ALGEBRA AND ARITHMETIC (ALGAR) 2024
BRAUER GROUPS IN ARITHMETIC AND GEOMETRY

Contents

1. Content description

Central simple algebras over fields and the Brauer group of a field are a classical topic of study in algebra and number theory. Their theory was developed in the 1920s, based on earlier study of division rings by Wedderburn and Dickson. The Brauer group of a field classifies finite-dimensional central division algebras over the field up to isomorphism. The earliest general discoveries on the Brauer group comprise the fact that the Brauer group of a finite field or the function field of a curve over $\mathbb{C}$ are trivial, as well as the computation of the Brauer group of a local or global number field.

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The study of central simple algebras over fields naturally extends to that of Azumaya algebras over commutative rings. Also the concept of the Brauer group extends to this setting. An even more general approach to the Brauer group is via Galois (and étale) cohomology. In this way one comes naturally to the notion of the Brauer group of a scheme.

On a more concrete level, the easiest examples of central simple (or Azumaya) algebras other than matrix algebras are *quaternion algebras*. Their study provides a natural link to quadratic form theory.

In this year's summer school, we will explore on the one hand the different ways to look at the Brauer group and the algebraic objects which are classified by it, and on the other hand we want to provide an introduction to the various ways in which the Brauer group can be applied to describe obstructions for certain varieties to show that they are not rational.

In one thread of lectures, *Nicolas Garrel* will provide a conceptual approach to the theory of Azumaya algebras over commutative rings and central simple algebras over fields, and hence to the definition of the Brauer group.

Julian Lyczak will discuss how Brauer groups are used to answer questions on the confluence of number theory and algebraic geometry.

Anne Quéguiner-Mathieu will shed a light on the ambiguous role of tensor products of quaternion algebras in the context of the 2-torsion part of the Brauer group of a field and the crucial difference in looking at central simple algebras up to isomorphism or up to Brauer equivalence.

Federico Scavia will present one of the first examples of a 3-dimensional unirational variety over $\mathbb{C}$ which is not rational, due to Artin and Mumford. The proof uses the unramified Brauer group as a crucial ingredient.

Throughout the summer school, we assume fluency with basic algebraic structures covered in most bachelor's programmes, and some familiarity with the following concepts:

- Tensor products of modules and algebras over commutative rings [7]
- Projective modules [7]
- Quaternion algebras – [5, Chap. 1]
- Central simple algebras – [5, Sections 2.1-2.4]
- Discrete valuations – [5, Appendix A.6]

2. Introductory days

The first two days, Thursday 22 and Friday 23 August, will give an introduction in the form of exercise workshops. We will solve exercises training the basic concepts of central simple algebras, quaternion algebras, tensor producs, projective modules and discrete valuations.

3. Main talks

3.1. **The Hasse principle (Julian Lyczak).** We consider the question of how to determine if a system of polynomial equations has a solution over $\mathbb{Q}$. Answering this question is in general very hard, but there are techniques which sometimes prove that there cannot be any solutions. For instance, the existence of a solution over $\mathbb{Q}$ implies the existence of *local solutions*. Local solutions are relatively easily understood (for example through *Hensel's lemma*) and if there are no local solutions then there cannot be a rational solution.

We will introduce the *Hasse principle*, which captures the idea that the converse also can hold: the existence of local solutions implies the existence of a rational solution.

We will discuss some postive results, such as the Hasse–Minkowski theorem, which states that the Hasse principle does hold for quadratic equations. We will then see that the Hasse principle can fail if one moves to systems with more equations or to equations of higher degree, and we highlight the role that quadratic reciprocity often plays in these cases.

3.2. **Morita theory (Nicolas Garrel).** The goal of this lecture is to introduce Azumaya algebras and the Brauer group of a commutative ring R, such that they emerge as naturally as possible from a minimal framework. The central context is that of a category in which the objects are the R-algebras and the morphisms from A to B are the A-B-bimodules. We call this the Morita category.

Then *Morita equivalences* between R-algebras are just the isomorphisms in this category, and the Brauer group of R is the group of those isomorphism classes which are invertible with respect to the tensor product. We also introduce the *Picard group of* R as the automorphism group of R itself.

3.3. **Rationality and the unramified Brauer group (Federico Scavia).** The simplest algebraic varieties are rational varieties, that is, those which are closest to projective space. In this lecture, we first define various degrees of rationality of varieties: *rational*, *stably rational*, and *unirational*. We state the classical results proving that these notions are not equivalent to each other in general, and in particular we recall the *Artin–Mumford example*: the first example of a unirational non-stably rational complex variety [3].

Let X be a complex variety, with function field $K/\mathbb{C}$. We define the *unramified Brauer group* of $K/\mathbb{C}$ as the subgroup of the Brauer group of K consisting of the Brauer classes with trivial residue at every discrete valuation ring $\mathbb{C} \subset A \subset K$ with fraction field K. If X is stably rational, the unramified Brauer group of K is trivial, that is, the non-vanishing of the unramified Brauer group is an *obstruction to stable rationality*. We give a formula to compute residues, but take the existence of residue maps as a given: we will return to this in Lecture 3.9.

3.4. **Azumaya algebras (Nicolas Garrel).** In this lecture we retrieve classical properties of Azumaya algebras over a commutative ring R from our abstract definition, relating it to other possible definitions in the process.

For instance, we prove that the inverse Brauer class of an Azumaya algebra A is given by the class of its opposite algebra A^{op}. We describe the ideals and the center of A and characterize Azumaya algebras as being central and separable. We also give the criterion that A is Azumaya if $A_{\mathfrak{p}}$ is a central simple algebras for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

To emphasize how the theory becomes more subtle over a commutative ring R compared to the situation where R is a field, we show that the classical Skolem-Noether theorem has the Picard group of R as an obstruction.

3.5. **The Brauer–Manin obstruction (Julian Lyczak).** We will change the problem from Lecture 3.1 into a geometric one, by considering varieties: the geometric object described by a system of polynomial equations.

Using the notion of unramified Brauer groups and residue maps, we will give explicit examples of elements in Brauer groups of varieties.

In this geometric terminology we will rephrase rational solutions and local solutions from the previous lecture into *rational points* and *adelic points*. We will show how the set of rational points embeds into the set of adelic points. We will describe the *Brauer–Manin obstruction*, which can explain why there are no rational points, even if there are adelic points.

3.6. **The Artin–Mumford example (Federico Scavia).** This lecture is devoted to the proof of the unirationality and stable irrationality of the Artin–Mumford example. We follow the proof of Colliot-Thélène and Ojanguren [4]. Let X be a conic bundle over complex projective plane, that is, the function field of X is isomorphic to the function field of a conic over $\mathbb{C}(x, y)$. We first give sufficient conditions for the unirationality of X and, most importantly, for the non-vanishing of the unramified Brauer group of the function field of X. Then, by explicit residue calculations, we show that these conditions are satisfied when X is the Artin–Mumford example.

3.7. **Descent and cohomology (Nicolas Garrel).** This lecture serves as a connecting point with the other courses, where the Brauer group has a more cohomological flavour. In order to do so, we introduce the technique of faithfully flat descent for modules and algebras, and show that Galois descent is a special case of that. We then explain how Azumaya algebras are the descents of endomorphism algebras of finite projective modules (the equivalent of matrix algebras over fields), and how the Brauer group embeds into the second cohomology group of the base ring. We also discuss the difference between the Brauer group and the torsion subgroup of this cohomology group.

3.8. **Milnor K-theory and the Brauer group (Anne Quéguiner-Mathieu).** Let F be a field of characteristic different from 2. In 1970, Milnor published his famous paper entitled *Algebraic K-theory and quadratic forms* [11]. Even though it is not formally stated in this way, the paper suggests the existence, for any field F of characteristic different from 2, of a commutative triangle of isomorphisms relating three graded rings, namely Milnor's K-theory modulo 2 defined by generators and relations, Galois cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, and the graded Witt ring, related to quadratic form theory. This conjecture was proved by Voevodsky [16] in 2003 and Orlov-Vishik-Voevodsky [12] in 2007.

In this lecture, we will study in details the degree 2 part of this triangle, and more specifically the relation between Milnor's K-theory modulo 2 and Galois cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ in degree 2. Since the second Galois cohomology group $H^2(F, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to the 2-torsion part of the Brauer group, the result, which was originally proved by Merkurjev [10], provides a description by generators and relations of the subgroup of the Brauer group which consists of Brauer classes of order 2. As we will explain, it leads to the observation that a central simple algebra of exponent 2 is always Brauer equivalent to a tensor product of quaternion algebras.

3.9. **Construction of the residue maps (Federico Scavia).** In Lecture 3.3, we defined the unramified Brauer group of a function field as the intersection of the kernels of certain residue maps. We gave a formula to compute the residue maps, but the existence of these maps was left as a black box. This lecture is devoted to sketching a definition of the residue maps. We begin with necessary preliminaries in group cohomology and Galois cohomology. We then construct the residue map for fields of Laurent series, and then use this to construct residue maps for function fields. When the ground field contains enough roots of unity, we recover the explicit formula used in Lecture 3.3.

3.10. **The Brauer group in arithmetic geometry (Julian Lyczak).** We will introduce some common techniques from the literature to compute the Brauer group and find explicit representations of its elements. We will work through some examples and use the Brauer–Manin obstruction to show that some equations do not admit rational solutions.

We give a brief overview of the leading conjectures in the area of arithmetic geometry, which show that understanding the Brauer–Manin obstruction is crucial for varieties which are geometrically not too complicated, but that for more complex varieties one does need additional techniques.

3.11. **Indecomposable algebras (Anne Quéguiner-Mathieu).** As in Lecture 3.8, we work over a base field F of characteristic different from 2. An F-central simple algebra D of degree 2 is isomorphic to a quaternion algebra. More precisely, given a quadratic subfield $F(\sqrt{a}) \subset D$, one has $D \simeq (a, x)_F$ for some $x \in F^\times$. By a classical theorem due to Albert [1, Thm. XI.9], a similar result holds in degree 4: Any central simple F-algebra D of degree 4 and exponent 2 contains a bi-quadratic commutative subfield $F(\sqrt{a}, \sqrt{b})$ and is isomorphic to a tensor product of two quaternion algebras $(a, x) \otimes (b, y)$ for some x and $y \in F^\times$. The situation is quite different in degree 8. Rowen [15] proved that a degree 8 and exponent 2 central simple algebra D over F always contains a tri-quadratic subfield $F(\sqrt{a}, \sqrt{b}, \sqrt{c})$, but Tignol noticed that it generally does not admit a decomposition as $(a, x) \otimes (b, y) \otimes (c, z)$ for some $x, y, z \in F^\times$. In 1979, elaborating on this example, Amitsur, Rowen and Tignol [2] constructed an example of a central simple algebra D of degree 8 and exponent 2 which is not isomorphic to a tensor product of three quaternion algebras. This construction, which is the main topic of this lecture, has proved useful to obtain various counterexamples in the theory of quadratic forms and algebras with involution.

4. Special talks

4.1. **The Brauer dimension of a field (Shilpi Mandal, Emory University).** Let K be a field. Recall that the *index* of a central simple algebra A over K is the degree (i.e. the square root of the dimension) of the unique division algebra over K that is equivalent to A, and that the order of the class of A in $\mathsf{Br}(K)$ is called its *period*. We denote them by $\mathsf{ind}(A)$ and $\mathsf{per}(A)$. It is well-known that $\mathsf{per}(A)$ divides $\mathsf{ind}(A)$, and that conversely $\mathsf{ind}(A)$ divides $\mathsf{per}(A)^d$, for some $d \in \mathbb{N}$; see [5, Section 4.5].

In the spirit of [13], we define the following for a field K. The *Brauer l-dimension of K* for a prime number l, denoted by $\mathsf{Br}_l\mathsf{dim}(K)$, is the smallest $d \in \mathbb{N} \cup \{\infty\}$ such that for every finite field extension L/K and every central simple L-algebra A of period a power of l, we have that $\mathsf{ind}(A)$ divides $\mathsf{per}(A)^d$. The supremum on $\mathsf{Br}_l\mathsf{dim}(K)$ where l runs over all primes is denoted by $\mathsf{Brdim}(K)$ and called the *Brauer dimension of K*.

If $\mathsf{Br}(K) = 0$, then we take the convention that $d = 0$. This holds in particular when K is algebraically closed, but also when K is a function field over an algebraically closed field, due to a theorem of Tsen [5, Theorem 6.2.8], or more generally any field of cohomological dimension 1.

If K is a number field or a local field (a finite extension of the field of p-adic numbers $\mathbb{Q}_p$, for some prime number p), then classical results from class field theory tell us that $\mathsf{Brdim}(K) = 1$, see [14, § 18.6].

It is expected that the growth of this invariant under a field extension is bounded by the transcendence degree. Some recent works in this area include that of Lieblich [8], Harbater-Hartmann-Krashen [6] for K a complete discretely valued field, in the good characteristic case. In the bad characteristic case, for such fields K, Parimala-Suresh have given some bounds in [13].

In my research, I am looking at the relation between the Brauer dimensions of a complete non-archimedean valued field and of its residue field.

4.2. **Orders in positive definite quaternion algebras (Jakub Krásensky, TU Prague).** Quaternion orders are nice discrete subrings of quaternion algebras, and they have many applications. For example, the simplest known proof of Lagrange's four-square theorem uses the order of so-called Hurwitz quaternions. This is the subset of (Hamilton) quaternions generated as $\mathbb{Z}$-module by 1, i, j and $(1+i+j+k)/2$. The ring of Hurwitz quaternions has many nice properties which are not shared by the seemingly more natural set of integral quaternions $\mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}k$. For example, it admits a certain kind of unique factorisation. In this talk, we will have a closer look at orders which can be embedded into the skew-field of Hamilton quaternions; we introduce some of their basic properties and look how they can be used to prove certain analogues of the four-square theorem.

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