective, they have certain disadvantages from an algebraic perspective. One therefore considers *henselian* valued fields, i.e. valued fields for which (the consequent of) the statement of Hensel's Lemma is satisfied. Many algebraic statements about complete rank-1 valued fields can be extended to henselian valued fields, but henselian valued fields can have arbitrary value groups (and hence arbitrary ranks) and can be more easily found via field-theoretic constructions. In Session 2.3, we discussed some of these basic properties.

In this lecture, we will discuss solvability of polynomial equations over henselian versus complete valued fields. Most of the talk will not be concerned with function fields, but we will conclude by showing how the local-global principle from Session 3.1 for function fields over complete valued base fields may, a posteriori, be extended in certain cases to include henselian valued base fields, following [9, Sect. 4]. For the main valuation theoretic ingredient comparing a henselian valued field with its completion, the talk takes inspiration from [25, Sect. 5.3], building on the pioneering work of Irving Kaplansky [24]. As a technical tool, we shall use an ingredient from model theory, namely the existence of $\aleph_1$-saturated elementary extensions; the interested reader may consult [28, Theorem 4.3.12] for a proof.

3.8. **The topology of the unit disc and its structure sheaf (K. Hübner).** The rational subsets $R\left(\frac{f_1,\ldots,f_n}{g}\right)$ provide a basis for the topology of the unit disk $\mathbb{D} = \mathrm{Spa}(K\langle T\rangle)$ (or a more general adic spectrum). They are defined in terms of inequalities involving the functions $f_1,\ldots,f_n,g \in K\langle T\rangle$. Examples of rational open subsets of $\mathbb{D}$ include smaller disks and annuli. Having endowed $\mathbb{D}$ (or more generally the spectrum of an affinoid K-algebra) with a topology, we can define its structure sheaf.

This completes the construction of affinoid rigid spaces that can now be glued to form global rigid spaces over K. We will discuss the examples of the affine and projective lines over K as rigid spaces that are glued from disks. References for this topic are [20] and [21, §2 and §3]

3.9. **Algebraic curves over a discretely valued field - II (Q. Liu).** In this session, we continue the discussions from Session 3.5.

3.10. **Computing quadratic form field invariants of semiglobal fields - of arbitrary rank (N. Daans).** In Session 3.4 we discussed how to use a local-global principle to compute field invariants of function fields in one variable over complete rank-1 valued fields. In this lecture, we discuss how, in certain circumstances, such results on field invariants can be extended to function fields over arbitrary henselian valued base fields, even if no analogous local-global principle is known in this generality. The lecture presents some of the results from [9] and [10].

3.11. **Formal models (K. Hübner).** As before, we fix a nonarchimedean field (K, K°). We start this session by introducing formal schemes over the valuation ring K°. The generic fiber of a formal scheme is a rigid space over K and the special fiber is a scheme over the residue field $K^\succ$ of K. If we start with a rigid space X/K, a formal scheme whose generic fiber is X is called a *formal model* of X. For instance, a formal model of the disk $\mathbb{D}$ is given by the affine line $\mathbb{A}^1_{K^\circ} = \mathrm{Spa}\, K^\circ\langle T\rangle$.

Formal models are far from being unique. If $\mathcal{X}$ is a formal model of the rigid space X/K, we can blow up a closed subscheme of the special fiber and obtain another formal model. In fact, using blowups we can construct a cofinal set of formal models of X. Our final goal is to show that the limit over all formal models of X recovers X. Everything about formal models can be found in [7], see [21, §8] for an overview.

3.12. **Valuations of semiglobal fields (D. Grimm).** We present a local-local principle which reduces the set of valuations needed to check whether a quadratic form is isotropic locally everywhere. Eventually we reduce this further to a local isotropy criterion that only takes into account certain local rings of some arithmetic model of the semiglobal field. Subsequently, we will consider the special case of two-dimensional quadratic forms: we show that, if the graph associated to the arithmetic model has no loops, then indeed every local square is a global square in F, and otherwise the failure of the local-global principle is controlled by the fundamental group of the graph. We also give a rough idea of why the local-global principle for isotropy of quadratic forms of dimension at least 3 holds independently of the topology of this graph.

3.13. **Reduction of algebraic curves (Q. Liu).** Let C be a projective smooth curve over a discrete valuation field K. To study the arithmetic and the rigid analytic properties of C, it is useful to extend C to an arithmetic surface over the valuation ring $\mathcal{O}_K$ of K. The arithmetic surface is called a *model of* C, and its special fiber (the fiber over the unique closed point of $\mathrm{Spec}(\mathcal{O}_K)$) a *reduction* of C. We will show how to compute some invariants of C from a reduction of C.

Sometimes a "nice model" of C exists only over a finite extension of K. The notion of being "nice" here means that the only possible singular points of the reduction are ordinary double points. Deligne and Mumford's stable reduction theorem asserts that there exists a finite extension L/K and a "nice" model of C_L over $\mathcal{O}_L$. We will draw some consequence of this theorem.

A reference for 3.5, 3.9 and 3.13 is [27, Chap. 8-10].

4. Special talks

4.1. **The ruled residue theorem for function fields (Sumit Ch. Mishra).** Let E be a field with a valuation v. In 1983, J. Ohm [30] proved that for any extension of v to the rational function field $E(X)$ in one variable, the corresponding residue field extension is either algebraic or ruled, i.e., it is the rational function field in one variable over a finite extension of the residue field of E. This is called the Ruled Residue Theorem. More generally, one can consider the function field F of a curve over E and ask if for all extensions of v to F, the corresponding residue field extension is either algebraic or ruled? If not, is there any bound on the number of extensions of v to F where this fails? I will mention known results for the function fields of conics [1]. Later on, I will discuss the case of function fields of elliptic curves (joint work with Karim J. Becher and Parul Gupta [2]) and hyperelliptic curves (joint work with Parul Gupta [8]).

4.2. **Rationality type problems in algebraic groups (Chayansudha Biswas).** We say that a variety X defined over a field k satisfies the local-global principle if having local solutions everywhere implies the existence of a global solution i.e. if $X(k_v) \neq \emptyset$ for all discrete valuations v of k (where k_v denotes the completion of (k, v)), then $X(k) \neq \emptyset$.

The quantitative version of this is the question of *approximation*. We may ask if local solutions can be simultaneously *approximated* by global solutions. We will see one way to make this question precise and look at some varieties for which it holds true. We say such varieties *have weak approximation*. The property of weak approximation can be seen as an extension of the Chinese Remainder Theorem for varieties defined over fields.

Recall that two varieties are birational if they have isomorphic function fields. Weak approximation is a birational property meaning that if two varieties are birational then one has weak approximation if and only if the other does. There is another birational property of varieties known as R-equivalence, which can be thought of as the algebro-geometric analog of the topological notion of path connectedness. We will study how these two birational properties interact with each other in the context of algebraic groups.

I will then talk about how studying the interaction between these two properties helped me resolve a conjecture proposed by Platonov around 1978 relating to weak approximation.

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